Approximability of nonlinear affine control systems

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ABSTRACT

This paper focuses on a strong approximability property for nonlinear affine control systems. We consider control processes governed by ordinary differential equations (ODEs) and study an initial system and the associated generalized system. Our theoretical approach makes it possible to prove a strong approximability result for the above dynamical systems. The latter can be effectively applied to some classes of variable structure and hybrid control systems. In particular, this paper deals with applications of the strong approximability property obtained to the conventional sliding mode processes and to hybrid control systems with autonomous location transitions. We also take into consideration some optimal control problems for the above class of hybrid systems.

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1. Introduction and motivation

Consider the following initial value problem for a system with an affine right-hand side:

\[ \dot{x}(t) = a(t, x(t)) + b(t, x(t))u(t) \quad \text{a.e. on } [0, t_f], \]
\[ x(0) = x_0, \]

where \( x_0 \in \mathbb{R}^n \) is an initial state, and the functions \( a : (0, t_f) \times \mathcal{R} \rightarrow \mathbb{R}^n \) and \( b : (0, t_f) \times \mathcal{R} \rightarrow \mathbb{R}^n \times m \) are uniformly bounded on an open set

\( (0, t_f) \times \mathcal{R} \subseteq \mathbb{R}^{n+1}, \)

measurable with respect to variable \( t \in (0, t_f) \) and uniformly Lipschitz continuous in \( x \) (with a common constant for almost all \( t \)). Recall that the above functions \( a \) and \( b \) are often denoted as Carathéodory functions (see [1–3] for theoretical details).

From the general existence/uniqueness theory for ordinary differential equations (ODEs) with Carathéodory right-hand sides, it follows that, for every \( u(\cdot) \) from \( L^1_{\text{loc}}(0, t_f) \), where \( L^1_{\text{loc}}(0, t_f) \) is the Lebesgue space of all \( m \)-valued integrable functions, problem (1) has a unique absolutely continuous solution \( x(\cdot) \). Motivated by numerous applications, let us consider system (1) over a set \( \mathcal{U} \) of bounded measurable control inputs. In this paper, we assume that the set of admissible controls \( \mathcal{U} \) has the following structure:

\[ \mathcal{U} := \{ v(\cdot) \in L^1_{\text{loc}}(0, t_f) \mid \|v(t)\| \leq U \text{ a.e. on } [0, t_f] \}, \]

where \( U \subseteq \mathbb{R}^m \) is a compact and convex set. We call a dynamical system of the type (1) a nonlinear affine control system (NACS).

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The aim of our investigations is to study the dynamical structure of the basic equation from (1) and to establish a strong approximability property of this system. Moreover, we are also interested in extending the approximability results for (1) to some classes of affine hybrid systems. Let us introduce the following approximability concept associated with (1).

**Definition 1.** We say that the NACS (1) possesses a strong approximability property (with respect to the control inputs) if every \( L^1_{\mathbb{R}}(0, t_f) \)-weakly convergent sequence \([u^k(\cdot)]\) of admissible control inputs \( u^k(\cdot) \in \mathcal{U} \) generates a uniformly convergent sequence of the corresponding trajectories.

Note that Definition 1 generalizes the conventional continuity concept studied in [1]. Recall that the classical Filippov continuity results are obtained under conditions of strongly convergent right-hand sides (see [1]). In this paper, we consider some general perturbations of controls and initial states and establish the uniform convergence properties of the solutions to (1). Moreover, we apply the abstract techniques of the set-valued analysis, rewrite the given affine control systems in the generalized form of Filippov, and study the approximability properties associated with the corresponding relaxed NACSs. This makes it possible to consider some special kind of robustness for conventional sliding mode control processes. We also apply the general theoretical approximability results obtained to a class of hybrid systems with autonomous location transitions.

The remainder of our paper is organized as follows. Section 2 contains a classical concept of sliding mode control processes and also indicates a class of hybrid systems under consideration. Section 3 is devoted to some basic mathematical results from the celebrated Filippov-type theory for discontinuous dynamical systems. Moreover, we present a useful abstract result. In Section 4, we present our main result; namely, we establish the strong approximability property of (1). In this section, we also give a specific overestimation for the dynamical state vector of system (1). The analytical facts from the above-mentioned Filippov-theory are applied in Section 4 to conventional sliding mode control processes. The theoretical results obtained are illustrated by a computational example. Applications of our main result to affine hybrid systems with autonomous location transitions and to the corresponding optimal control problems are discussed in Section 5. Section 7 summarizes the article.

### 2. Mathematical models of conventional sliding mode and hybrid control processes

In this section, we discuss the formal mathematical models of the dynamical systems under consideration, namely, of sliding mode and hybrid control processes. Driven by engineering requirements, there has been increasing interest in general structural properties of sliding mode, of hybrid and, in general, of variable structure systems [4]. In particular, continuity and the approximability properties of a control system with respect to controls and initial states play an important role in mathematical control theory. These properties can first be interpreted from the point of view of possible consistent numerical (for example, discrete) approximations of dynamical systems. They can also be applied to a specific robustness analysis of the affine systems under consideration.

The NACS introduced in Section 1 represents the main dynamical model for classical sliding mode control theory (see, e.g., [5,6]). The sliding mode based control design for system (1) is usually related to a bounded feedback control strategy of the form

\[ w(t, x) := \tilde{w}(\sigma(t, x), \dot{x}(t), x, \ldots, \sigma^{(l-1)}(t, x)), \]

where \( \tilde{w} : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a bounded measurable (feedback) control function and \( \sigma : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^l \) is a smooth s-valued output of the system under consideration. The number \( l \in \mathbb{N} \) is a relative degree associated with the given system (see, e.g., [7,8] for details). In this case, one also deals with a time-dependent control function that is determined on trajectories of the system as a composite function \( u_w(t) \equiv w(t, x(t)) \) for a trajectory \( x(\cdot) \) of (1). Since \( w(\cdot) \) is a bounded and \( x(\cdot) \) an absolutely continuous function, there exists a compact convex set \( \mathcal{U} \subseteq \mathbb{R}^m \) such that \( u_w(\cdot) \in \mathcal{U} \). The sliding mode control technique has become a modern application focus of nonlinear control theory [5,8–10]. During the last two decades, there has been considerable effort to develop theoretical and computational frameworks for classical and high-order sliding modes (see, e.g., [5,9]).

In parallel to the classical NACS (1) and to the above-mentioned sliding mode processes, we also study a hybrid version of the above NACSs. In this paper, we restrict our consideration to nonlinear affine hybrid systems with autonomous location transitions.

**Definition 2.** A nonlinear affine hybrid system (NAHS) is a 7-tuple

\[ \{ \mathcal{Q}, \mathcal{X}, \mathcal{U}, \mathcal{A}, \mathcal{B}, \mathcal{U}, \Psi \}, \]

where

- \( \mathcal{Q} \) is a finite set of discrete states (called locations);
- \( \mathcal{X} = \{ \mathcal{X}_q \}, q \in \mathcal{Q} \), is a family of state spaces such that \( \mathcal{X}_q \subseteq \mathbb{R}^n \);
- \( \mathcal{U} \subseteq \mathbb{R}^m \) is a set of admissible control input values (called the control set);
An admissible hybrid trajectory associated with an NAHS is a triple
\[ x(t) \in \mathbb{R}^n, \tau \in \mathbb{R}^m, \]
where \( x(t) \) is a function in \( \mathbb{R}^n \) that is continuously differentiable for almost all \( t \in [0, t_f] \) and \( x(0) \) is a location in \( \mathbb{R}^n \). We denote by \( \mathcal{A} = \{ a_q(\cdot, \cdot) \}, \mathcal{B} = \{ b_q(\cdot, \cdot) \}, q \in \mathcal{Q} \) are families of uniformly bounded on open set \( (0, t_f) \times \mathcal{R} \) Caratheodory functions.

\[ a_q, b_q : (0, t_f) \times \mathcal{R} \to \mathbb{R}^n, \mathbb{R}^{n \times m}; \]
\( \mathcal{U} \) is the set of admissible control functions introduced above;
\( \Psi \) is a subset of \( \mathcal{X} := \{(x, q', x') : q, q' \in \mathcal{Q}, x \in \mathcal{X}_q, x' \in \mathcal{X}_{q'} \} \).

We suppose that smooth functions \( m_{q, q'} : \mathbb{R}^n \to \mathbb{R} \), where \( q, q' \in \mathcal{Q} \), are given such that the surfaces \( M_{q, q'} := \{ x \in \mathbb{R}^n : m_{q, q'}(x) = 0 \} \) are pairwise disjoint. The surfaces \( M_{q, q'} \) represent switching sets at which a switch from location \( q \) to location \( q' \) takes place. We say that a location switching from \( q \) to \( q' \) occurs at a switching time \( t_{\text{switch}} \in [0, t_f] \). We now consider an NAHS with \( r \in \mathbb{N} \) switching times \( 0 = t_0 < t_1 < \cdots < t_{r-1} < t_r = t_f \). Note that the above sequence of switching times \( \{ t_i \} \) is not defined a priori. A hybrid control system remains in location \( q_i \in \mathcal{Q} \) for all \( t \in [t_{i-1}, t_i) \), where \( i = 1, \ldots, r \). We refer to [11,12,4] for some alternative concepts of hybrid systems and note that the theoretical approach proposed in this paper can also be applied to wide classes of hybrid/switched systems with the above nonlinear affine structure.

**Definition 3.** An admissible hybrid trajectory associated with an NAHS is a triple \( X := (x(\cdot), \{ q_i \}, \tau) \), where \( x(\cdot) \) is a \( \alpha \)-continuous part of the trajectory, \( \{ q_i \}_{i=1}^r \) is a finite sequence of locations, and \( \tau := \{ t_i \}, i = 1, \ldots, r \) is the corresponding sequence of switching times such that

\[ x(0) = x_0 \notin \bigcup_{q_i, q_{i+1} \in \mathcal{Q}} M_{q_i, q_{i+1}}, \]

and for each \( i = 1, \ldots, r \) and every admissible control \( u(\cdot) \in \mathcal{U} \) we have
\begin{itemize}
  \item \( x_i(\cdot) = x(\cdot)|_{[t_{i-1}, t_i]} \) is an \( \alpha \)-continuous function on \( (t_{i-1}, t_i) \) continuously prolongable to \( [t_{i-1}, t_i] \), \( i = 1, \ldots, r \);
  \item \( x_i(t) = a_{q_i}(t, x_i(t)) + b_{q_i}(t, x_i(t))u_i(t) \) for almost all times \( t \in [t_{i-1}, t_i] \), where \( u_i(\cdot) \) is a restriction of the chosen control function \( u(\cdot) \) on the time interval \( [t_{i-1}, t_i] \).
\end{itemize}

Note that the pair \( (q, x(t)) \) represents the hybrid state at time \( t \), where \( q \) is a location \( q \in \mathcal{Q} \) and \( x(t) \in \mathbb{R}^n \). **Definition 3** describes the dynamics of the given NAHS. Since \( x(\cdot) \) from **Definition 3** is a \( \alpha \)-continuous function, this concept describes a class of hybrid systems without impulse components of the (continuous) trajectories. Therefore, the corresponding switching sets \( M_{q, q'} \) are defined for \( x_i(t_i) = x_{i+1}(t_i), i = 1, \ldots, r-1 \). Under the above assumptions, for each admissible control \( u(\cdot) \in \mathcal{U} \) and for every interval \( [t_{i-1}, t_i] \) (for every location \( q_i \in \mathcal{Q} \)) there exists a unique absolutely continuous solution of the nonlinear affine differential equations from **Definition 3**. This means that for each \( u(\cdot) \in \mathcal{U} \) we have a unique absolute continuous trajectory of the NAHS under consideration. Moreover, the switching times \( \{ t_i \} \) and the discrete trajectory \( \{ q_i \} \) for an NAHS are also uniquely defined. The evolution equation for the trajectory \( x(\cdot) \) of an NAHS can also be represented as follows:

\[ \dot{x}(t) = \sum_{i=1}^r \chi_{[t_{i-1}, t_i]}(t)(a_{q_i}(t, x(t)) + b_{q_i}(t, x(t))u_i(t)) \quad \text{a.e. on } [0, t_f], \]
\[ x(0) = x_0, \]
where \( \chi_{[t_{i-1}, t_i]}(\cdot) \) is the characteristic function of the interval \( [t_{i-1}, t_i] \) for \( i = 1, \ldots, r \). Note that for classes of hybrid systems of the NAHS type the function \( \chi \) also depends on the trajectory of the given system. Evidently, this dependence is characterized by the specific switching mechanism introduced above. As mentioned above, a sequence \( \tau \) is not defined a priori. However, for every \( \tau \) from **Definition 3** the structure of the initial value problem (3) is the same.

**3. On the Filippov approach to differential equations with discontinuous right-hand sides**

We now provide some relevant mathematical facts that will be used in the subsequent sections. Consider the initial value problem given by

\[ \dot{z}(t) = g(z(t)), \quad \text{a.e. on } [0, t_f], \]
\[ z(0) = z_0, \]
where \( g : \mathbb{R}^n \to \mathbb{R}^n \) is Lebesgue measurable and locally essentially bounded (bounded on a neighborhood of every point, excluding sets of measure zero [1]). The Filippov set-valued map \( \mathcal{K}[g] : \mathbb{R}^n \to 2^{\mathbb{R}^n} \) related to problem (4) is defined as follows:

\[ \mathcal{K}[g](z) := \bigcap_{\delta > 0} \bigcap_{\delta} \mathcal{C}(g(V_\delta(z)) \setminus \delta). \]

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where \( V_\delta(z) \) is an open ball centered at \( z \) with radius \( \delta \). Here \( \mu(\delta) = 0 \). Note that by \( \mu \) we denote a Lebesgue measure and \( \mathcal{C} \) denotes a closure of a convex hull. Dynamical systems of the form

\[
\dot{z}(t) \in \mathcal{K}[g](z(t)) \quad \text{a.e. on } [0, t_f],
\]

\[
z(0) = z_0,
\]

are called differential inclusions (see [1,3]). An absolutely continuous function \( z : [0, t_f] \rightarrow \mathbb{R}^n \) that satisfies (5) is said to be Filippov solution of the original initial value problem (4). As a consequence of the general theory of differential inclusions, we obtain the existence of a set \( \delta_g \subset \mathbb{R}^n \) of measure zero such that, for every set \( W \subset \mathbb{R}^n \) of measure zero,

\[
\mathcal{K}[g](z) \equiv \mathcal{C} \left\{ \lim_{j \to \infty} g(z_j) \mid z_j \rightarrow z, \ z_j \not\in \delta_g \bigcup W \right\}.
\]

Since the set-valued map \( \mathcal{K}[g] \) introduced above is upper semi-continuous with nonempty, convex and compact values, and it is also locally bounded (see [1]), it follows that a Filippov solution to (4) exists. Generally, a Filippov solution to a dynamical system of type (5) is not necessarily unique.

Let us now recall that the celebrated Filippov selection lemma (see, e.g., [13]) gives rise to an explicit parameterization of the convex-valued differential inclusion of type (5). We formulate here a consequence of this result, and also refer to [14,3] for theoretical details.

**Proposition 1.** A function \( z(\cdot) \) is a solution of (5) if and only if it is a solution of the Gamkrelidze system

\[
\dot{\eta}(t) = \sum_{j=1}^{n+1} \alpha^j(t) \gamma_j(t, \eta(t), u^j(t)) \quad \text{a.e. on } [0, t_f],
\]

\[
\eta(0) = z_0, \quad \alpha(\cdot) \in \Lambda(n + 1),
\]

where \( \gamma : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuous, \( \gamma(t, \cdot, \cdot) \) is a continuously differentiable function with the bounded derivative \( \gamma_\eta \),

\[
\alpha(\cdot) := (\alpha^1(\cdot), \ldots, \alpha^{n+1}(\cdot))^T, \quad u^j(\cdot) \in U,
\]

and

\[
\Lambda(n + 1) := \left\{ \alpha(\cdot) | \alpha^j(\cdot) \in \mathbb{R}^n_+(0, t_f), \ \alpha^j(t) \geq 0, \ \sum_{j=1}^{n+1} \alpha^j(t) = 1 \ \forall t \in [0, t_f] \right\},
\]

is a set of admissible multipliers \( \alpha(\cdot) \).

The right-hand side of (6) (the function \( \gamma \) ) can be evaluated constructively for every concrete \( \mathcal{K}[g] \) from (5). For example, the initial affine system (1) leads to the Gamkrelidze system (6) with the following right-hand side:

\[
\gamma(t, \eta(t), u^j(t)) = a(t, \eta(t)) + \sum_{j=1}^{n+1} \alpha^j(t)b(t, \eta(t))u^j(t).
\]

Let us also note that in this case we have

\[
\mathcal{K}[g](z(t)) = a(t, z(t)) + b(t, z(t))U.
\]

**Proposition 1** establishes a parameterization of the convex-valued inclusion (5) by an equivalent control system, namely, by system (6). Clearly, every trajectory of (5) is also a trajectory of (6). Evidently, the vector function \( \alpha(\cdot) \) introduced above takes its values from the simplex of the space \( \mathbb{R}^{n+1} \). It is also possible to prove that every \( \alpha^j(\cdot), j = 1, \ldots, n + 1 \) takes only two values, namely 0 and 1, and can be interpreted as a characteristic function of a measurable subset of \([0, t_f]\) (see [14] for details). Note that a pair

\[
u(\cdot) := ((\alpha^1(\cdot), \ldots, \alpha^{n+1}(\cdot))^T, (u^1(\cdot), \ldots, u^{n+1}(\cdot))^T)\]

from the Cartesian product \( \Lambda(n + 1) \times U^{n+1} \) can be considered as a control vector for the relaxed system (6).

Finally, let us present the following abstract result.

**Theorem 1.** Let \( \mathcal{X} \) be a separable Banach space and let \( \{S, \Sigma, \mu\} \) be a measurable space with a probability measure \( \mu \). Let \( C \subset \mathcal{X} \) be closed and convex. If \( h : S \rightarrow C \) is a \( \mu \)-measurable function, then \( \int h(\tau)\mu(d\tau) \in C \).

---

1 Note that an initial value problem for a nonstationary ordinary differential equation \( \dot{z}(t) = \tilde{g}(t, z(t)) \) can be reduced to the above problem (4) by introducing the additional equation \( f = 1 \), the extended vector \( z := (z^T, t)^T \), and the right-hand side \( g := (g^T, 1)^T \).
Evidently, there is a radius $R$ such that $C \cap V(\beta, R) = \emptyset$. Using a separating theorem from convex analysis (see, e.g., [15]), we obtain a nontrivial $L \in \mathcal{X}^*$ with $\|\xi - \beta\|_X < R$ for all variables $\xi \in V(\beta, R)$. $\psi \in C$. By $\mathcal{X}^*$ we have denoted the (topological) dual space to $\mathcal{X}$. Thereby we have the inequality

$$\sup_{\|\xi\|_X \leq 1} (\mathcal{L}(\beta + R\xi)) = \mathcal{L}\beta + R|\mathcal{L}| \leq \mathcal{L}(h(\tau)), \quad \tau \in S,$$

and by integration with respect to $\mu$ – the corresponding inequality

$$\mathcal{L}\beta + R|\mathcal{L}| \leq \int \mathcal{L}(h(\tau))\mu(d\tau). \quad (7)$$

Because

$$\int \mathcal{L}(h(\tau))\mu(d\tau) = \mathcal{L} \int h(\tau)\mu(d\tau) = \mathcal{L}\beta,$$

(7) leads to $\mathcal{L}\beta + R|\mathcal{L}| \leq \mathcal{L}\beta$, contradicting the fact that $\mathcal{L}$ is nontrivial. Therefore $\beta$ belongs to $C$. The proof is finished. \(\square\)

We will use Theorem 1 in the next section and derive an overestimation of the state variables of the NACSs under consideration.

4. Strong approximability of conventional affine control systems

This section is devoted to the strong approximability properties of the NACSs introduced in Section 1. The original NACS (1) is studied over the set of measurable control functions (the elements of $L^1_m(0, t_f)$). In parallel with this main control space, we also consider its (topologically) dual, namely, the space $L^\infty_m(0, t_f)$ of all essentially bounded $m$-valued functions. For an element $v(\cdot) \in L^1_m(0, t_f)$, $\psi(\cdot) \in L^\infty_m(0, t_f)$, the corresponding duality pairing is denoted by $\langle v(\cdot), \psi(\cdot) \rangle_{1, \infty}$. Recall that a sequence $\{u^k(\cdot)\}$ from $L^1_m(0, t_f)$ converges weakly to a function $u(\cdot) \in L^1_m(0, t_f)$ if

$$\lim_{k \to \infty} \langle u^k(\cdot), v(\cdot) \rangle_{1, \infty} = \langle u(\cdot), v(\cdot) \rangle_{1, \infty}$$

for all $v(\cdot) \in L^\infty_m(0, t_f)$. We use the notation $C_n(0, t_f)$ for the Banach space of all $n$-valued continuous functions on $[0, t_f]$ equipped with the usual sup-norm $\|\cdot\|_{C_n(0, t_f)}$. We now are able to present our main theoretical result. This result establishes a fundamental continuity property of the NACSs under consideration.

**Theorem 2.** Consider the initial value problem (1), an associated sequence $\{u^k(\cdot)\}$, $k \in \mathbb{N}$ of control functions $u^k(\cdot) \in L^1_m(0, t_f)$, and a sequence of vectors $\{x^k_0\}$ from $\mathbb{R}^n$ such that $\{u^k(\cdot)\}$ converges weakly to an element $u(\cdot) \in L^1_m(0, t_f)$ and $\lim_{k \to \infty} x^k_0 = x_0$. Then, for all $k \in \mathbb{N}$, the problem

$$\begin{align*}
\dot{x}(t) &= a(t, x(t)) + b(t, x(t))u^k(t) \quad \text{a.e. on } [0, t_f], \\
x(0) &= x^k_0
\end{align*} \quad (8)$$

has a unique (absolutely continuous) solution $x^k(\cdot)$ on the time interval $[0, t_f]$ and, moreover,

$$\lim_{k \to \infty} \|x(\cdot) - x^k(\cdot)\|_{C_n(0, t_f)} = 0,$$

where $x(\cdot)$ is the solution of (1) corresponding to $u(\cdot)$.

**Proof.** Evidently, the graph

$$\text{Graph}(x(\cdot)) := \{(t, x(t)) \mid t \in (0, t_f)\}$$

of the absolutely continuous solution $x(\cdot)$ to (1) is a relatively compact subset of the set $(0, t_f) \times \mathbb{R}$. For small $\epsilon > 0$, a closed $\epsilon$-neighborhood $\mathcal{V}_\epsilon$ of Graph($x(\cdot)$) also belongs to $(0, t_f) \times \mathbb{R}$. Without loss of generality, we can assume that functions $a$ and $b$ are also uniformly bounded on $\mathcal{V}_\epsilon$, measurable with respect to variable $t$ and uniformly Lipschitz continuous in $x$ (for all $(t, x) \in \mathcal{V}_\epsilon$). From the Carathéodory existence theorem (see [1]), it follows that for all $k \in \mathbb{N}$ there exists a $\delta_k > 0$ such that the initial value problem (8) has a unique solution on the interval $[0, \delta_k]$. This solution can be extended to a time interval $I_k := [0, \Delta_k]$, where

$$\Delta_k \leq \min\{\tau \in (0, t_f) \mid x(\tau) \in \partial \text{Pr}[\mathcal{V}_\epsilon]\}$$

and $\partial \text{Pr}[\mathcal{V}_\epsilon]$ is the boundary of the projection of $\mathcal{V}_\epsilon$ on the space $\mathbb{R}^n$. Define

$$\tilde{x}^k(\cdot) := x^k(\cdot) - x(\cdot), \quad \tilde{u}^k(\cdot) := u^k(\cdot) - u(\cdot), \quad \tilde{x}^k_0 := x^k_0 - x_0.$$
For a.e. \( t \) from \( I_k \), \( k \in \mathbb{N} \), we have the following differential equation:

\[
\dot{x}^k(t) = \dot{x}(t) - a(t, x(t) + \dot{x}(t)) + b(t, x(t) + \dot{x}(t))u(t) - [a(t, x(t)) + b(t, x(t))]u(t)
\]

where \( \dot{x}(t) = \dot{x}^k_0 \). Let \( L \) be a common Lipschitz constant for the above functions \( a \) and \( b \). Then for \( \|\dot{x}\|_{\mathbb{P}^n} \leq \varepsilon \) we have

\[
\|a(t, x + \dot{x}) - a(t, x)\| \leq L\|\dot{x}\|, \\
\|b(t, x + \dot{x}) - b(t, x)\| \leq L\|\dot{x}\|. \\
\forall x \in \mathbb{P}_v. 
\]

Since \( \{u^k(\cdot)\} \) converges weakly to \( u(\cdot) \), we have the weak convergence of the sequence \( \{\dot{u}^k(\cdot)\} \) to the zero element \( 0_{C_m(0, t_j)} \) of the space \( L^1(0, t_j) \). Using the Dunford–Pettis Theorem (see, e.g., [16]), we conclude that all the functions from \( \{u^k(\cdot)\} \) have a common modulus of absolute continuity of the Lebesgue integral, i.e., there exists a nondecreasing continuous function \( \nu : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \lim_{\nu \to 0^+} \nu(\varepsilon) = 0 \) and

\[
\int_0^t \|\dot{u}^k(\tau)\|d\tau \leq \nu(\varepsilon) 
\]

for all \( k \in \mathbb{N} \) and for every (Lebesgue) measurable subset \( E \) of \([0, t_j]\) with measure \( \text{mes}\{E\} \leq \varepsilon \). Let \( K > 0 \) be a boundedness constant for \( a \) and \( b \). Since \( \{\dot{u}^k(\cdot)\} \) converges weakly to \( 0_{C_m(0, t_j)} \), we conclude that

\[
\lim_{k \to \infty} \int_0^t b(\tau, x(\tau))\dot{u}^k(\tau)d\tau = 0 \\
\forall \tau \in [0, t_j].
\]

The obtained pointwise convergence of the above integral in combination with (11) implies the uniform convergence

\[
\lim_{k \to \infty} \int_0^t b(\tau, x(\tau))\dot{u}^k(\tau)d\tau = 0. \\
\| C_m(0, t_j) = 0. \\
\]

Moreover, from (11) it follows that

\[
\int_0^t b(\tau, x(\tau))\dot{u}^k(\tau)d\tau - \int_0^s b(\tau, x(\tau))\dot{u}^k(\tau)d\tau \leq K \int_0^t \|\dot{u}^k(\tau)\|d\tau \leq K \nu(t_1 - t^2)
\]

for all \( t, t^2 \in [0, t_j] \). Therefore, the function \( K\nu(\varepsilon) \) is a common modulus of absolute continuity for all functions \( \int_0^s b(\tau, x(\tau))\dot{u}^k(\tau)d\tau \) on the time interval \([0, t_j]\). We now rewrite the resulting differential equation from (9) in an integral form:

\[
\dot{x}^k(t) = \dot{x}^k_0 + \int_0^t \left[ a(\tau, x(\tau) + \dot{x}^k(\tau)) - a(\tau, x(\tau)) \right]d\tau + \int_0^t \left[ b(\tau, x(\tau) + \dot{x}^k(\tau)) - b(\tau, x(\tau)) \right]u^k(\tau)d\tau
\]

for all \( t \in [0, t_j] \). Let \( M := \max\{K, L\} \). Using (10) and (11), we derive the following estimate for the sup-norm of solution \( \dot{x}^k(\cdot) \) on the time interval \( I_k = [0, \Delta_k] \):

\[
\|\dot{x}^k(\cdot)\|_{C_m(0, \Delta_k)} \leq \|\dot{x}^k_0\| + M\Delta_k\|\dot{x}^k(\cdot)\|_{C_m(0, \Delta_k)} + 2\nu(\Delta_k)M\|\dot{x}^k(\cdot)\|_{C_m(0, \Delta_k)} + \left| \int_0^t b(\tau, x(\tau))\dot{u}^k(\tau)d\tau \right|_{C_m(0, \Delta_k)}.
\]

The last inequality implies that

\[
\|\dot{x}^k(\cdot)\|_{C_m(0, \Delta_k)} \leq \|\dot{x}^k_0\| + \int_0^\Delta b(\tau, x(\tau))\dot{u}^k(\tau)d\tau \leq \|\dot{x}^k_0\| + \int_0^\Delta b(\tau, x(\tau))\dot{u}^k(\tau)d\tau \leq \|\dot{x}^k_0\| + \left| \int_0^t b(\tau, x(\tau))\dot{u}^k(\tau)d\tau \right|_{C_m(0, \Delta_k)}.
\]

For a suitable constant \( 0 < c < 1 \), one can put \( \Delta_k = \Delta \) for all sufficiently large \( k \in \mathbb{N} \), where \( \Delta \) is a solution of the following equation:

\[
M\Delta + 2\nu(\Delta)M = c.
\]

Using the properties of the function \( \nu(\cdot) \) introduced above, we conclude that the last equation has a solution \( \Delta \). From (12), (13), and from the given convergence of the initial points \( \lim_{k \to \infty} \|\dot{x}^k_0\| = 0 \), we finally deduce that

\[
\|\dot{x}^k(\cdot)\|_{C_m(0, \Delta_k)} \leq \frac{1}{(1 - c)} \left( \|\dot{x}^k_0\| + \int_0^\Delta b(\tau, x(\tau))\dot{u}^k(\tau)d\tau \right). 
\]
and that \( \lim_{k \to \infty} \| \tilde{x}^k(\cdot) \|_{C_0(0, \Delta_k)} = 0 \) on a common fixed time interval \( \tilde{t}_k := [0, \Delta] \). We now define a new initial time instant \( \tilde{t}_1 := \Delta \) and consider the next interval \( [\tilde{t}_1, \tilde{t}_1 + \Delta] \) of the length \( \Delta \). Since all the above inequalities/estimates are still valid on this new time interval, the resulting differential equation from (9) also has a solution on the interval \( [\tilde{t}_1, \tilde{t}_1 + \Delta] \) which converges to zero. Further, consider the initial point \( \tilde{t}_j := j \Delta \) for \( j = 2, \ldots \). In a finite number of \( j \)-steps we will cover the full time interval \( [0, t_f] \). Therefore, the initial value problem for the resulting differential equation from (9) with the corresponding initial value \( \tilde{x}_0 \) and the original initial value problem (8) have unique solutions. These solutions are uniquely prolongable on the full time interval \( [0, t_f] \). Moreover, we have

\[
\lim_{k \to \infty} \| \tilde{x}^k(\cdot) \|_{C_0(0, \Delta_k)} = 0
\]
on \( [0, t_f] \). This implies that \( \lim_{k \to \infty} \| x(\cdot) - \tilde{x}^k(\cdot) \|_{C_0(0, t_f)} = 0 \).

Note that Theorem 2 can be interpreted as follows: for the NACS (1) the weak convergence of controls and the convergence of the initial conditions cause the uniform convergence of the corresponding state variables. Clearly, this result can also be interpreted as a kind of “robustness” of NACS (1) with respect to perturbations of controls and initial state variables. We now apply the abstract convexity result from Section 2, namely, Theorem 1, and derive an overestimation for solutions of the initial value problem (1).

It is evident that from the point of view of practical applications it would be interesting to estimate a minimal \( k^* \in \mathbb{N} \) such that for all \( k > k^* \) the norm \( \| x(\cdot) - \tilde{x}^k(\cdot) \|_{C_0(0, t_f)} \) considered in Theorem 2 is less then a given constant. A possible answer to the question can be given when one can prove a uniform convergence in Theorem 2. In our opinion, such a uniform result can be obtained only under some further technical assumptions/ restrictions.

We now use the abstract Theorem 1 from Section 3 and derive a specific estimation for the state vector of (1).

**Theorem 3.** Consider the initial value problem (1) under the conditions of Section 1. Assume that \( a(t, x), b(t, x) \in \mathbb{C} \) for all \( (t, x) \in \mathbb{R}^{n+1} \), where \( C \) is a compact convex subset of \( \mathbb{R}^n \). Let \( U \subseteq \mathbb{R}^m \) be a convex compact set containing 0. Then \( x(t) \in x_0 + tC[1 + \text{diam}(U)] \), where diam(\( U \)) is a diameter of the set \( U \).

**Proof.** Consider the Lebesgue measure \( \mu \) on \( [0, t] \subseteq [0, t_f] \) scaled by 1/t. Then we apply Theorem 1 to our affine control system and compute the state of (1) at a time \( t \in [0, t_f] \):

\[
x(t) = x_0 + t \left[ \int_0^t a(\tau, x(\tau)) \mu(d\tau) + \int_0^t b(\tau, x(\tau)) u(\tau) \mu(d\tau) \right].
\]

Therefore, \( x(t) \in x_0 + tC + tC[1 + \text{diam}(U)] \).

The result presented makes it possible to estimate the state vector of an NACS of type (1) under some general assumptions. In particular, from Theorem 3 it follows that the reachable sets of system (1) on \( [0, t_f] \) belong to the compact convex set \( x_0 + t_f C[1 + \text{diam}(U)] \).

5. Applications to sliding mode dynamics

Let us now apply the theoretical results form the previous section to classical sliding mode dynamics. Returning to control systems of type (1) with the feedback control strategy (2), we consider the corresponding closed-loop system:

\[
\begin{align*}
\dot{x}(t) &= a(t, x(t)) + b(t, x(t)) w(t, x(t)) \quad \text{a.e. on } [0, t_f], \\
x(0) &= x_0,
\end{align*}
\]

where \( w(\cdot, \cdot) \) is a bounded measurable function. We now investigate the continuity/approximability properties of the closed-loop system from (14) under perturbations of controls and initial states. Evidently, some “stable” properties of (14) in the above sense can be interpreted as a kind of robustness for the corresponding control systems with sliding mode regimes. Generally, the state equation from the resulting system (14) is a differential equation with a discontinuous right-hand side. The discontinuity here is determined by the specific form of the chosen feedback control strategy. Function (2) usually guarantees some strong stability properties of the generic sliding mode regimes, for example, asymptotic Lyapunov-type or finite-time stability with respect to the (smooth) sliding surface \( \sigma(t, x) = 0 \) (see [1,9]). Since the closed-loop system from (14) is a special case of (4), we apply the Filippov approach from Section 2 and consider the associated differential inclusion,

\[
\begin{align*}
\dot{\tilde{x}}(t) &\in \mathcal{K}[a, b](\tilde{x}(t)) \quad \text{a.e. on } [0, t_f], \\
\tilde{x}(0) &= (x_0, 0),
\end{align*}
\]

where \( \tilde{x} := (x^T, t)^T \) is the extended state vector,

\[
\mathcal{K}[a, b](\tilde{x}) := \overline{co} \left\{ \lim_{j \to \infty} [a(\tilde{x}_j) + b(\tilde{x}_j) w(\tilde{x}_j)] \mid \tilde{x}_j \to \tilde{x}, \tilde{x}_j \notin \delta \bigcup W \right\},
\]

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Consider the initial value problem (1) under the conditions of Section 2 and the corresponding system (14). Let \( \{w^k(\cdot, \cdot)\} \), \( k \in \mathbb{N} \) be a sequence of some bounded measurable \( L^1_m(0, t_f; \mathbb{R}^n) \)-convergent feedback controls \( w^k : [0, t_f] \times \mathbb{R}^n \rightarrow \mathbb{R}^m \) with

\[
\lim_{k \to \infty} \|w^k(\cdot, \cdot) - w(\cdot, \cdot)\|_{L^1_m(0, t_f; \mathbb{R}^n)} = 0.
\]

Also, let \( \lim_{k \to \infty} x^k_0 = x_0 \). Then, for all \( k \in \mathbb{N} \), the approximating initial value problem

\[
\dot{x}(t) = a(t, x(t)) + b(t, x(t))w^k(t, x(t)) \quad \text{a.e. on } [0, t_f],
\]

where \( x(0) = x^k_0 \), have an absolutely continuous solution \( x^k(\cdot) \) on \([0, t_f]\), and

\[
\lim_{k \to \infty} \|x^k(\cdot) - x(\cdot)\|_{C^0(0, t_f)} = 0,
\]

where \( x(\cdot) \) is a solution generated by the closed-loop system (14).

**Proof.** For every \( k \in \mathbb{N} \), we define

\[
\mathcal{K}^k[a, b](\bar{x}) := \overline{\left\{ \lim_{j \to \infty} [a(\bar{x}_j) + b(\bar{x}_j)w^k(\bar{x}_j)] \mid \bar{x}_j \rightarrow \bar{x}, \bar{x}_j \not\in \delta \cup W \right\}},
\]

and consider the corresponding sequence of approximating initial value problems:

\[
\begin{align*}
\dot{x}^k(t) &\in \mathcal{K}^k[a, b](\bar{x}(t)), \\
\dot{x}^k(0) &\in (x^k_0, 0).
\end{align*}
\]

Since for every \( k \in \mathbb{N} \) the set-valued map \( \mathcal{K}^k[a, b] \) is upper semi-continuous with nonempty, convex and compact values, and it is also locally bounded, it follows that a Filippov solution \( x^k(\cdot) \) to an approximating initial value problem exists (see [1]). On the given trajectories of the closed-loop systems under consideration, we now determine the following functions:

\[
u_w(t) := w(t, x(t)) \quad \text{and} \quad u^k_w(t) := w^k(t, x^k(t)).
\]

Since the measurable functions \( w \) and \( w^k \) are bounded and \( x(\cdot) \), \( x^k(\cdot) \) are absolutely continuous functions, the functions \( u_w \) and \( u^k_w \) are also measurable and bounded. Moreover, there exists a compact convex set \( U \subseteq \mathbb{R}^m \), \( 0 \notin U \) such that \( u_w(t), u^k_w(t) \in U \) for almost all \( t \in [0, t_f] \). For this control set \( U \), we introduce the following differential inclusions:

\[
\begin{align*}
\dot{z}(t) &\in \mathcal{F}_U(z(t)) := a(t, z(t)) + b(t, z(t))U, \\
\dot{z}^k(t) &\in \mathcal{F}_U(z^k(t)) := a(t, z^k(t)) + b(t, z^k(t))U.
\end{align*}
\]

Since the inclusions from (17) have a convex right-hand side, these inclusions are equivalent to the corresponding Gankrelidze system (see Proposition 1). Introduce

\[
v(\cdot) := ((a^1(\cdot), \ldots, a^{n+1}(\cdot))^T, (u_w(\cdot), \ldots, u^{n+1}(\cdot))^T)
\]

and

\[
v^k(\cdot) := ((a^1(\cdot), \ldots, a^{n+1}(\cdot))^T, (u^k_w(\cdot), \ldots, u^{n+1}(\cdot))^T)
\]

for every \( k \in \mathbb{N} \). Evidently, the convergence property

\[
\lim_{k \to \infty} \|w^k(\cdot, \cdot) - w(\cdot, \cdot)\|_{L^1_m(0, t_f; \mathbb{R}^n)} = 0
\]

implies weak convergence of the sequence \( \{u^k_w(t)\} \) to the control function \( u_w(\cdot) \) and also implies weak convergence of \( \{v^k(\cdot)\} \) to \( v(\cdot) \). Therefore, from Theorem 2 it follows that

\[
\lim_{k \to \infty} \|\eta^k(\cdot) - \eta(\cdot)\|_{C^0(0, t_f)} = 0,
\]

where \( \eta(\cdot) \) and \( \eta^k(\cdot) \) are solutions to the corresponding initial value problems for Gankrelidze systems with \( (x_0, v(\cdot)) \) and \( (x^k_0, v^k(\cdot)) \). From the definitions of the right-hand sides of all the inclusions under consideration, we deduce that \( \mathcal{K}[a, b](\bar{x}) \subseteq \mathcal{F}_U(z) \) and \( \mathcal{K}^k[a, b](\bar{x}) \subseteq \mathcal{F}_U(z^k) \) for all \( x \in \mathbb{R}^n \). This means that the set of solutions to (17) contains all the solutions to the initial value problems (15) and (16). In the particular case with

\[
v(\cdot) := ((1, \ldots, 0)^T, (u_w(\cdot), \ldots, 0)^T), \quad v^k(\cdot) := ((1, \ldots, 0)^T, (u^k_w(\cdot), \ldots, 0)^T),
\]

we obtain

\[
\lim_{k \to \infty} \|x(\cdot) - x^k(\cdot)\|_{C^0(0, t_f)} = 0.
\]
It is well known that some applied control strategies are caused by problems with incomplete information. Practically implementable sliding mode control schemes usually contain disturbances in the actuator channel [6,8,9,5], and are in fact observer-based schemes (see, e.g., [6]). This fact motivates the following example.

Example 1. Under the assumptions of possible model uncertainties, an adequate feedback control law of type (2) contains the corresponding errors. This situation can be described, for example, by some bounded additive errors \( \delta_k^k(\cdot) \), \( p = 1, \ldots, (l - 1) \) in the state/ derivatives estimations. The above-mentioned feedback control strategy can now be presented in the following form:

\[
\begin{align*}
  w^k(t, x(t)) := & \tilde{w}(\sigma(t), x(t) + \delta_1^k(t)), \\
  \dot{x}_1(t) := & \tilde{w}(\sigma(t), x(t) + \delta_1^k(t)), \\
  \dot{x}_2(t) := & \dot{x}_1(t) - x_2(t) + w(x(t)),
\end{align*}
\]

where \( \{w^k\} \) is a sequence of feedback control laws generated by a current \( k \)-estimation of the state and the sliding surface. Note that (18) represents a specific type of the norm bounded perturbations of the state vector. As mentioned above, the additive disturbances in (18) constitute a dynamical effect related to differential equations of type (1) with some unknown additive uncertainties in the right-hand sides (see, e.g., [3]).

Assume that all the functions \( \delta_p^k(\cdot) \) are elements of the space \( L^1(0, t_f) \) (recall that \( \sigma(t), x \) from (2) is an \( s \)-dimensional vector). Moreover, let us assume that \( \lim_{k \to \infty} \|\delta_p^k\|_{L^1(0, t_f)} = 0 \) for all \( p = 1, \ldots, (l - 1) \). Evidently, a bounded sequence \( \{w^k\} \) from (18) satisfies the conditions of Theorem 4. Then the trajectories generated by the feedback control strategies \( \{w^k\} \) with uncertainties \( \delta_p^k \) converge uniformly to a trajectory generated by the error-free control (2).

Example 2. Let us now make Example 1 concrete and consider the following two-dimensional control system:

\[
\begin{align*}
  \dot{x}_1(t) &= x_2(t) \\
  \dot{x}_2(t) &= x_1(t) - x_2(t) + w(x(t))
\end{align*}
\]

with \( x_0 = (0.1, 0.1)^T \). We study this system on the time interval \([0, 10]\). The control law in (19) is chosen in feedback form:

\[ w(x) = -3\text{sign}(\sigma(x)), \quad x := (x_1, x_2)^T, \]

where \( \sigma(x) = (5x_1 + x_2) \) is a one-dimensional output of the system under consideration (see Section 1). We now consider additive errors \( \delta_k^k(\cdot) = 1/k^2, \ k \in \mathbb{N} \) such that the disturbed feedback control function \( w^k(x(t)) \) has the following form:

\[ w^k(t, x(t)) := \tilde{w}(\sigma(x(t) + \delta_k^k(t))) \]

(see Example 2). Moreover, let

\[ x_0^k := x_0 + (1/k, 1/k)^T, \quad k \in \mathbb{N} \]

be a disturbance of the initial vector \( x_0 \) (corresponding to Theorem 4). Using standard MATLAB tools for numerical solutions of ordinary differential equations, we obtain the trajectory of the initial system in the absence of disturbances (the black line in Fig. 1) and also trajectories of the disturbed systems (19) with \( x_0^k, \ w^k \) for \( k = 1, \ldots, 4 \) (the green, blue, red, and magenta lines in Fig. 1). The light blue line in Fig. 1 represents the trajectory of the disturbed system (19) for \( k = 10 \).

The corresponding values of the sup-norm

\[ \|x(\cdot) - x^k(\cdot)\|_{C(0, t_f)}, \quad k = 1, \ldots, 4, \quad k = 10, \]

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where $x(\cdot)$ is a solution of the initial system and $x^k(\cdot)$ are solutions of the disturbed systems, are presented in Fig. 2. As we can see, the sup-norm $\|x(\cdot) - x^k(\cdot)\|_{C_n(0,10)}$ examined is less than 0.1 for $k = 10$. Fig. 3 shows the evolution of the Euclidean norms $\|x(t) - x^k(t)\|$, where $k = 1, \ldots, 4$ and $k = 10$, for $t \in [0,10]$.

Finally, note that the presented dynamical behavior of the initial and perturbed systems from the above examples is a numerical illustration of our main continuity result for the sliding mode type of closed-loop systems (Theorem 4).

6. Nonlinear affine hybrid systems

Let us now consider hybrid systems, namely, the NAHSs introduced in Section 1 (Definitions 2 and 3). It is well known that some important structural properties of hybrid and switched dynamical systems cannot be expressed by the same properties as those of active subsystems (locations). For example, the Lyapunov stability of a switched system is generally not a consequence of the corresponding stability properties of all the subsystems (see, e.g., [12]). The same is also true, for example, with respect to optimality (see [17, 4, 11] for details). Otherwise, the dynamical structure of the NACSs make it possible to extend the strong approximability result from the main theorem (Theorem 2) to hybrid control systems under consideration.

**Theorem 5.** For a sequence of locations $\{q_i\} \subset \mathcal{Q}$, where $i = 1, \ldots, r$, consider the initial value problem (3), and sequences of controls $\{u^i(\cdot)\}$, where $u^i(\cdot) \in L^1_m(0, t_f)$, $k \in \mathbb{N}$, such that every $\{u^i(\cdot)\}$ converges weakly to an element $u_i(\cdot) \in L^1_m(0, t_f)$. 

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Moreover, let \( \{x^k_0\} \) from \( \mathbb{R}^n \) be a convergent sequence of the initial vectors with \( \lim_{k \to \infty} x^k_0 = x_0 \). Then, for all \( k \in \mathbb{N} \), the following initial value problem,

\[
\dot{x}(t) = \sum_{i=1}^{r} \chi_{[t_{i-1},t_i)}(t)(a_{q_i}(t, x(t)) + b_{q_i}(t, x(t))u_i(t)) \quad \text{a.e. on } [0, t_f],
\]

\[
x(0) = x^k_0,
\]

has a unique (absolutely continuous) solution \( x^k(\cdot) \) on \([0, t_f]\) and

\[
\lim_{k \to \infty} \|x(\cdot) - x^k(\cdot)\|_{C^0(0,t_f)} = 0,
\]

where \( x(\cdot) \) is the solution of (3) corresponding to \( u(\cdot) \).

Note that Theorem 5 is an immediate consequence of our main theorem (Theorem 2). The corresponding proof contains two additional facts. First, for a given sequence of locations, the structure of the initial value problem (3) is similar to the structure of NACS (1). Moreover, every control \( u_i(\cdot) \) associated with the location \( q_i \) is considered as a reduction of an admissible general control function \( u(\cdot) \) (see Definition 3). Therefore, the weak convergence of all sequences \( \{u^k(\cdot)\} \) from Theorem 5 also implies the weak convergence of the function \( u_i(\cdot) \) to an admissible control function \( u(\cdot) \). As we can see, in contrast to the above-mentioned stability and optimality properties for general hybrid systems, the established continuity property of the NAHS under consideration follows from the same continuity properties of the given subsystems. Evidently, Theorem 5 also determines a robustness-like behavior of the NAHS with respect to possible disturbances of control functions.

We now apply our general approximability result to an optimal control problem associated with the above NAHSs. Note that, with the exception of certain special cases, the solution to an optimal problem in a general setting remains a challenging task (see, e.g., [4,18–21,11,22–24]). This is due to the fact that the two aspects of system behavior, i.e., discrete and continuous, are tightly linked, to such an extent that they cannot be decoupled in an effective and simple way. Thus, approximate numerical algorithms are inevitable in solving these problems. Our main approximability result makes it possible to study the consistency of some usual numerical schemes for hybrid optimal control problems with NAHSs.

Let \( f^0_q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \), where \( q \in \mathcal{Q} \), be continuously differentiable functions. We also assume that the control set has the simple box structure:

\[
U := \{u \in \mathbb{R}^m : v^-_j \leq u_j \leq v^+_j, \; j = 1, \ldots, m\},
\]

where \( v_j^-, v_j^+, j = 1, \ldots, m \) are some constants. Given an NACS we now formulate the following optimal control problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} f^0_{q_i}(t, x(t))dt \\
\text{over all trajectories } X & \text{ of the NAHS.}
\end{align*}
\]

We assume that problem (20) has an optimal solution \( (u^{opt}(\cdot), X^{opt}(\cdot)) \). It is well known that the main tools toward the construction of optimal hybrid trajectories is the Pontryagin-type hybrid maximum principle (see [25,22,23]). The application of the above hybrid maximum principle to problem (20) involves the existence of the adjoint variable \( \psi(\cdot) \) that is a solution of the corresponding boundary value problem (see [4]),

\[
\begin{align*}
\dot{\psi}(t) &= -\sum_{i=1}^{r} \chi_{[t_{i-1},t_i)}(t) \frac{\partial H_{q_i}(t, x^{opt}(t), u^{opt}(t), \psi(t))}{\partial x} \quad \text{a.e. on } [0, t_f], \\
\psi_{r}(t_f) &= 0,
\end{align*}
\]

where

\[
H_{q_i}(t, x, u, \psi) := \langle \psi, (a_{q_i}(t, x)) + b_{q_i}(t, x)u \rangle - f^0_{q_i}(t, x)
\]

is a “partial” Hamiltonian for the location \( q_i \in \mathcal{Q} \). \( x^{opt} \) is a solution to (3) associated with \( u^{opt}(\cdot) \), and \( \langle \cdot, \cdot \rangle \) denotes the scalar product in \( \mathbb{R}^n \). Note that in contrast to conventional optimal control problems the adjoint function \( \psi(\cdot) \) determined by (21) is not an absolutely continuous function (it has “jumps” at switching times \( t_i, \; i = 1, \ldots, r \)). We assume that problem (20) is regular (nonsingular) in the following sense: \( \psi(t) \neq 0 \) for all \( t \in [0, t_f] \setminus \mathcal{Y} \), where \( \mathcal{Y} \) is a subset of \([0, t_f]\) of measure zero. When solving sophisticated optimal control problems based on some necessary optimality conditions one can obtain a singular solution. Recall that there are two possible scenarios for a singularity: the irregularity of the Lagrange multiplier associated with the cost functional [26] and the irregularity of the Hamiltonian. In the latter case, the Hamiltonian is not an explicit function of the control function during a time interval. Various supplementary conditions (constraint qualifications) have been proposed under which it is possible to assert that the Lagrange multiplier rule (and the corresponding maximum principle) holds in “normal” form, i.e., that the first Lagrange multiplier is not equal to zero. In this case, the corresponding minimization problem is called regular. We refer to [27,28] for theoretical details. Note that some regularity conditions for general constrained optimal control problems can be formulated as controllability conditions for the linearized system [28].
Evidently, the differential equation from (21) can be rewritten as follows:

$$\dot{\psi}(t) = -\sum_{i=1}^{r} \chi_{[t_{i-1}, t_i)}(t) \left( \begin{bmatrix} \partial a_{t_i}(t, x^{opt}(t)) \partial x \\ \partial b_{t_i}(t, x^{opt}(t)) \partial x \end{bmatrix} + (u^{opt}(t))^T \begin{bmatrix} \partial b_{t_i}(t, x^{opt}(t)) \partial x \end{bmatrix} \right) \dot{\psi}(t) + \frac{\partial f_{q_i}(t, x^{opt}(t))}{x}. \quad (22)$$

The maximality condition of the above-mentioned hybrid maximum principle (see [25]) implies the “bang–bang” structure of the optimal control $u^{opt}(\cdot)$ given by components

$$u_{t_i}^{opt}(t) = 1([\psi^T(t)b_{t_i}(t, x(t))])v_i^+ + (1 - 1([\psi^T(t)b_{t_i}(t, x(t))]))v_i^-, \quad \forall t \in [t_{i-1}, t_i), \; j = 1, \ldots, m, \quad (23)$$

where $1(z) \equiv 1$ if $z \geq 0$ and $1 \equiv 0$ if $z < 0$ for a scalar variable $z$. As we can see, the optimal control is a function of $\psi(\cdot)$ and functions $b_{t_i}(\cdot, \cdot), \; q_i \in \mathcal{Q}$. A practical solution of boundary value problem (21) is usually based on an iterative numerical scheme. We refer to [2,3] for some corresponding computational approaches. Using (22), the $(l+1)$th iteration $\psi^{(l)}(\cdot)$ for (21) can be generally written as

$$\psi^{S,(l+1)}(t) = -L_{S} \left( \sum_{i=1}^{r} \chi_{[t_{i-1}, t_i)}(t) \left( \begin{bmatrix} \partial a_{t_i}(t, x^{opt}(t)) \partial x \\ \partial b_{t_i}(t, x^{opt}(t)) \partial x \end{bmatrix} \right)^T \psi^{S,(l)}(t) + \frac{\partial f_{q_i}(t, x^{opt}(t))}{x} \right), \quad (24)$$

where the components of $u^{opt}(\cdot)$ are given by (23) replacing $\psi(\cdot)$ by $\psi^{S,(l)}(\cdot), \; l \in \mathbb{N}$. Here, $L_{S}(w(\cdot))$ is a sequence of Riemann sums

$$L_{S}(w(\cdot)) := \frac{t}{S} \sum_{s=1}^{S} w \left( \frac{t}{S} \right), \quad s \in \mathbb{N},$$

for the integral $L(w(\cdot)) := \int_{0}^{t} w(t)dt$ of a piecewise continuous function $w(\cdot)$. It is well known (see, e.g., [29]) that a sequence $\{L_{S}\}$ converges pointwise (weakly) to $L$. To put it another way, we have a pointwise convergence of the approximations given by (24) to an exact solution $\psi(\cdot)$ of (21). Since a weak convergence in the space of piecewise continuous functions coincides with the weak convergence, we conclude that a sequence of controls $\{u^{S,(l)}(\cdot)\}$, where every $j$-component of $u^{S,(l)}(\cdot)$ is defined as follows,

$$u_{t_i}^{S,(l)}(t) = 1([\psi^{S,(l)}(t)^Tb_{t_i}(t, x(t))])v_i^+ + (1 - 1([\psi^{S,(l)}(t)^Tb_{t_i}(t, x(t))]))v_i^-, \quad \forall t \in [t_{i-1}, t_i), \; j = 1, \ldots, m,$$

converges weakly to $u^{opt}(\cdot)$. From our main continuity result for hybrid systems, namely, from Theorem 5, we now deduce the strong convergence of the sequence $x^{S,(l)}(\cdot)$ (the trajectories associated with $[u^{S,(l)}(\cdot)]$) to an optimal trajectory $x^{opt}$. That means a numerical consistence (in the sense of a strong convergence of trajectories) of the usual computational schemes applied to boundary value problem (21) associated with the optimality conditions for the hybrid optimal control problem (20). Finally, note that the above-mentioned Theorem 5 can play a fundamental role in accuracy analysis of various numerical algorithms for hybrid optimization problems of type (20).

7. On the Filippov continuity result

This section is devoted to a short comparative analysis between the approximability concept introduced above (see Definition 1) and the classical continuity property of differential equations with discontinuous right-hand sides (see [1]). Recall that the continuity properties of solutions to differential inclusions associated with conventional control systems constitute a classical area of investigations in the theory of dynamical systems. Consider the initial value problem (4) and assume that $g(\cdot)$ and a function $g^*(\cdot)$ are piecewise continuous functions in a finite open domain $G \subset \mathbb{R}^n$. Assume that $G$ consists of a finite number of subdomains $G_i, i = 1, \ldots, L$, such that in each $G_i$ the above functions $g(\cdot), \; g^*(\cdot)$ are continuous up to the boundary of $G_i$. Note that in the case of a time-dependent right-hand side of system (4) one needs some additional technical assumptions related to the above domains $G_i$ (see [1] for details). For $g(\cdot)$ and $g^*(\cdot)$ we now recall the well-known Filippov boundedness condition (see [1]).

Definition 4. Two piecewise continuous functions $g(\cdot)$ and $g^*(\cdot)$ satisfy the Filippov boundedness condition if for each point of continuity $z_1 \in G$ of $g^*(\cdot)$ there exists a point of continuity $z_2 \in G$ of $g(\cdot)$ such that $\|z_1-z_2\| \leq \gamma$ and $\|g(z_2)-g^*(z_1)\| \leq \gamma$ for positive $\gamma$.

Under some additional assumptions, the above Filippov boundedness concept guarantees a continuity property of solutions to (14). Let us present the corresponding continuity result from [1].
Proposition 2. Let all the solutions of the initial value problem (4) exist on the time interval \([0, \tau_f]\) and let their graph be contained in \(G\). Then, for any \(\varepsilon > 0\), there exists \(\gamma > 0\) such that, for any \(g^*(\cdot)\) satisfying the above Filippov boundedness condition, and for any \(z_0^*\) with \(||z_0 - z_0^*|| \leq \gamma\), each solution \(z^*(\cdot)\) of the following initial value problem,

\[
\dot{z}(t) = g^*(z(t)), \quad \text{a.e. on } [0, \tau_f], \\
z(0) = z_0^*,
\]

exists on the same time interval \([0, \tau_f]\). Moreover, \(||z(\cdot) - z^*(\cdot)|| \leq \varepsilon\), where \(z(\cdot)\) is a solution to the initial value problem (4).

Evidently, systems (4) and (25) can be interpreted as particular closed-loop realizations of a general control system of the following type:

\[
\dot{z}(t) = f(z(t), \omega(z(t))), \quad \text{a.e. on } [0, \tau_f], \\
z(0) = z_0,
\]

where \(g(z) = f(z, u(z))\) and \(f(\cdot, \cdot)\) is a smooth enough function. \(\omega(\cdot) \in L^1_{\text{loc}}(0, \tau_f, \mathbb{R}^n)\) is an admissible feedback control. Therefore, Proposition 2 represents an approximability property of a closed-loop system of type (26). In the particular case of a closed-loop variant of the given affine system (1), we have \(f(z, u(z)) = a(z) + b(z)w(z)\) with \(z := (t, x)\), and the Filippov boundedness condition for some chosen piecewise continuous functions \(w(\cdot)\) and \(w^*(\cdot)\) can be made concrete as follows:

\[
||z_1 - z_2|| \leq \gamma, \quad ||w(z_2) - w^*(z_1)|| \leq \gamma.
\]

Let us now discuss the relationship between the two approximability results presented in this paper, namely, between Theorem 4 and Proposition 2. Consider a sequence \(\{w^k(\cdot)\}\) of functions \(w^k : \mathbb{R}^n \to \mathbb{R}^m, k \in \mathbb{N}\) that converges \(L^1_{\text{loc}}(0, \tau_f, \mathbb{R}^n)\)-weakly to \(w(\cdot)\). By \(L^1_{\text{loc}}(0, \tau_f, \mathbb{R}^n)\) we denote here a Lebesgue space of all \(m\)-valued measurable functions on \(\mathbb{R}^n\). Condition (27) written for some \(w^*(z) \equiv w^k(z), k \in \mathbb{N}\) implies a strong convergence of the given sequence \(\{w(\cdot)\}\) in the sense of the \(L^1_{\text{loc}}(0, \tau_f, \mathbb{R}^n)\)-norm. This strong convergence of controls is considered at least in the interior of every subdomain \(G_t, t = 1, \ldots, L\) (see Definition 4). The finite structure of the domain \(G\) from Definition 4 implies the following fact: the set of points of discontinuity associated with every admissible control function \(w(\cdot)\) has a measure zero. Note that this statement is also typical for the right-hand sides of conventional discontinuous control systems. Therefore, in the case of the affine system (25) with piecewise continuous control function \(w(\cdot)\), the above Filippov boundedness condition corresponds to a strong \(L^1_{\text{loc}}(0, \tau_f, \mathbb{R}^n)\)-convergence of the approximating sequence \(\{w^k(\cdot)\}\). Since the trajectories of (20) are absolutely continuous functions, the sequence \(\{u^k_w(\cdot)\}\) of composite functions \(u^k_w(t) \equiv w(z(t))\) is also a \(L^1_{\text{loc}}(0, \tau_f)\)-strongly convergent sequence.

The above strong convergence is considered almost everywhere in \(G\). Under this condition, Proposition 2 guarantees a uniform convergence of the associated trajectories. In contrast to this fact, our main approximability result, namely, Theorem 4, implies a uniform convergence of trajectories under assumptions of the \(L^1_{\text{loc}}(0, \tau_f, \mathbb{R}^n)\)-weak convergence of the sequence \(\{w^k(\cdot)\}\). Since the measurable functions \(w(\cdot)\) and \(w^k(\cdot)\) are bounded and the associated trajectories are absolutely continuous functions, the composite functions \(u^k_w(\cdot)\) and \(u^k_w(\cdot)\), where \(u^k_w(t) \equiv w(x(t))\), are also measurable and bounded. Moreover, in this case the sequence \(\{u^k_w(\cdot)\}\) of composite functions of type \(w^k(x(\cdot))\) is evidently a \(L^1_{\text{loc}}(0, \tau_f)\)-weakly convergent sequence that satisfies the conditions of Theorem 4.

Summarizing, we can conclude that Theorem 4 generalizes the classical Filippov continuity result (Proposition 2) for the special case of affine dynamical systems, as represented by (25).

8. Conclusions

In this contribution, we have studied various types of affine control systems and have established their strong approximability properties. These results are direct consequences of the given system structure and can also be interpreted as a kind of robustness-like behavior with respect to possible disturbances of the admissible controls. Using the facts obtained for affine systems, we next examined models with sliding mode regimes and specified the above general result for affine control systems with a specific feedback controls. Our main approach was also applied to hybrid systems with autonomous location transitions and to the corresponding optimal control problems. We have established the strong approximability properties of the above hybrid systems. Moreover, we also applied our theoretical result to the consistence problem of suitable numerical algorithms for a class of hybrid optimal control problems.

Finally, note that the basic theoretical techniques and results presented in this paper can be extended to some other classes of hybrid and switched systems and also applied to numerical analysis of the associated optimization problems.

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References


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