Hybrid LQ-Optimization Using Dynamic Programming

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***Abstract*— In this paper we study the linear quadratic optimal control problem for linear hybrid systems in which transitions between different discrete locations occur autonomously when the continuous state intersects given switching surfaces. In particular, we make an explicit connection between the newly developed, Pontryagin-type Hybrid Maximum Principle and the Bellman Dynamic Programming approach. As a consequence, we extend the classic Riccati-formalism, derive the associated Riccati-type equations, and prove the discontinuity of the full "hybrid" Riccati matrix. Finally, we discuss some computational aspects of the obtained theoretical results and propose a numerical algorithm in the framework of an optimal feedback control law.**

I. Introduction

During the last decade, a vast body of research on hybrid control systems has been produced, drawing its motivation from the fact that many modern application domains involve complex systems, in which sub-system interconnections, mode-transitions, and heterogeneous computational devices are present. And, hybrid models, in which continuous and discrete dynamical components interact, have proved useful for capturing these types of phenomena. As a consequence, discrete-continuous dynamical interactions have emerged as a major challenge in the controls community.

In this paper, we study optimal control of hybrid systems, and despite significant progress in this area over the last few years, the ability to operate hybrid systems in an optimal manner remains a challenging task as the computational complexity associated with such problems often prove to be a bottleneck. Indeed, in the general setting of hybrid systems, one has to deal not only with the infinite dimensional optimization problems related to the continuous dynamics, but also with a potential combinatorial explosion related to the discrete part. In this context, and with focus on particular classes, many schemes have been proposed to address the problem. Some are based on a newly elaborated condition of optimality see e.g., [1], [2], [4], [5], [7], [8], [12], [13], [20], [22], [23], [24], [25], others are more related to semi-classical approaches see e.g., [9], [15], [16], [21], [27].

During the last few years, there has been a revival of the first-order optimization techniques and related numerical schemes based on the Pontryagin-type hybrid maximum principle (HMP) (see e.g., [1], [2], [4], [5], [18].) This fact is due to their intuitive interpretation in combination with the existence of well established consistence and convergence results. On the other hand, dynamic programming (DP) based approaches have not been sufficiently advanced to the linear hybrid systems setting, as well as to the corresponding LQ-type hybrid optimal control problems (OCP), beyond the initial work done in [14]. And, this development is exactly the topic under investigation in this paper.

For a classical feedback OCP, one of the main tools toward the construction of optimal trajectories is the celebrated Bellman DP method. It is also well-known that for a conventional OCP the DP approach is equivalent to the techniques based on the usual Pontryagin Maximum Principle (see e.g., [11], [19]). The aim of our contribution is to study a possible relationship between the DP and HMP in the case of a hybrid linear quadratic (HLQ) problem, and to deduce the corresponding Riccati-formalism similar to the classic LQ-theory. And, it should be noted already at this point that the conventional theory of linear systems can not be formally applied in the hybrid systems setting (see [27] for details). Therefore, it is necessary to extend this theory in the context of some concrete classes of hybrid systems.

This paper is organized as follows: Section 2 contains the problem formulation together with the necessary basic concepts and preliminary facts. Section 3 is devoted to the application of the HMP to the HLQ-control processes governed by linear hybrid systems with autonomous location transitions. Moreover, in this section we present the main result, namely, the *discontinuity property of the Riccati matrix in the hybrid setting*. In Section 4, we briefly discuss some computational aspects of the obtained "hybrid" extension of the classic Riccati-based approach to OCPs, while Section 5 concludes the paper.

II. Optimization of Linear Hybrid Systems

Let us start by introducing a variant of the general concept of a linear hybrid system with autonomous location transitions [2], [4], [5], [12], [13], [24].

Definition 1: A *linear hybrid system* is a 7-tuple

$$\{\mathcal{Q}, \mathcal{X}, U, A, B, \mathcal{U}, \mathcal{S}\},$$

where

- $\mathcal{Q}$ is a finite set of discrete states (called *locations*);
- $\mathcal{X} = \{\mathcal{X}_q\}, q \in \mathcal{Q}$, is a family of state spaces such that $\mathcal{X}_q \subseteq \mathbb{R}^n$;
- $U \subseteq \mathbb{R}^m$ is a set of admissible control input values (called *control set*);

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- $A = \{A_q(\cdot)\},\ B = \{B_q(\cdot)\},\ q \in \mathcal{Q}$ are families of continuously differentiable matrix-functions
$$A_q : \mathbb{R} \to \mathbb{R}^{n\times n},\ B_q : \mathbb{R} \to \mathbb{R}^{n\times m};$$
- $\mathcal{U}$ is the set of all admissible control functions;
- $\mathcal{S}$ is a subset of Ξ, where
$$\Xi := \{(q, x, q', x') \ :\ q, q' \in \mathcal{Q}, x \in \mathcal{X}_q,\ x' \in \mathcal{X}_{q'}\}.$$

A linear hybrid system from Definition 1 is defined on the finite time-interval $[0, t_f]$. We refer to [22], [23], [24], [25] for some abstract concepts of hybrid systems. Let U be a convex and closed set. We assume that
$$\mathcal{U} := \{u(\cdot) \in \mathbb{L}^\infty_m(0, t_f) \mid u(t) \in U \text{ a.e. on } [0, t_f]\},$$
where $\mathbb{L}^\infty_m(0, t_f)$ is the standard Lebesgue space of measurable and essentially bounded functions. In this paper, we suppose that affine functions $m_{q,q'} : \mathbb{R}^n \to \mathbb{R},\ q,\ q' \in \mathcal{Q}$, $m_{q,q'}(x) = b_{q,q'}x + c_{q,q'}$ are given such that the hyperplanes $M_{q,q'} := \{x \in \mathbb{R}^n \ :\ m_{q,q'}(x) = 0\}$ are pairwise disjoint. Here $b_{q,q'} \in \mathbb{R}^n$ and $c_{q,q'} \in \mathbb{R}$ for every $q,\ q' \in \mathcal{Q}$. The given hyperplanes $M_{q,q'}$ represents the (affine) *switching sets* at which a switch from location q to location q' can take place. We say that a location switching from q to q' occurs at a *switching time* $t^{switch} \in [0, t_f]$. We now consider a linear hybrid system with $r \in \mathbb{N}$ switching times $0 = t_0 < t_1 < ... < t_{r-1} < t_r = t_f$. Note that the above sequence of switching times $\{t_i\}$ is not defined a priory. A hybrid control system remains in location $q_i \in \mathcal{Q}$ for all $t \in [t_{i-1}, t_i),\ i = 1, ..., r$. In the following, we recall the notion of hybrid trajectory of the systems under consideration (see e.g., [4]).

Definition 2: An admissible *hybrid trajectory* associated with a given linear hybrid system from Definition 1 is a triple $\mathbf{X} = (x(\cdot), \{q_i\}, \tau)$, where $x(\cdot)$ is a continuous part of trajectory, $\{q_i\}_{i=1,...,r}$ is a finite sequence of locations and τ is the corresponding sequence of switching times such that $x(0) = x_0 \notin \bigcup_{q,q' \in \mathcal{Q}} M_{q,q'}$ and for each $i = 1, ..., r$ and every admissible control $u(\cdot) \in \mathcal{U}$ we have

- $x_i(\cdot) = x(\cdot)|_{(t_{i-1},t_i)}$ is an absolutely continuous function on (t_{i-1}, t_i) continuously prolongable to $[t_{i-1}, t_i],\ i = 1, ..., r$;
- $\dot{x}_i(t) = A_{q_i}(t)x_i(t) + B_{q_i}(t)u_i(t)$ for almost all times $t \in [t_{i-1}, t_i]$, where $u_i(\cdot)$ is a restriction of the chosen control function $u(\cdot)$ on the time interval $[t_{i-1}, t_i]$.

A linear hybrid system in the sense of Definition 1 and Definition 2 that satisfies all above assumptions is denoted by $\mathcal{LHS}$ (see Figure 1 for an illustration).

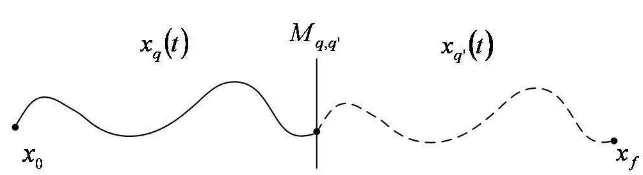

Fig. 1. Dynamical behavior of a hybrid system.

Note that the pair $(q, x(t))$ represents the hybrid state at time t, where q is a location $q \in \mathcal{Q}$ and $x(t) \in \mathbb{R}^n$. Definition 2 describes the dynamic of a hybrid control system $\mathcal{LHS}$. Since $x(\cdot)$ is a continuous function, Definition 2 describes a class of hybrid systems without impulse components of the (continuous) trajectories. Therefore, sets $M_{q,q'}$ are defined for $x(t_i) = x(t_{i+1}),\ i = 1, ..., r-1$.

Under the above assumptions, for each admissible control $u(\cdot) \in \mathcal{U}$ and for every interval $[t_{i-1}, t_i]$ (for every location $q_i \in \mathcal{Q}$) there exists a unique absolutely continuous solution of the linear differential equations from Definition 2. This means that for each $u(\cdot) \in \mathcal{U}$ we have a unique absolute continuous trajectory of $\mathcal{LHS}$. Moreover, the switching times $\{t_i\}$ and the discrete trajectory $\{q_i\}$ for a hybrid control system $\mathcal{LHS}$ are also uniquely defined. Note that the evolution equation for the trajectory $x(\cdot)$ of a given linear hybrid system $\mathcal{LHS}$ can also be represented as follows
$$\begin{aligned} \dot{x}(t) = \sum_{i=1}^{r} \beta_{[t_{i-1},t_i)}(t) \times \\ \big(A_{q_i}(t)x_i(t) + B_{q_i}(t)u_i(t)\big) \text{ a.e. on } [0, t_f], \end{aligned} \tag{1}$$
where $x(0) = x_0$ and $\beta_{[t_{i-1},t_i)}(\cdot)$ is the characteristic function of the interval $[t_{i-1}, t_i)$. Let $S_f : \mathbb{R} \to \mathbb{R}^{n\times n}$, $S_q : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $R_q : \mathbb{R} \to \mathbb{R}^{m\times m}$, where $q \in \mathcal{Q}$. Assume that S_f is symmetric and positive semidefinite, and that for every time instant $t \in [0, t_f]$ and every $q \in \mathcal{Q}$ the matrix $S_q(t)$ is also a symmetric and positive semidefinite matrix. Moreover, let $R_q(t)$ be a symmetric and positive definite for every $t \in [0, t_f]$ and every $q \in \mathcal{Q}$. We also assume that the given matrix-functions $S_q(\cdot),\ R_q(\cdot)$ are continuously differentiable. Given a system $\mathcal{LHS}$ we consider the following HLQ problem:
$$\begin{aligned} &\text{minimize } J(u(\cdot), x(\cdot)) := \frac{1}{2}(x_r^T(t_f)S_f x_r(t_f)) + \\ &\frac{1}{2}\sum_{i=1}^{r}\int_{t_{i-1}}^{t_i} \big(x_i^T(t)S_{q_i}(t)x_i(t) + u_i^T(t)R_{q_i}(t)u_i(t)\big)dt \\ &\text{over all admissible trajectories } \mathbf{X} \text{ of } \mathcal{LHS}. \end{aligned} \tag{2}$$

Evidently, (1) is the problem of minimizing the quadratic Bolza cost functional J over all trajectories of the given linear hybrid system. Note that we study the hybrid OCP (1) in the absence of possible target and state constraints. Throughout the paper we assume that the HLQ problem (2) has an optimal solution $(u^{opt}(\cdot), \mathbf{X}^{opt}(\cdot))$, where $u^{opt}(\cdot) \in \mathcal{U}$ and $\mathbf{X}^{opt}(\cdot)$ belongs to the set of admissible trajectories from Definition 2. It is necessary to stress that the existence of an optimal pair $(u^{opt}(\cdot), \mathbf{X}^{opt}(\cdot))$ for a HLQ problem of the above type follows from the general existence theory for linear quadratic OCPs with a convex closed control set U (see e.g., [19]). We now apply the HMP (see [4]) to the HLQ problem under consideration and formulate the corresponding necessary optimality conditions. For general optimality conditions in the form of a HMP see also [4], [13], [23], [24], [25].

Theorem 1: [4] Let $(u^{opt}(\cdot), \mathbf{X}^{opt}(\cdot))$ be an optimal solution of the regular OCP (2). Then there exist absolutely continuous functions $\psi_i(\cdot)$ on the time intervals $(t_{i-1}^{opt}, t_i^{opt})$, where $i = 1, ..., r$, and a nonzero vector of Lagrange multipliers $a = (a_1, ..., a_{r-1})^T \in \mathbb{R}^{r-1}$ such that

$$\begin{aligned} &\dot{\psi}_i(t) = -A_{q_i}^T(t)\psi_i(t) + \\ &S_{q_i}(t)x_i^{opt}(t) \text{ a.e. on } [t_{i-1}^{opt}, t_i^{opt}], \\ &\psi_r(t_f) = -S_f x_r^{opt}(t_f), \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\psi_i(t_i^{opt}) = \psi_{i+1}(t_i^{opt}) + \\ &a_i \frac{dm_{q_i,q_{i+1}}(x_i^{opt}(t_i^{opt}))}{dx_i} = \psi_{i+1}(t_i^{opt}) + a_i b_{q_i,q_{i+1}}, \end{aligned} \tag{4}$$

where $i = 1, ..., r-1$. Moreover, for every admissible control $u(\cdot) \in \mathcal{U}$ the partial Hamiltonian

$$\begin{aligned} &H_{q_i}(t, x, u, \psi) := \langle \psi_i, A_{q_i}(t)x_i + B_{q_i}(t)u_i \rangle - \\ &\frac{1}{2}\big(x_i^T S_{q_i}(t)x_i + u_i^T R_{q_i}(t)u_i\big). \end{aligned}$$

satisfies the following maximization conditions

$$\begin{aligned} &\max_{u \in U} H_{q_i}(t, x^{opt}(t), u, \psi(t)) = \\ &H_{q_i}(t, x^{opt}(t), u^{opt}(t), \psi(t)), \ \ t \in [t_{i-1}^{opt}, t_i^{opt}), \end{aligned} \tag{5}$$

where $i = 1, ..., r$ and $\psi(t) := \sum_{i=1}^r \beta_{[t_{i-1}^{opt}, t_i^{opt})}(t)\psi_i(t)$ for all $t \in [0, t_f]$.

Note that the *adjoint variable* $\psi(\cdot)$ is an absolutely continuous function on every open time intervals $(t_{i-1}^{opt}, t_i^{opt})$ for $i = 1, ..., r$ but discontinuous at the switching points $t_i^{opt} \in \tau^{opt}$. On the other hand, we are able to establish the continuity properties of the ”full” optimal Hamiltonian

$$\tilde{H}^{opt}(t) := \sum_{i=1}^r \beta_{[t_{i-1}^{opt}, t_i^{opt})}(t) H_{q_i}(t, x^{opt}(t), u^{opt}(t), \psi(t))$$

computed for optimal pair $(u^{opt}(\cdot), \mathbf{X}^{opt}(\cdot))$ and for the corresponding adjoint variable $\psi(\cdot)$.

Theorem 2: Under assumptions of Theorem 1, the ”full” optimal Hamiltonian $\tilde{H}^{opt}(\cdot)$ introduced above is a continuous function on $[0, t_f]$.

Proof. Consider the time interval $[t_{i-1}^{opt}, t_i^{opt}]$ and the associated partial Hamiltonian $H_{q_i}(t, x, u, \psi)$. Evidently, the function $\tilde{H}^{opt}(t)$ is continuous on the open time intervals $(t_{i-1}^{opt}, t_i^{opt})$, $i = 1, ..., r$. For an optimal pair $(u^{opt}(\cdot), \mathbf{X}^{opt}(\cdot))$ we have

$$\begin{aligned} &H_{q_i}(t_i^{opt}, x^{opt}(t_i^{opt}), u^{opt}(t_i^{opt}), \psi(t_i^{opt})) = \\ &-\frac{\partial J(u^{opt}(\cdot), x^{opt}(\cdot))}{\partial t_i^{opt}}. \end{aligned}$$

Using (3)-(4) and the well-known formula for variation of the costs functional J (see e.g., [11], [24]), we can compute

$$\begin{aligned} &H_{q_i}(t_i^{opt}, x^{opt}(t_i^{opt}), u^{opt}(t_i^{opt}), \psi(t_i^{opt})) = \\ &H_{q_{i+1}}(t_i^{opt}, x^{opt}(t_i^{opt}), u^{opt}(t_i^{opt}), \psi(t_i^{opt})) + \\ &a_i \frac{\partial m_{q_i,q_{i+1}}(x^{opt}(t_i^{opt}))}{\partial t_i^{opt}} = \\ &H_{q_{i+1}}(t_i^{opt}, x^{opt}(t_i^{opt}), u^{opt}(t_i^{opt}), \psi(t_i^{opt})) + \\ &a_i \frac{\partial [b_{q_i,q_{i+1}} x^{opt}(t_i^{opt}) + c_{q_i,q_{i+1}}]}{\partial t_i^{opt}} = \\ &H_{q_{i+1}}(t_i^{opt}, x^{opt}(t_i^{opt}), u^{opt}(t_i^{opt}), \psi(t_i^{opt})), \end{aligned} \tag{6}$$

where $i = 1, ..., r-1$. Clearly, from the obtained relation (6) follows the continuity of the introduced function $\tilde{H}^{opt}(t)$ not only on the open time intervals $(t_{i-1}^{opt}, t_i^{opt})$, $i = 1, ..., r$, but also for all switching times $t_i^{opt} \in \tau^{opt}$, where τ^{opt} is the optimal sequence of switching times from $\mathbf{X}^{opt}$. □

Note that a similar result is obtained in [24] for general hybrid systems with controlled location transitions (Theorem 2.2, p. 1590) and for some classes of nonlinear hybrid systems with autonomous location transitions. From Theorem 2 it follows the continuity of the Hamiltonian for (2) computed for optimal state and control variables and for the corresponding discontinuous adjoint variables. Note that the similar result can also be proved for general nonlinear optimal control processes governed by hybrid systems with autonomous location transitions (see [4], [24]). The corresponding proof is based on a generalization of the classic needle variations and on the associated formula for variation of the costs functional in the hybrid OCP under consideration.

III. THE EXTENSION OF THE RICCATI-FORMALISM TO HYBRID LINEAR QUADRATIC OCPS

In this section we extend the well know Bellman DP techniques for conventional LQ problems to the HLQ optimization problems of the type (2). Let us consider the linear boundary value problem (1), (3) for $U \equiv \mathbb{R}^m$. The maximization condition (5) from the above HMP (Theorem 1) implies that $u_i^{opt}(t) = R_{q_i}^{-1}(t)B_{q_i}^T(t)\psi_i(t)$ for $t \in [t_{i-1}^{opt}, t_i^{opt})$. Using this representation of an optimal control and the basic facts from the theory of linear differential equations, we now compute (similarly to [11], [19]) an optimal control $u^{opt}(\cdot)$ for (2) in the form of an optimal *partially linear* feedback control law

$$\begin{aligned} &u^{opt}(t) = -C(t)x^{opt}(t) = \\ &-\sum_{i=1}^r \beta_{[t_{i-1}, t_i)}(t) C_i(t) x_i^{opt}(t), \end{aligned} \tag{7}$$

where $C_i(t) := R_{q_i}^{-1}(t)B_{q_i}^T(t)P_i(t)$ is a *partial gain matrix* and $P_i(\cdot)$ is the *partial Riccati matrix* associated with every location $q_i^{opt} \in \mathcal{Q}$. Analogously to the classic case, for every location $q_i^{opt} \in \mathcal{Q}$ and for almost all $t \in (t_{i-1}^{opt}, t_i^{opt})$ we obtain the differential equation

$$\begin{aligned}&\dot{P}_i(t)+P_i(t)A_{q_i}(t)+A_{q_i}^T(t)P_i(t)-\\&P_i(t)B_{q_i}(t)R_{q_i}^{-1}(t)B_{q_i}^T(t)P_i(t)+S_{q_i}(t)=0,\end{aligned} \tag{8}$$

known as the Riccati matrix differential equation. We call this equation the partial Riccati equation. Evidently, every matrix $C_i(\cdot)$ and every matrix $P_i(\cdot)$ and the corresponding partial Riccati equation (8) are associated with a current location $q_i \in \mathcal{Q}$ of the given $\mathcal{LHS}$. We also can deduce the usual relations

$$\psi_i(t) = -P_i(t)x_i^{opt}(t) \tag{9}$$

for $t \in [t_{i-1}^{opt}, t_i^{opt})$ and $i = 1, ..., r$. A symmetric (for all variables $t \in [0, t_f]$) *hybrid Riccati matrix*

$$P(t) := \sum_{i=1}^{r} \beta_{[t_{i-1}^{opt}, t_i^{opt})}(t)P_i(t)$$

which satisfies all equations (8) and the boundary (terminal) condition $P(t_f) = S_f$ gives rise to the optimal feedback dynamics of (1) determined by the above partially linear feedback control function (7).

It is necessary to stress that the partial Riccati equation (8) can also be derived with help of the general Bellman equation (see [14]). Analogously to the classic optimal control theory it is possible to determine a *partial value function* associated with every location of a hybrid system. Replacing the control variable by $u_i^{opt}(t)$ in the above-mentioned hybrid Bellman equation for (2), one can obtain a hybrid version of the well known differential equation for the partial value function in a LQ problem (see [11] for details). Following [17], one can also prove that this partial value function for the HLQ under consideration can be chosen (similarly to the classic LQ-problem) as a quadratic function with a shifting vector defined for every location $q_i \in \mathcal{Q}$. Using this shifted-quadratic partial value function and the above differential equation, one obtains (8).

The investigation of the family of equations (8) on the full time interval $[0, t_f]$ involves the continuity question associated with the above-introduced hybrid Riccati matrix $P(\cdot)$. Evidently, the continuity/smoothness of a value function is a question of general interest also in the context of other classes of OCPs governed by linear or nonlinear hybrid systems. Related to the above presented optimization theory for a $\mathcal{LHS}$ we are now able to formulate our main theoretical result, namely, the discontinuity of the hybrid Riccati matrix $P(\cdot)$.

Theorem 3: Under assumptions of Theorem 1, the hybrid Riccati matrix $P(\cdot)$ is a discontinuous function on $[0, t_f]$.

Proof. Assume that $P(\cdot)$ is continuous on the time interval $[0, t_f]$. In particular, this means that $P_i(t_i^{opt}) = P_{i+1}(t_i^{opt})$ for all numbers $i = 1, ..., r-1$. Using the above continuity assumption for $P(\cdot)$, the continuity of $x(\cdot)$ and the formula for the adjoint variable, we deduce that

$$\psi_i(t_i^{opt}) = -\lim_{t \uparrow t_i^{opt}} P_i(t)x_i^{opt}(t),$$

$$\psi_{i+1}(t_i^{opt}) = -\lim_{t \downarrow t_i^{opt}} P_{i+1}(t)x_{i+1}^{opt}(t).$$

Then from (9) and from the jump conditions (4) for the adjoint variables $\psi(\cdot)$ we obtain the following relation

$$-P_i(t_i^{opt})x^{opt}(t_i^{opt}) = -P_{i+1}(t_i^{opt})x^{opt}(t_i^{opt}) + a_i b_{q_i,q_{i+1}},$$

where $i = 1, ..., r-1$.. Hence

$$\left[P_{i+1}(t_i^{opt}) - P_i(t_i^{opt})\right]x_i(t_i^{opt}) = a_i b_{q_i,q_{i+1}}. \tag{10}$$

Since $x^{opt}(\cdot)$ is continuous and the (optimal) Lagrange multipliers $a = (a_1, ..., a_{r-1})^T$ are nontrivial, the function $P(\cdot)$ is a discontinuous function on $[0, t_f]$. The obtained contradiction completes the proof. □

From (10) it follows that the *jump* of the hybrid Riccati matrix $P(\cdot)$ to an optimal switching time $t_i^{opt} \in \tau^{opt}$ is proportional to the associated Lagrange multiplier a_i and to the vector $b_{q_i,q_{i+1}}$ which characterizes the corresponding switching hyperplane $M_{q_i,q_{i+1}}$.

Note that Theorem 3 is a consequence of the continuity of the optimal Hamiltonian $\tilde{H}^{opt}(\cdot)$ (Theorem 2). Similarly to the above proof let us assume that the function $P(\cdot)$ is continuous. Consider now the continuity condition for two partial Hamiltonians $H_{q_i}^{opt}$ and $H_{q_{i+1}}^{opt}$, namely, the result of Theorem 2 for some locations $q_i, q_{i+1} \in \mathcal{Q}$

$$\begin{aligned}&H_{q_i}(t_i^{opt}, x^{opt}(t_i^{opt}), u^{opt}(t_i^{opt}), \psi(t_i^{opt})) =\\&H_{q_{i+1}}(t_i^{opt}, x^{opt}(t_i^{opt}), u^{opt}(t_i^{opt}), \psi(t_i^{opt}))\end{aligned} \tag{11}$$

Using (11) and the above relations for $\psi_i(t_i^{opt})$ and $\psi_{i+1}(t_i^{opt})$, we deduce that

$$\begin{aligned}&-\left\langle P_i(t_i^{opt})x^{opt}(t_i^{opt}), A_{q_i}(t_i^{opt})x^{opt}(t_i^{opt})+\right.\\&\left.B_{q_i}(t_i^{opt})u^{opt}(t_i^{opt})\right\rangle-\\&\frac{1}{2}\big((x^{opt})^T(t_i^{opt})S_{q_i}(t_i^{opt})x^{opt}(t_i^{opt})+\\&(u^{opt})^T(t_i^{opt})R_{q_i}(t_i^{opt})u^{opt}(t_i^{opt})\big)=\\&-\left\langle P_{i+1}(t_i^{opt})x^{opt}(t_i^{opt}), A_{q_{i+1}}(t_i^{opt})x^{opt}(t_i^{opt})+\right.\\&\left.B_{q_{i+1}}(t_i^{opt})u^{opt}(t_i^{opt})\right\rangle-\\&\frac{1}{2}\big((x^{opt})^T(t_i^{opt})S_{q_{i+1}}(t_i^{opt})x^{opt}(t_i^{opt})+\\&(u^{opt})^T(t_i^{opt})R_{q_{i+1}}(t_i^{opt})u^{opt}(t_i^{opt})\big).\end{aligned} \tag{12}$$

Since $P(\cdot)$ is assumed to be continuous in general, this function is also continuous for a special case of a $\mathcal{LHS}$ indicated by the following relations

$$\begin{aligned}&S_{q_{i+1}}(t_i^{opt}) = S_{q_i}(t_i^{opt}),\ R_{q_{i+1}}(t_i^{opt}) = R_{q_i}(t_i^{opt}),\\&B_{q_i}(t_i^{opt}) = B_{q_{i+1}}(t_i^{opt}),\ A_{q_i}(t_i^{opt}) \neq A_{q_{i+1}}(t_i^{opt})\end{aligned} \tag{13}$$

for all $i = 1, ..., r-1$. Note that under assumptions (13) we have a continuous (on the full time interval $[0, t_f]$) optimal control function $u^{opt}(\cdot)$ from (7). In particular, $u^{opt}(\cdot)$ has no jumps at $t = t_i^{opt}$. Then from (12) we deduce the following relations $P_i(t_i^{opt})A_{q_i}(t_i^{opt}) = P_{i+1}(t_i^{opt})A_{q_{i+1}}(t_i^{opt})$ and $A_{q_i}(t_i^{opt}) = A_{q_{i+1}}(t_i^{opt})$. This is a contradiction with respect to assumed conditions (13). This means that even in the special case of a hybrid OCP (2) given by assumptions (13) we have discontinuity conditions $P_i(t_i^{opt}) \neq P_{i+1}(t_i^{opt})$ for some $i = 1, ..., r-1$ and the contradiction with the continuity assumption for $P(\cdot)$ on the full time interval $[0, t_f]$. It is necessary to stress that the hybrid Riccati matrix $P(\cdot)$ is a discontinuous function considered on the full time interval $[0, t_f]$. On the other hand, for some (but not for all) locations $q_i^{opt}, q_{i+1}^{opt} \in \mathcal{Q}$ we can have $P_i(t_i^{opt}) = P_{i+1}(t_i^{opt})$. In this case from (10) it follows that the corresponding Lagrange multipliers $a_i \in a \neq 0 \in \mathbb{R}^{r-1}$ is equal to zero. Let us note that the hybrid Riccati matrix $P(\cdot)$ is a completely (for all $q_i \in \mathcal{Q},\ i = 1, ..., r$) continuous function only in the case of a special HLQ problem characterized by

$$S_{q_{i+1}}(t_i^{opt}) = S_{q_i}(t_i^{opt}),\ R_{q_{i+1}}(t_i^{opt}) = R_{q_i}(t_i^{opt}),\\ B_{q_i}(t_i^{opt}) = B_{q_{i+1}}(t_i^{opt}),\ A_{q_i}(t_i^{opt}) = A_{q_{i+1}}(t_i^{opt}) \tag{14}$$

for all $i = 1, ..., r-1$. It is evident, that under conditions (14) the given $\mathcal{LHS}$ can be rewritten as a conventional linear control system (by introduction of the new continuous system matrices and new continuous matrices in the costs functional). Therefore, the corresponding HLQ problem (2) with (14) is equivalent (in this special case) to the classic LQ-type OCP. Finally, note that similarly to the conventional LQ problems the closed-loop $\mathcal{LHS}$ associated with the partially linear feedback (7) also possesses stability properties (in the sense of the classic Lyapunov concept) on the infinite time horizon. This fact can be established using the above-mentioned quadratic partial value functions as candidate Lyapunov functions for the corresponding stability analysis (see [10] for details).

IV. Optimal Partially Linear Feedback Control Law: Computational Aspects

Our main theoretical result, namely Theorem 3, can also provide a basis for the constructive design of the optimal feedback control strategy in the framework of the above-formulated LQ-type hybrid OCP (2). Evidently, in the context of the presented advanced Riccati-formalism from Section III the main difficulties in computing the optimal partially linear feedback control (7) are caused by jumps of the hybrid Riccati matrix $P(\cdot)$ at some $t_i^{opt} \in \tau^{opt}$. Note that the discontinuity property of the hybrid Riccati matrix is a new effect in relation to the conventional LQ-theory. Let us now study this discontinuity effect from the numerical point of view. By $u_i^{opt}(\cdot)$ we denote the restriction of the control function $u^{opt}(\cdot)$ on the time interval $[t_{i-1}, t_i)$. Evidently, $u_i^{opt}(t) = \beta_{[t_{i-1},t_i)}(t) C_i(t) x_i^{opt}(t)$. Assume that for two given locations $q_i^{opt}, q_{i+1}^{opt} \in \mathcal{Q}$ we can compute the value of the partial Riccati matrix $P_{i+1}(t)$ for every time instants t from the closed interval $[t_i^{opt}, t_{i+1}^{opt}]$, $i = 0, ..., r-1$. Using the continuity property of the function $\tilde{H}^{opt}(\cdot)$ (Theorem 2), we obtain the following nonspecific algebraic Riccati equation with respect to the unknown matrix $P_i(t_i^{opt})$.

$$\begin{aligned}&\frac{3}{2} P_i(t_i^{opt}) B_{q_i}(t_i^{opt}) R_{q_i}^{-1}(t_i^{opt}) B_{q_i}^T(t_i^{opt}) P_i(t_i^{opt}) +\\ &P_i(t_i^{opt}) A_{q_i}(t_i^{opt}) - \frac{1}{2}\big(S_{q_{i+1}}(t_i^{opt}) - S_{q_i}(t_i^{opt})\big) -\\ &\frac{3}{2} P_{i+1}(t_i^{opt}) B_{q_{i+1}}(t_i^{opt}) R_{q_{i+1}}^{-1}(t_i^{opt}) \times\\ &B_{q_{i+1}}^T(t_i^{opt}) P_{i+1}(t_i^{opt}) - P_{i+1}(t_i^{opt}) A_{q_{i+1}}(t_i^{opt}) = 0.\end{aligned} \tag{15}$$

If we now transpose (15) and combine with itself we get the system of the linear (Lyapunov-type) equation and the symmetric Riccati equation

$$\begin{aligned}&P_i(t_i^{opt}) A_{q_i}(t_i^{opt}) - A_{q_i}^T(t_i^{opt}) P_i(t_i^{opt}) -\\ &P_{i+1}(t_i^{opt}) A_{q_{i+1}}(t_i^{opt}) + A_{q_{i+1}}^T(t_i^{opt}) P_{i+1}(t_i^{opt}) = 0,\\ &3 P_i(t_i^{opt}) B_{q_i}(t_i^{opt}) R_{q_i}^{-1}(t_i^{opt}) B_{q_i}^T(t_i^{opt}) P_i(t_i^{opt}) +\\ &P_i(t_i^{opt}) A_{q_i}(t_i^{opt}) + A_{q_i}^T(t_i^{opt}) P_i(t_i^{opt}) + S_{q_i}(t_i^{opt}) -\\ &3 P_{i+1}(t_i^{opt}) B_{q_{i+1}}(t_i^{opt}) R_{q_{i+1}}^{-1}(t_i^{opt}) B_{q_{i+1}}^T(t_i^{opt}) P_{i+1}(t_i^{opt}) -\\ &P_{i+1}(t_i^{opt}) A_{q_{i+1}}(t_i^{opt}) - A_{q_{i+1}}^T(t_i^{opt}) P_{i+1}(t_i^{opt}) - S_{i+1}(t_i^{opt}) = 0.\end{aligned} \tag{16}$$

This system (16) defines the value of the partial Riccati matrix, namely $P_i(t_i^{opt})$, which can be used as the necessary start condition for solving the Riccati matrix differential equation (8) on the next time interval $[t_{i-1}^{opt}, t_i^{opt})$. Note that for $i = r-1$ we have the final condition for the last partial Riccati matrix $P_r(t_r) = P(t_f) = S_f$. From this terminal condition, we can obtain the inverted-time solution of the differential Riccati equation (8) for the interval $[t_{r-1}^{opt}, t_f]$ and determine the value $P_r(t_{r-1}^{opt})$. This value is used in system (16) for $i = r-1$ and one calculates $P_{r-1}(t_{r-1}^{opt})$. It is necessary to stress that the jumps in the Riccati matrices at the time instants $t = t_i^{opt}$ are given by the solutions of system (**??**) and the resulting optimal feedback control $u^{opt}(\cdot)$ from (7) is a discontinuous piecewise linear control function.

We now are able to summarize a general conceptual computational algorithm for the numerical treatment of the optimal partially linear feedback control in the given HLQ problem (2). Note that in the algorithm presented below an approximating trajectory $x^{appr}(\cdot)$ to $x^{opt}(\cdot)$ and the corresponding sequence τ^{appr} to τ^{opt} are assumed to be given. The elements of τ^{appr} approximate the optimal switching times $t_i^{opt} \in \tau^{opt}$ for every $i = 1, ..., r-1$. A trajectory $x^{appr}(\cdot)$, a sequence τ^{appr} and the associated sequence of the corresponding locations can be obtained in various ways, for instance, with help of the *gradient-based* algorithms proposed in [2], [3], or using the *optimality zone algorithms* from [12], [24].

Conceptual Algorithm 1:

1) Consider an approximating trajectory $x^{appr}(\cdot)$, the corresponding sequence τ^{appr}, the sequence of locations and the terminal condition $P(t_f) = S_f$ for a given $\mathcal{LHS}$. Set $k = 1$ and $l = 1$.
2) With help of the inverted-time integrating procedure, compute the value $P_r(t_{r-1}^{opt})$ of the partial Riccati matrix P_r. Using (16) calculate the Riccati matrix $P_{r-1}(t_{r-1}^{opt})$.
3) By the inverted-time integrating solution define $P_{r-k}(t_{r-k-1}^{opt})$, increase k by one. If $k = r-1$, then go to Step 4. Otherwise go to Step 2.
4) Complete all partial Riccati matrices $P_i(\cdot)$ and define the corresponding partial gain matrices

$$C_i(t) = R_{q_i}^{-1}(t) B_{q_i}^T(t) P_i(t).$$

Compute the quasi-optimal (in the sense of the above approximations) piecewise feedback control function from (7).

5) Using the obtained quasi-optimal feedback control low, compute the corresponding trajectory $x^l(\cdot)$ of the $\mathcal{LHS}$ under consideration. Determine the new approximating sequence τ^l from the conditions

$$t_i^l := \min\{t \in [0, t_f] \ : \ x^l(t) \bigcap M_{q_i, q_{i+1}} \neq \emptyset\}.$$

where $i = 1, ..., r-1$. Finally, increase l by one and go to Step 2.

We are also able to prove the following convergence result for the presented Conceptual Algorithm 1.

Theorem 4: Under assumptions of Theorem 1, there exists an initial approximating trajectory $x^{appr}(\cdot)$ such that the sequence of hybrid trajectories generated by Algorithm 1 is a minimizing sequence for (2).

The proof of Theorem 4 is based on the convexity arguments (see [6]) for the HLQ problem under consideration and on the convergence properties of the general first-order optimization methods in real Hilbert spaces.

We now consider some examples of the HLQ problems and illustrate the numerical approach proposed above.

Example 1: First, let us examine the example from [26]

$$\dot{x} = u, \text{ for } q_1$$
$$\dot{x} = -x + u \text{ for } q_2$$

where $x_0 = 0.9$ and the switching manifold has the affine-linear structure $b_{1,2}x + c_{1,2} = 0$ with $b_{1,2} = 1$ and $c_{1,2} = -1$. The quadratic cost functional has the following easy form

$$J(u(\cdot), x(\cdot)) = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt$$

Using Algorithm 1, we evaluate the optimal trajectory and the corresponding optimal control for the HLQ problem under consideration. The optimal behavior of the given $\mathcal{LHS}$ (the pair $(x_1(t), u_1(t))$) is presented in comparison with the classic optimal dynamics of the first subsystem (indicated by $(x_0(t), u_0(t))$) (see Fig. 2).

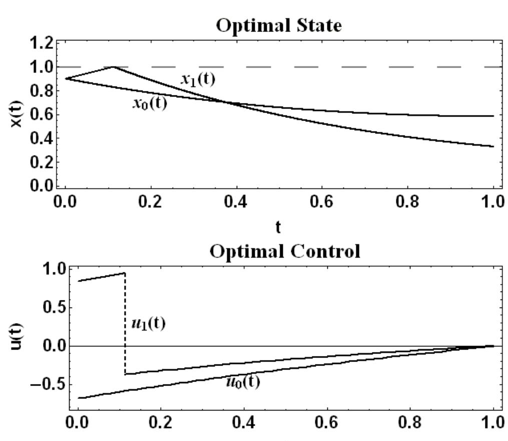

Fig. 2. Optimal behavior of the $\mathcal{LHS}$.

As one can see, an optimal HLQ dynamics is given here by a discontinuous (in time) partially linear feedback strategy. Unlike to the classic Riccati matrix function $P_0(\cdot)$ (for the first subsystem), the corresponding hybrid Riccati matrix $P_1(\cdot)$ is a discontinuous function (see Fig. 3).

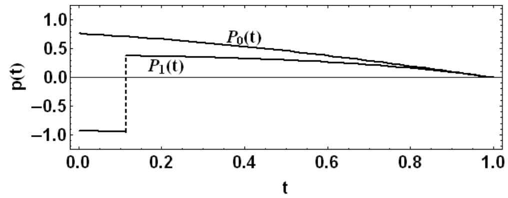

Fig. 3. Optimal behavior of the $\mathcal{LHS}$.

For the computed switching time instant $t_1^{opt} = 0.1066$ we apply (16) and obtain the following values of the hybrid Riccati "matrix": $P_1^-(t_1^{opt}) = -0.9897$, $P_1^+(t_1^{opt}) = 0.2315$. The computed optimal cost in the given HLQ problem is $J^{opt} = 0.4215$. Finally note that the optimal cost in the above classic LQ problem formulated for the first location is equal to 0.6169.

Let us now study our next example of a $\mathcal{LHS}$ with 3 locations.

Example 2: The dynamics of the hybrid system is given by the following linear equations associated with the corresponding locations.

$$\begin{aligned} \dot{x}_1 &= -x_1(t) + 4u_1(t) && \forall t \in [0, t_1] \\ \dot{x}_2 &= -2x_2(t) + 2u_2(t) && \forall t \in [t_1, t_2] \\ \dot{x}_3 &= -4x_3(t) - 5u_3(t) && \forall t \in [t_2, t_f] \end{aligned}$$

where $x_0 = 5$ and t_1, t_2 are (unknown) switching times. The switching manifolds are affine-linear manifolds

$$M_{1,2}(x) = x + 3, \ M_{2,3}(x) = x - 8$$

Our aim is to minimize the quadratical cost function J from (2), where

$$S_f = 0, \ S_{q_1} = S_{q_2} = S_{q_3} = 1, \ R_{q_1} = R_{q_2} = R_{q_3} = 1.$$

Applying Algorithm 1 we obtain a trajectory showed in Fig. 4 and the computed optimal cost is $J^{opt} = 168.6059$.

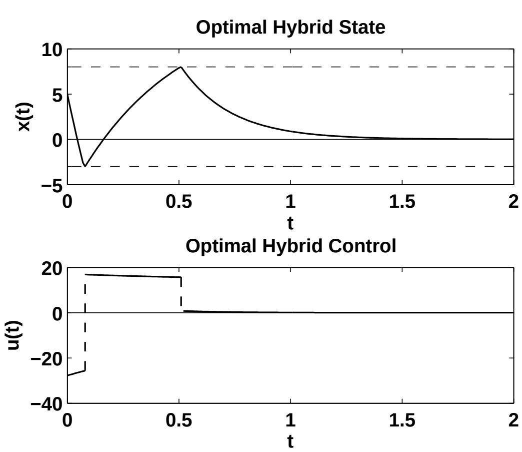

Fig. 4. Optimal behavior of the $\mathcal{LHS}$.

The above computational results are obtained using the standard Mathematica and MATLAB packages.

Unlike to the conventional LQ theory the first closed-loop subsystem from Example 1 is an unstable system (the Riccati "matrix" $P_1(t)$ is negative for all $t \in [0, t_1^{opt})$). Otherwise the "full" closed-loop systems from Example 1 and Example 2 are asymptotically stable (in the Lyapunov sense). This computational fact is in accordance with the theoretical consequences of the hybrid LQ-theory developed in Section III.

Finally, let us note that the theoretic and computational results from [28] are obtained under the incorrect and non-proved basic continuity assumption for the Riccati matrices under consideration. Therefore, the numerical results for the optimal feedback and the optimal value from [28] can be improved by using the above analytical techniques and the implementable Conceptual Algorithm 1 proposed in our paper.

V. Concluding Remarks

In this paper, we have developed a new theoretical and computational approach to a class of hybrid linear quadratic optimal control problems. This approach is based on the extension of the Maximum Principle and Dynamic Programming techniques to control processes governed by linear hybrid systems with autonomous location transitions. Using the continuous structure of the given class of hybrid control systems, we established continuity property of the optimal Hamiltonian function from the HMP. On the other hand, we prove the discontinuous property of the Riccati matrix and find the explicit expression for the jumps of this matrix-function at the optimal switching points. This makes it possible to construct an implementable computational algorithm for the numerical treatment of the optimal partially linear feedback control strategy. Using this algorithm, one can generate an optimal feedback control low analogously to the conventional LQ-problem. It is necessary to stress that in contrast to the classic linear quadratic OCP, the optimal feedback for the HLQ (2) is a discontinuous piecewise linear function.

The approach proposed in this paper can be extended to some other classes of hybrid OCPs, for instance, to linear impulsive hybrid OCPs introduced in [5]. Finally, note that it seems to be possible to prove the discontinuous property of the general value function in the nonlinear setting and study the DP-based theoretical and computational techniques for nonlinear hybrid OCPs.

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