On the Optimal Control of Volterra Integro-Differential Equations

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Abstract—This contribution constitute an theoretic work devoted to the class of optimal control problems (OCPs) involving a specific dynamics described by Volterra integro-differential equations. We study OCPs associated with the Volterra integro-differential systems and establish the solvability property of this class of problems. A special structure of the abstract dynamic optimization problem under consideration makes it possible to interpret the initially given sophisticated OCP as a separate convex optimization problem in a suitable Hilbert space. This fact implies effective splitting type solution schemes in combination with the first-order computational methods for the numerical treatment of the initially given OCPs. We concretely consider the celebrated Armijo-type gradient approach for this purpose. Finally, we give a mathematically rigorous proof of the numerical consistency of splitting algorithm applied to the main OCP.

I. INTRODUCTION

Volterra-type integro-differential dynamic systems arise in many real-world applications such as biology, bio-medical and social sciences, economy and financial engineering (see e.g., [1], [11], [20], [23]). The dynamic behaviour of the Volterra integro-differential models constitutes an adequate formal modelling approach to various problems from the modern biomathematics, population dynamics and right body mechanics [25], [24], [29], [38], [41]. Let us also refer to some interesting applications of the Volterra integro-differential equations in neural networking [28] and in modeling of consumer systems [38]. Originated in some classic works of V. Volterra [35], [36], [37] the integro-differential equations of convolution type constitute nowadays a useful and appropriate modelling framework for control systems with a persistent process "memory".

A new class of control processes under consideration naturally involves the corresponding OCPs associated with this class of systems. In our paper, we study a formal model which describes the optimally controlled development of two interacting biological species (see e.g., [36], [37] and [13]). Note that optimal control of biological populations constitutes a challenging, important and relative new part of the modern control theory (see e.g., [33] and references therein). However, the recent theoretic development in optimal control of general integro-differential systems is not sufficiently advanced by the implementable numerical solution approaches.

We consider here a specific OCP associated with a Volterra-type integro-differential dynamic system and establish some basic analytic properties of this sophisticated optimization problem. A specific structure of the OCP we study makes it possible to consider this non-standard dynamic optimization problem in the context of a separate convex optimization program in a real Hilbert space. Finally, we propose a gradient approach of the Armijo type for a constructive numerical treatment of the initially given sophisticated OCP (see also [5], [6], [7], [8], [9], [34]).

The remainder of this paper is organized as follows: Section II contains a mathematically rigorous introduction to the separate convex optimization and to the class of dynamic systems under consideration. In Section III we establish some necessary properties of the Volterra integro-differential equations of convolution type. Section IV contains the main OCP involving the general Volterra integro-differential dynamic system. We prove here a formal existence result. Section V deals with the numerical analysis of the resulting optimal solution procedure. We establish the main equivalence result and show that the initially given complicated OCP with the controlled Volterra integro-differential equations is equivalent to a separate convex program in a suitable Hilbert space. This section also presents the Armijo type gradient method as a conceptual solution procedure for the OCP under consideration. Section VI summarizes our paper.

II. ELEMENTS OF THE CONVEX PROGRAMMING

Let $H = H_1 \otimes H_2$ be a Cartesian product of two real Hilbert spaces. Consider the following minimization / maximization problem

\[
\begin{align*}
\text{extremize } & J(v_1, v_2) \\
\text{subject to } & (v_1, v_2) \in \mathcal{V}_1 \otimes \mathcal{V}_2 \subset H
\end{align*}
\]

(1)

where $\mathcal{V}_1 \subset H_1$, $\mathcal{V}_2 \subset H_2$ are bounded, convex sets and $J(\cdot, v_2) : H_1 \to [-\infty, \infty]$, $J(v_1, \cdot) : H_2 \to [-\infty, \infty]$ are proper convex or concave (i.e., one can be convex while the other is concave) functionals for $v_1 \in H_1$, $v_2 \in H_2$, respectively. We next use the following natural notation

\[ v := (v_1, v_2), \quad \mathcal{V} := \mathcal{V}_1 \otimes \mathcal{V}_2. \]

Recall that a convex functional $J(\cdot, v_2)$ is called "proper" if $J(\cdot, v_2) \neq -\infty$ (over $H_1$) and its effective domain

\[ \text{dom}[J(\cdot, v_2)] := \{ v_1 \in H_1 \mid J(v_1, v_2) < \infty \} \]
is a non-empty set. Evidently, the same concept can also be applied to \( J(v_1, \cdot) \). We next omit the word "proper", because only such convex functionals will be considered in this paper.

Let us note that the optimization problem (1) belongs to the "separate convex programming" class and constitutes an abstract framework for many practically oriented optimization problems (see e.g., [4], [16], [17], [18]). In addition to the above formal conditions, we next suppose that \( J(\cdot) \) is bounded on a set
\[
\mathcal{Y} + \varepsilon \mathcal{B} \subset \text{int}\{\text{dom}\{J(\cdot, \cdot)\}\},
\]
where \( \varepsilon > 0 \) and \( \mathcal{B} \) is the open unit ball of \( H \). The "separate" functionals \( J(v_1, \cdot) \) and \( J(\cdot, v_2) \) can be convex or concave. Note that the "partial" convexity (or concavity) of \( J(\cdot, v_2) \) and \( J(v_1, \cdot) \) does not imply the "global" convexity (concavity) property of \( J(\cdot) \). For example, the bilinear functional
\[
J(v) = \langle v_1, v_2 \rangle_H,
\]
where \( \langle \cdot, \cdot \rangle_H \) denotes a scalar product in \( H \), is a non-convex functional. In the case of a convex \( J(\cdot, v_2) \) and concave \( J(\cdot, v_2) \) we call (1) a convex-concave problem. We next denote the basic problem (1) as a "minimization problem" if "extremize" in (1) is replaced by "minimize". Consequently, the "maximization version" of (1) corresponds to "maximize" instead of "extremize".

Since \( J(\cdot) \) is assumed to be bounded on \( \mathcal{Y} + \varepsilon \mathcal{B} \), we conclude that this objective functional is continuous on this set (see [18], [26], [31]). Consequently, it is also Lipschitz continuous on the set \( \mathcal{Y} \) (see [31], Thm. 10.4) and the existence of an optimal solution
\[
\psi^{\text{opt}} := (v_1^{\text{opt}}, v_2^{\text{opt}}) \in \mathcal{Y}
\]
to problem (1) is guaranteed by application of the Weierstrass Theorem (see e.g., [15], [26]).

In the next section, based on the previous observations, we study a concrete OCP associated with a specific linear Volterra integro-differential equations. The abstract optimization problem (1) provides a necessary theoretic framework for this concrete OCP. The elements (objective functional, restrictions) of the main OCP we next consider have the same properties as the corresponding elements of the abstract problem (1). This section contains some prerequisite facts that are useful for the further analysis of optimal control processes governed by the Volterra integro-differential equations of convolution type.

Let us recall the following fundamental concept from Variational Analysis [26], [30], [32].

**Definition 1:** A sequence \( \{v^k\} \subset H, \ k = 0, 1, \ldots \) is called a minimizing (or maximizing) sequence for (1) if
\[
\lim_{k \to \infty} J(v^k) = \min_{v \in \mathcal{Y}} J(v) \quad \text{(or max} J(v))\).
\]

We also need the following simple convexity property of a combined functional (see e.g., [9] or [31]).

**Lemma 1:** Let \( \psi^1 : \mathbb{R}^n \to \mathbb{R} \) be a convex functional and \( \psi^2 : H \to \mathbb{R}^n \) be an affine function on \( H \). Then, the combined functional \( \psi : H \to \mathbb{R} \) given by \( \psi(v) := \psi^1(\psi^2(v)) \) is convex.

Note that Lemma 1 can be proved by a direct calculation (see e.g., [9]). We now consider the "minimization" version of the basic optimization problem (1) and introduce the following function:
\[
F(\hat{v}_2) := \min_{v_1 \in \mathcal{Y}} J(v_1, \hat{v}_2), \quad \hat{v}_2 \in \mathcal{Y}.
\]
Here \( \hat{v}_2 \in \mathcal{Y} \) is a fixed element. Consider now the auxiliary minimizing problem
\[
\text{minimize } F(\hat{v}_2)
\]
subject to \( \hat{v}_2 \in \mathcal{Y} \).

Since (2) and (3) constitute the conventional convex (or concave) programs, the auxiliary function \( F(\cdot) \) in (2) is well-defined and the existence of an optimal solution \( \hat{v}_2^0, v_2^0 \in \mathcal{Y} \) to the above problems is guaranteed (see [31]). Evidently, \( v_2^0 \) depends on a concrete selection \( \hat{v}_2 \in \mathcal{Y} \) in (2). The next fundamental result provides a theoretic basis for the proof of numerical consistence of a computational algorithm we finally propose. This implementable computational scheme we next develop makes it possible an effective numerical treatment of the main OCPs under consideration.

**Lemma 2:** The value
\[
\theta := \min_{\hat{v}_2 \in \mathcal{Y}} F(\hat{v}_2)
\]
in (3) is the overall minimal value for the initially given problem (1). A solution set \( \text{Argmin}_{v \in \mathcal{Y}} J(v) \) of (1) is given as follows
\[
\text{Argmin}_{v \in \mathcal{Y}} J(v) = J^{-1}(\theta).
\]

In the case \( (v_1^{\text{opt}}, v_2^{\text{opt}}) \in \text{Argmin}_{v \in \mathcal{Y}} J(v) \) we also have
\[
(v_1^{\text{opt}}, v_2^{\text{opt}}), \ (v_1^{\text{opt}}, v_2^{\text{opt}}) \in \text{Argmin}_{v \in \mathcal{Y}} J(v).
\]

A formal proof of Lemma 2 can be found in [9]. Finally note that Lemma 2 provides an analytic basis for various "splitting" methods in the general mathematical programming (see e.g., [16], [17], [21] and references therein). The main difficulty of this "nested optimization" approach consists in a constructive determination of an expression \( \hat{v}_2^{\text{opt}} \). For a convex (or concave, or convex-concave) case Lemma 2 constitutes a natural solution approach.

### III. On a Class of Volterra Integro-Differential Equations

We now consider the following initial value problem for a scalar Volterra integro-differential equation with the bounded time delay
\[
\dot{z}(t) = a(t)z(t) + \int_0^t \phi(t-s)z(s)ds + u(t), \quad t \geq 0, \quad z(0) = z_0 \in \mathbb{R}.
\]

Here \( z(\cdot) \) is a state variable and \( u(\cdot) \in L_2(\mathbb{R}_+, \mathbb{R}) \) is a control input. By \( L_2(\mathbb{R}_+, \mathbb{R}) \) we denote a space of all square integrable scalar functions. We next assume that \( a(\cdot), \ \phi(\cdot) \) are a continuous functions. Under the presented hypotheses there exists a unique (absolutely continuous) solution \( z^d(\cdot) \) of
associated with a concrete control input \( u(\cdot) \in L^2(\mathbb{R}_+; \mathbb{R}) \) (see e.g. [13] and references therein). Let us also note that a simple integration applied to the both sides of (4) leads to a generic Volterra integral equation with a convolution-type kernel. The integral term in (4) evidently expresses the generic process "memory".

In parallel with (4) we introduce a homogeneous (non-controlled) version

\[
\dot{\xi}(t) = a(t)\xi(t) + \int_0^t \phi(t-s)\xi(s)ds, \quad \xi(0) = 1, \quad (5)
\]

of the above Volterra integro-differential system. Let \( \xi(\cdot) \) be a solution of (5). We next establish the following technical result.

**Lemma 3**: Let \( u(\cdot) \in L_2(\mathbb{R}_+; \mathbb{R}) \) and \( \xi(\cdot) \) be a solution to (5). Then \( z^{\mu}(\cdot) \) is determined by the following equation:

\[
z^{\mu}(t) - \int_0^t z^{\mu}(s)\left[ a(s) - a(t-s) \right] \xi(t-s)ds = \int_0^t d\xi(t-s)z^{\mu}(s)ds.
\]

Using (7) and the differential equations in (4)-(5), we deduce

\[
\int_0^t d\xi(t-s)z^{\mu}(s)ds = \int_0^t \left[ -a(t-s)\xi(t-s) - \int_0^{t-s} \phi(t-s-r)\xi(r)dr \right] z^{\mu}(s) + \int_0^t \phi(t-s)\xi(t-s)u(s)ds.
\]

This implies

\[
z^{\mu}(t) - \xi(t)z_0 = \int_0^t z^{\mu}(s)\left[ a(s) - a(t-s) \right] \xi(t-s)ds + \int_0^t \xi(t-s)u(s)ds.
\]

and we get relation (6). The proof is completed. \( \square \)

As we can see Lemma 3 determines an affine relationship between a control input \( u(\cdot) \) and the generated trajectory \( z^{\mu}(\cdot) \) of system (4). That means the trajectory \( z^{\mu}(\cdot) \) of (4) generated by an admissible control \( u(\cdot) \) is an affine function of this control input. Let us also mention a direct corollary of Lemma 3.

**Lemma 4**: Let \( u(\cdot) \in L_2(\mathbb{R}_+; \mathbb{R}) \) and \( z^{\mu}(\cdot) \) be the corresponding solution to (5). Then the following functional\n
\[
P : L_2(\mathbb{R}_+; \mathbb{R}) \to \mathbb{R}, \quad P(u(\cdot)) := z^{\mu}(t)
\]

is affine for every fixed \( t \in \mathbb{R}_+ \). For the case of a constant function \( a(t) = \text{const} \) we obtain the explicit formulae

\[
z^{\mu}(t) = \xi(t)z_0 + \int_0^t \xi(t-s)u(s)ds.
\]

Lemma 4 can be proven by taking into consideration the basic relation (6) and the linearity of the conventional integral and differential operators.

**IV. OPTIMAL CONTROL OF VOLterra INTEGRO-DIFFERENTIAL SYSTEMS**

Let us now consider a celebrated formal model describing the controlled development of two interacting biological species due originally to Volterra [36], [37]. We also refer to [13], [20] for some extensions and further mathematical details. The dynamic model we consider contains the nonhomogeneous (controlled) Volterra integro-differential equations of the type (4)

\[
x_1(t) = (a_1 - b_1x_2(t))x_1(t) - \int_0^t \varphi_1(t-s)x_1(s)x_2(t-s)ds + c_1u_1(t),
\]

\[
x_2(t) = (a_2 + b_2x_1(t))x_2(t) + \int_0^t \varphi_2(t-s)x_1(s)x_2(t-s)ds + c_2u_2(t),
\]

\[
t \in [0, T], \quad T \in \mathbb{R}_+,
\]

\[
x_1(0) = x_{1,0} \in \mathbb{R}, \quad x_2(0) = x_{2,0} \in \mathbb{R}.
\]

Here \( x_1(t) \) denotes the number of biological species at time \( t \) with the nutrition "controlled" by a constrained variable \( u_1(t) \in U_1 \subset \mathbb{R} \). Moreover, \( x_2(t) \) is a number of species that depends on \( x_1(t) \) for its development. The nutrient concentration for this second specie is formalized by the input variable \( u_2(t) \in U_2 \subset \mathbb{R} \). The numbers

\[
a_1, \ b_1, \ c_1, \ i = 1, 2
\]

are some positive system parameters. The nonnegative continuous functions \( \varphi_1(\cdot), \varphi_2(\cdot) \) in (8) represent the biologically motivated "heredity characteristics" (or process "memory" factors).

We next assume that \( u_1(\cdot), u_2(\cdot) \in L_2(\mathbb{R}_+; \mathbb{R}) \) and consider compact and convex admissible control sets \( U_1, \ U_2 \). Additionally we introduce the following useful notation:

\[
\nu(t) := (u_1(t), u_2(t))^T, \quad x(t) := (x_1(t), x_2(t))^T, \quad t \in \mathbb{R}_+,
\]

\[
\mathcal{W} := \mathcal{W}_1 \otimes \mathcal{W}_2, \quad \mathcal{W}_i := \{w(\cdot) \in L_2(\mathbb{R}_+; \mathbb{R}) \mid w(t) \in U_i \},
\]

where

\[
\mathcal{W}_i := \{w(\cdot) \in L_2(\mathbb{R}_+; \mathbb{R}) \mid w(t) \in U_i \},
\]

and \( i = 1, 2 \). From Section II it follows that for every admissible control input \( \nu(\cdot) \in \mathcal{W} \) there exists a unique (absolutely continuous) solution

\[
x(\cdot) := (x_1(\cdot), x_2(\cdot))^T
\]

to system (8). Note that every component of the above solution to system (8) depends on a complete control input \( \nu(\cdot) \).

We are now ready to formulate the main OCP for the Volterra integro-differential system (8). Let

\[
L : W^{1,1}(\mathbb{R}_+; \mathbb{R}) \times W^{1,1}(\mathbb{R}_+; \mathbb{R}) \to \mathbb{R},
\]
be a convex-concave functional (convex in the first argument and concave with respect to the second argument). Here \( W^{1,1} (\mathbb{R}_+, \mathbb{R}) \) is a Sobolev space of all absolutely continuous functions. In general, we assume that \( L(\cdot, \cdot) \) is a Mayer-type functional (see e.g., [26]), namely,

\[
L(x_1(\cdot), x_2(\cdot)) := h(x_1(T), x_2(T)),
\]

where \( h : \mathbb{R}^2 \to \mathbb{R} \) is a continuously differentiable convex-concave function. For example, functional \( L(\cdot, \cdot) \) can have the following quadratic structure:

\[
L(x_1(\cdot), x_2(\cdot)) = h(x_1(T), x_2(T)) := k_1 x_1^2(T) - k_2 x_2^2(T),
\]

where \( k_1, k_2 \) are positive constants. Recall that \( T \in \mathbb{R}_+ \) in (8)-(10) is a terminal time. Consider the following (main) OCP

\[
\text{minimize } L(x_1(\cdot), x_2(\cdot))
\]

subject to (8), \( v(\cdot) \in \mathcal{W} \).

We first establish the existence result for (10).

**Theorem 1:** Consider OCP (11) with

\[
\mathcal{W} := \mathcal{W}_1 \otimes \mathcal{W}_2, \quad \mathcal{W}_i := \{ w(\cdot) \in L_2(\mathbb{R}_+, \mathbb{R}) \mid w(t) \in U_i \}
\]

for \( i = 1, 2 \). Assume that \( \phi_1(\cdot) \) and \( \phi_2(\cdot) \) are a continuous functions and \( a_i, b_i, c_i \), \( i = 1, 2 \) are positive constants. Moreover, let \( h : \mathbb{R}^2 \to \mathbb{R} \) be a continuously differentiable convex-concave function. Then the main OCP (11) possesses an optimal solution.

**Proof.** Since \( U_1, U_2 \) are assumed to be compact and convex, the set of admissible controls \( \mathcal{W} \) is norm-bounded closed and convex [15], [27]. These properties imply the weak-compactness of this set in the Hilbert space \( L_2(\mathbb{R}_+, \mathbb{R}^2) \) (see e.g., [2], [27]). Since the following Volterra control-operator in (4)

\[
u(\cdot) \to a(\cdot) z(t) + \int_0^T \phi(t-s) z(s) ds + u(t)
\]

is a linear bounded operator, it is continuous with respect to the corresponding control input. This implies continuity of \( \tilde{z}(\cdot) \) with respect to \( u(\cdot) \in L_2(\mathbb{R}_+, \mathbb{R}) \). The continuity of the function \( h(\cdot, \cdot) \) and the established continuity property of \( x_i^*(\cdot), i = 1, 2 \) with respect to \( v(\cdot) \) imply the continuity property of the combined functional

\[
\bar{L}(\nu(\cdot)) := L(x_1^*(\cdot), x_2^*(\cdot)).
\]

Here \( \nu(\cdot) \in \mathcal{W} \) and \( (x_1^*(\cdot), x_2^*(\cdot))^T \) are the corresponding solutions of system (8). This fact implies continuity of \( \bar{L}(u_1(\cdot), u_2(\cdot)) \) in the \( L_2(\mathbb{R}_+, \mathbb{R}) \) - weak topology. Using the proved weak-compactness of \( \mathcal{W} \) and the celebrated Weierstrass Theorem (see e.g., [15], [26]) we next deduce the existence of an optimal solution to the minimization problem

\[
\text{minimize } \bar{L}(\nu(\cdot))
\]

subject to \( \nu(\cdot) \in \mathcal{W} \).

The minimization problem (13) constitutes the equivalently rewritten main OCP (11). The proof is completed. \( \Box \)

We next denote by \( (v^{opt}(\cdot), x^{opt}(\cdot)) \) an optimal solution (optimal pair) of the main OCP (11). Let us also note that the equivalent representation (13) of the initially given OCP (11) does not include the Volterra integro-differential dynamic system (8). This fact is a simple consequence of the specific (implicit) definition (12) of the objective functional \( \bar{L}(\cdot) \) that involves a solution \( (x_1^*(\cdot), x_2^*(\cdot)) \) to system (8).

Recall that various dynamic systems of the Lotka-Volterra type as well as the corresponding optimal control problems constitute an important formal modelling approach to real-world biological and social processes. We refer to [11], [20], [24], [29], [41] for the necessary results related to the various predator-prey equations, population growth and competing species models. The Volterra type approach is also actively used in the biological theory of mutualism and symbiosis [35], [38], [39]. Since the formal Lotka-Volterra-type systems describe adequately the natural competing processes, many researchers extended the initially biologically inspired Volterra modelling framework to numerous applications in optimal economical behaviour (see e.g., [23] and references therein). Some interesting applications of the Lotka-Volterra equations to financial engineering and specifically to optimal hedging strategies can be found in [1], [23].

The consideration of the time-delays in the Volterra integro-differential system (8) finally implies a higher level of adequateness of the resulting biological social and financial models. Clearly, several types of time-delays are inevitable in these systems (biological, social, economical). The same observation is also true with respect to the “process memory” or “heredity characteristic” modelled by the convolution-type integral terms in equations (8).

Finally note that the main OCP (11) with a general convex-concave functional (9) or (10) expresses a possible optimal behaviour of the second competing “biological specie” (or of a competing financial institution) described by \( x_2 \) under the “degrading” dynamics of the first “biological specie” given by \( x_2 \). Let us also mention that various alternative practically motivated formulations of optimal control problems are conceivable in the context of system (8). For example, one can examine several types of minimax (or maximin) type optimization problems associated with the (delayed) Volterra integro-differential system (8).

**V. NUMERICAL ASPECTS**

We now use the analytical facts and methods from Section II and develop an Armijo-type gradient numerical approach to the sophisticated OCP (11). Following the “nested optimization” concept expressed in Lemma 2 (see Section II) we next fix an admissible control input \( \tilde{u}_2(\cdot) \in \mathcal{W}_2 \) and consider first equation in (8). The solution of this equation for an admissible \( u_1(\cdot) \in \mathcal{W}_1 \) is evidently parametrized by \( x_2(\cdot) \in W^{1,1}(\mathbb{R}_+, \mathbb{R}) \) and we use the following notation

\[
x_1^*(\cdot) \cdot x_2(\cdot)
\]

for this parametric solution. Note that the above “parametrization” is formally determined by the following
Volterra operator
\[ x_2(\cdot) \rightarrow (a_1 - b_1x_2(t))x_1(t) - \int_0^t \phi_1(t-s)x_1(t-s)x_2(s)ds \]
which is a compact operator (see e.g., [10]). Since every compact operator is continuous (see e.g., [27]), the \(x_2(\cdot)\)-parametrization \(x_2^0(\cdot)[x_2(\cdot)]\) introduced above is a continuous parametrization.

**Theorem 2:** Assume that the conditions of Theorem 1 are satisfied. Then the main OCP (11) constitutes a convex-concave version of the abstract extremal problem (1).

**Proof.** We have to specify all the components of the abstract optimization problem (1) in the concrete OCP (10). Evidently, we can put
\[ H_1 \equiv H_2 = L_2(\mathbb{R}_+, \mathbb{R}). \]
The "separated" objective functional (12)
\[ \tilde{L}_1(u_1(\cdot), \hat{u}_2(\cdot)) := L(x_1^u(\cdot)[x_2(\cdot)], x_2) \]
corresponds to the abstract objective function \(J(\hat{v}_1, \cdot)\) in (2). Applying Lemma 3 (formulae (6)) to system (8), we get next
\[ x_1^u(t)[x_2(\cdot)] - \int_0^t x_1^u(s)[x_2(\cdot)][(a_1 - b_1 x_2(s)) - (a_1 - b_1 x_2(t-s))] \xi_1(t-s)ds = \]
\[ \xi_1(t)x_{1,0} + \int_0^t \xi_1(t-s)u_1(s)ds = \]
\[ \xi_1(t) = (a_1 - b_1 x_2(t)) \xi_1(t) + \int_0^t \phi_1(t-s)x_2(t-s)\xi_1(s)ds, \]
(14)
Here \(\xi_1(\cdot)\) is a solution to the non-controlled system associated with the first equation in (8) (similar to (5))
\[ \xi_1(t) = (a_1 - b_1 x_2(t)) \xi_1(t) + \int_0^t \phi_1(t-s)x_2(t-s)\xi_1(s)ds, \]
(15)
Here \(\xi_1(\cdot)\) is the solution of the second equation from (8) also parameterized by \(x_1(\cdot) \in W^{1,1}(\mathbb{R}_+, \mathbb{R})\). The proof is completed.

Let us observe that the homogeneous equation (15) for \(\xi_1(\cdot)\) and the corresponding equation for \(\xi_2(\cdot)\) (when an admissible \(\hat{u}_1(\cdot)\) is fixed) can be numerically solved by application of the conventional Laplace transform (see e.g., [10]). We will discuss the Laplace transform based numerical solution approach to (15) in an example studied below.

Theorem 2 makes it possible to apply the basic numerical Lemma 2 to the main OCP (11) involved the Volterra integro-differential system (8). In conformity to the abstract auxiliary problems (2.2)-(2.3) we now formulate the corresponding auxiliary problems for the sophisticated OCP (3.4):
\[ \text{minimize } \tilde{L}_1(u_1(\cdot), \hat{u}_2) \]
\[ \text{subject to } u_1(\cdot) \in U_1 \]
(16)
and
\[ \text{minimize } L_2(\hat{u}_1(\cdot), u_2(\cdot)) \]
\[ \text{subject to } u_2(\cdot) \in U_2 \]
(17)
Taking into consideration the basic assumptions from Section III and the convex-concave character of the objective functional \(L(x_1(\cdot), x_2(\cdot))\), we easy conclude that problem (16) is a convex program and problem (17) is a concave program. Since the trajectory \(x(\cdot) = (x_1^u(\cdot), x_2^u(\cdot))\) of system (8) is Fréchet differentiable (with respect to the control input \(v(\cdot) \in U\), see e.g., [15], [26]) and function \(h(\cdot)\) in (9) is assumed to be continuously differentiable, the objective functional \(L(v(\cdot))\) in (12) is Fréchet differentiable. This fact evidently implies the Fréchet differentiability of the specific objective functionals in problems (16) and (17).

Let us now consider the celebrated gradient-based methodology for an effective numerical treatment both of the above auxiliary problems. We discuss shortly the projected gradient approach and also refer to the fundamental work [3] for the necessary mathematical details and proofs. Let \(\mathcal{P}_U\) be the operator of projection on to convex set \(U_i, i = 1, 2\). The Armijo-type projected gradient method for (16) has the following formal expression:
\[ u_1^{(0)} \in U_1, \]
\[ u_1^{(l+1)} = \gamma_1 \mathcal{P}_U[u_1^{(l)}(\cdot)] - \alpha_1 \nabla_{u_1} L_1(u_1(\cdot), \hat{u}_2(\cdot)) ] + (1 - \gamma_1) u_1^{(l)}(\cdot), \]
(18)
\[ u_2^{(l+1)} = \gamma_2 \mathcal{P}_U[u_2^{(l)}(\cdot)] + \alpha_2 \nabla_{u_2} L_2(\hat{u}_1(\cdot), u_2(\cdot)) ] + (1 - \gamma_2) u_2^{(l)}(\cdot), \]
(19)
Taking into account that \(-L_2(\hat{u}_1(\cdot), u_2(\cdot))\) is convex we can apply the same gradient-type method to problem (17):
\[ u_1^{(0)} \in U_1, \]
\[ u_1^{(l+1)} = \gamma_1 \mathcal{P}_U[u_1^{(l)}(\cdot)] + \alpha_1 \nabla_{u_1} L_1(u_1(\cdot), \hat{u}_2(\cdot)) ] + (1 - \gamma_1) u_1^{(l)}(\cdot), \]
\[ u_2^{(l+1)} = \gamma_2 \mathcal{P}_U[u_2^{(l)}(\cdot)] + \alpha_2 \nabla_{u_2} L_2(\hat{u}_1(\cdot), u_2(\cdot)) ] + (1 - \gamma_2) u_2^{(l)}(\cdot), \]
(18)
Here \(\alpha_1, \alpha_2, \gamma_1, \gamma_2, i = 1, 2\) are sequences of some suitable step sizes. By \(V\) we denote here the Fréchet derivative of the corresponding functional.

The projected gradient method (19) for the second auxiliary problem (17) and the possible Armijo-type step size selection can be accomplished by the same manner as in the case of method (18) for problem (16). Recall that under some weak assumptions on the optimization problems the projected gradient iterations (18)-(19) generate minimizing sequences for the auxiliary optimization problems under consideration (see also [12], [14], [40]).
VI. CONCLUSION

In this paper, we have initiated a theoretical consideration of a class of OCPs associated with the control systems described by Volterra integro-differential equations. Our contribution has a conceptual nature and is devoted to the necessary analytic foundations of the numerical approaches to optimal control problems involving the complex Volterra type dynamics.

Using the classic properties of the Volterra integro-differential systems, we firstly represent the initially given sophisticated Volterra type OCP as a separate convex program in a suitable Hilbert space. Application of the extended splitting type approach in combination with the Armijo gradient method makes it possible to develop an effective solution procedure for the initially given sophisticated OCP.

The main theoretical and numerical approaches proposed in this paper can also be applied to the Volterra type integro-differential systems with unbounded delays (see [13]) and to the corresponding Volterra type OCPs. Let us finally note that the theoretic solution schemes we propose need to be really extended by the comprehensively examples, case studies and by the corresponding numerical simulations.

REFERENCES