Continuity Properties of Nonlinear Affine Control Systems: Applications to Hybrid and Sliding Mode Dynamics

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Abstract: This paper focuses on the continuity and approximability properties for nonlinear affine control systems. We consider dynamical systems governed by ordinary differential equations and establish the continuity properties of the given and relaxed (in the sense of Filippov) systems with respect to controls and initial state variables. The approach based on the set-valued analysis makes it possible to study discontinuous models in the abstract setting and to obtain general theoretical results. The latter can be effectively applied to vide classes of variable structure control systems. In particular, this paper deals with applications of the above-mentioned continuity and approximability to some hybrid control systems and to the classical sliding mode control processes.

Keywords: nonlinear affine control systems, differential inclusions, hybrid systems, sliding mode control, robustness.

1. INTRODUCTION AND MOTIVATION

Consider the following initial value problem

$$\begin{aligned} &\dot{x}(t) = a(t, x(t)) + b(t, x(t))u(t) \text{ a.e. on } [0, t_f], \\ &x(0) = x_0, \end{aligned} \tag{1}$$

where $x_0 \in \mathbb{R}^n$ is a fixed initial state, the functions $a,\ b : (0, t_f) \times \mathcal{R} \to \mathbb{R}^n$ are uniformly bounded on an open set $(0, t_f) \times \mathcal{R} \subseteq \mathbb{R}^{n+1}$, measurable with respect to variable $t \in (0, t_f)$ and uniformly Lipschitz continuous in $x \in \mathcal{R} \subseteq \mathbb{R}^n$ (with a common constant for almost all t). Recall that the above functions a and b are often denoted as Caratheodory functions (see (9; 1; 18) for theoretical details). From the general existence/uniqness theory for ordinary differential equations with Caratheodory right hand sides it follows that for every $u(\cdot) \in \mathbb{L}^1_m(0, t_f)$, where $\mathbb{L}^1_m(0, t_f)$ is the Lebesgue space of all m-valued integrable functions, problem (1) has a unique absolutely continuous solution $x(\cdot)$.

The aim of our investigations is to study the affine structure of the basic equation from (1) and to establish some continuity/approximability properties of the corresponding conventional control system. First, we examine some convergence properties of 1 under perturbations of controls and initial states. Second, we apply the abstract techniques of the set-valued analysis, rewrite the given affine control systems in the generalized form of Filippov and establish the corresponding approximability for the relaxed NACSs. This make it possible to consider some special kind of robustness for sliding mode control processes. Moreover, we also apply the obtained general theoretical results to some classes of hybrid systems with autonomous location transitions.

Motivated by numerous applications, let us consider a system (1) over a set $\mathcal{U}$ of bounded measurable control inputs. In this paper we assume that the set of admissible controls $\mathcal{U}$ has the following structure

$$\mathcal{U} := \{v(\cdot) \in \mathbb{L}^1_m(0, t_f) \mid v(t) \in U \text{ a.e. on } [0, t_f]\},$$

where $U \subseteq \mathbb{R}^m$ is a compact and convex set. We call a dynamical system of the type (1) a nonlinear affine control system (NACS). Note that under some additional smoothness assumptions the NACS introduced above represents the main mathematical model for the classical sliding mode control theory (see e.g., (20; 4)). The sliding mode based control design for system (1) is usually related to a bounded feedback control strategy of the form

$$w(t, x) := \tilde{w}(\sigma(t, x), \dot{\sigma}(t, x), ..., \sigma^{(l-1)}(t, x)), \tag{2}$$

where $\tilde{w} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^m$ is a bounded measurable function and $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^s$ is a smooth s-valued output of the system under consideration. The number $l \in \mathbb{N}$ is a relative degree associated with the given system (see e. g., (13; 14) for details). In this case one also deals with a time-depending control function that is determined on trajectories of the system as a composite function $u_w(t) \equiv w(t, x(t))$ for a trajectory $x(\cdot)$ of (1). Since w is a bounded and $x(\cdot)$ an absolutely continuous function, there exists a compact convex set $U \subseteq \mathbb{R}^m$ such that $u_w(\cdot) \in \mathcal{U}$. The sliding mode control technique has become a modern application focus of nonlinear control theory (20; 10; 14; 15). During the last two decades, there has been considerable effort to develop theoretical and computational frameworks for classical and hight order sliding modes (see e.g., (20) and (15)).

In parallel to the classical NACS (1) we also study here a hybrid version of the above NACSs and introduce a variant of the general hybrid systems concept. In this paper, we restrict our consideration to nonlinear affine hybrid systems with autonomous location transitions.

Definition 1. A *nonlinear affine hybrid system* (NAHS) is a 7-tuple $\{\mathcal{Q}, \mathcal{X}, U, \mathcal{A}, \mathcal{B}, \mathcal{U}, \Psi\}$, where

- $\mathcal{Q}$ is a finite set of discrete states (called *locations*);
- $\mathcal{X} = \{\mathcal{X}_q\}, q \in \mathcal{Q}$, is a family of state spaces such that $\mathcal{X}_q \subseteq \mathbb{R}^n$;
- $U \subseteq \mathbb{R}^m$ is a set of admissible control input values (called *control set*);
- $\mathcal{A} = \{a_q(\cdot,\cdot)\}$, $\mathcal{B} = \{b_q(\cdot,\cdot)\}$, $q \in \mathcal{Q}$ are families of uniformly bounded on an open set $(0, t_f) \times \mathcal{R}$ Caratheodory functions $a_q, \ b_q : (0, t_f) \times \mathcal{R} \to \mathbb{R}^{n \times m}$;
- $\mathcal{U}$ is the set of admissible control functions introduced above;
- Ψ is a subset of Ξ, where

$$\Xi := \{(q, x, q', x') \ : \ q, q' \in \mathcal{Q}, x \in \mathcal{X}_q, \ x' \in \mathcal{X}_{q'}\}.$$

We suppose that smooth functions $m_{q,q'} : \mathbb{R}^n \to \mathbb{R}$, where $q, \ q' \in \mathcal{Q}$, are given such that the surfaces

$$M_{q,q'} := \{x \in \mathbb{R}^n \ : \ m_{q,q'}(x) = 0\}$$

are pairwise disjoint. The given surfaces $M_{q,q'}$ represent switching sets at which a switch from location q to location q' can take place. We say that a location switching from q to q' occurs at a switching time $t^{switch} \in [0, t_f]$. We now consider a NAHS with $r \in \mathbb{N}$ switching times $\{t_i\}$, $i = 1, ..., r$, where $0 = t_0 < t_1 < ... < t_{r-1} < t_r = t_f$. Note that the above sequence of switching times $\{t_i\}$ is not defined a priory. A hybrid control system remains in location $q_i \in \mathcal{Q}$ for all $t \in [t_{i-1}, t_i)$, $i = 1, ..., r$. We refer to (5; 7; 16; 3) for some alternative concepts of hybrid systems and note that the theoretical approach proposed in this paper can also be applied to wide classes of hybrid/switched systems with the above nonlinear affine structure. In the following, we recall the notion of hybrid trajectory of systems under consideration (see e.g., (3)).

Definition 2. An admissible *hybrid trajectory* associated with a NAHS is a triple $\mathbf{X} = (x(\cdot), \{q_i\}, \tau)$, where $x(\cdot)$ is a continuous part of trajectory, $\{q_i\}_{i=1,...,r}$ is a finite sequence of locations and τ is the corresponding sequence of switching times such that $x(0) = x_0 \notin \bigcup_{q \in \mathcal{Q}} M_q$ and for each $i = 1, ..., r$ and every admissible control $u(\cdot) \in \mathcal{U}$ we have

- $x_i(\cdot) = x(\cdot)|_{(t_{i-1}, t_i)}$ is an absolutely continuous function on (t_{i-1}, t_i) continuously prolongable to $[t_{i-1}, t_i]$, $i = 1, ..., r$;
- $\dot{x}_i(t) = a_{q_i}(t, x_i(t)) + b_{q_i}(t, x_i(t))u_i(t)$ for almost all times $t \in [t_{i-1}, t_i]$, where $u_i(\cdot)$ is a restriction of the chosen control function $u(\cdot)$ on the time interval $[t_{i-1}, t_i]$.

Note that the pair $(q, x(t))$ represents the hybrid state at time t, where q is a location $q \in \mathcal{Q}$ and $x(t) \in \mathbb{R}^n$. Definition 2 describes the dynamic of the given NAHS. Since $x(\cdot)$ from Definition 2 is a continuous function, this concept describes a class of hybrid systems without impulse components of the (continuous) trajectories. Therefore, the corresponding switching sets $M_{q,q'}$ are defined for $x(t_i) = x(t_{i+1})$, $i = 1, ..., r-1$. Under the above assumptions, for each admissible control $u(\cdot) \in \mathcal{U}$ and for every interval $[t_{i-1}, t_i]$ (for every location $q_i \in \mathcal{Q}$) there exists a unique absolutely continuous solution of the nonlinear affine differential equations from Definition 2. This means that for each $u(\cdot) \in \mathcal{U}$ we have a unique absolute continuous trajectory of NAHS under consideration. Moreover, the switching times $\{t_i\}$ and the discrete trajectory $\{q_i\}$ for a NAHS are also uniquely defined. Note that the evolution equation for the trajectory $x(\cdot)$ of a NAHS can also be represented as follows

$$\begin{aligned} \dot{x}(t) = \sum_{i=1}^{r} \chi_{[t_{i-1}, t_i)}(t) \big(a_{q_i}(t, x_i(t)) + \\ b_{q_i}(t, x_i(t)) u_i(t) \big) \ \ \text{a.e. on } [0, t_f], \ \ x(0) = x_0 \end{aligned} \quad (3)$$

where $\chi_{[t_{i-1}, t_i)}(\cdot)$ is the characteristic function of the interval $[t_{i-1}, t_i)$ for $i = 1, ..., r$. Note that for classes of hybrid systems of the NAHS-type the function χ also depends on the trajectory of the given system. Evidently, this dependence is characterized by the specific switching mechanism introduced above.

Driven by engineering requirements, there has been increasing interest in general structural properties of sliding mode, hybrid and in general, of variable structure systems (3). In particular, a continuity and the corresponding approximability of a control system with respect to controls and initial dates play an important role in mathematical control theory. This property can firstly be interpreted from the point of view of possible consistent numerical (for example, discrete) approximations of dynamical systems. Otherwise, it also can be applied to a specific robustness analysis of the affine systems under consideration.

The reminder of our paper is organized as follows. Section 2 contains the relevant mathematical results. In particular, we consider the necessary basic facts from the celebrated Filippov theory for discontinuous dynamical systems. In Section 3, the conventional NACSs are considered. Moreover, we also examine the corresponding differential inclusions. We study the general continuity property and establish some over estimations for the above dynamical systems. Section 4 is devoted to some classes of hybrid systems and also to the conventional sliding mode control methodology. In this section we apply our main analytical results from the previous sections to hybrid and sliding mode dynamics. Section 5 summarizes the article and also points possible directions for further research.

2. PRELIMINARIES

We first provide some relevant concepts and facts. Consider the initial value problem given by

$$\dot{z}(t) = g(z(t)), \ \text{a.e. on } [0, t_f], \ z(0) = z_0, \quad (4)$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a Lebesgue measurable and locally essentially bounded (bounded on a neighborhood of every point, excluding sets of measure zero (9)). The following Filippov set-valued map $\mathcal{K}[g] : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ related to problem (4) is defined as follows:

$$\mathcal{K}[g](z) := \bigcap_{\delta>0} \bigcap_{\mu(\mathcal{S})=0} \overline{\text{co}}\{g(\mathcal{V}_\delta(z)) \setminus \mathcal{S}\},$$

where $\mathcal{V}_\delta(z)$ is an open ball centered at z with radius δ and μ is a Lebesgue measure. Dynamical systems of the form

$$\dot{z}(t) \in \mathcal{K}[g](z(t)) \text{ a.e. on } [0, t_f],\ z(0) = z_0 \qquad (5)$$

are called differential inclusions (see (9; 1; 18) for details). An absolutely continuous function $z : [0, t_f] \to \mathbb{R}^n$ that satisfies (5) is said to be Filippov solution of the original initial value problem (4). As a consequence of the general theory of differential inclusions (see e.g., (17), Theorem 1) we obtain the existence of a set $\mathcal{S}_g \subset \mathbb{R}^n$ of measure zero such that for every set $\mathcal{W} \subset \mathbb{R}^n$ of measure zero,

$$\mathcal{K}[g](z) \equiv \overline{\text{co}}\{\lim_{j\to\infty} g(z_j) \mid z_j \to z,\ z_j \notin \mathcal{S}_g \bigcup \mathcal{W}\}.$$

Since the set-valued map $\mathcal{K}[g]$ introduced above is upper semi-continuous with nonempty, convex and compact values, and it is also locally bounded (see (9)), it follows that Filippov solution to (4) exists. Generally, a Filippov solution to a dynamical system of the type (5) is not necessary unique. [1]

Let us now recall that the celebrated Filippov Selection Lemma (see e.g., (12)) gives rise to an explicit parametrization of the convex-valued differential inclusion of the type (5). We formulate here a consequence of this result and also refer to (11; 8; 18) for theoretical details.

Proposition 3. A function $z(\cdot)$ is a solution of (5) if and only if it is a solution of the Gamkrelidze system

$$\dot{\eta}(t) = \sum_{j=1}^{n+1} \alpha^j(t)\gamma(t, \eta(t), u^j(t)) \text{ a.e. on } [0, t_f], \qquad (6)$$
$$\eta(0) = x_0,\ \alpha(\cdot) \in \Lambda(n+1),$$

where $\gamma : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous, $\gamma(t, \cdot, \cdot)$ is a continuously differentiable function with the bounded derivative γ_η, $\alpha(\cdot) := \left(\alpha^1(\cdot), ...\alpha^{n+1}(\cdot)\right)^T$, $u^j(\cdot) \in \mathcal{U}$ and

$$\Lambda(n+1) := \{\alpha(\cdot)\ |\alpha^j(\cdot) \in \mathbb{L}^1_1(0, t_f),\ \alpha^j(t) \geq 0,$$
$$\sum_{j=1}^{n+1} \alpha^j(t) = 1\ \ \forall t \in [0, t_f]\},$$

is a set of admissible multipliers $\alpha(\cdot)$.

Proposition 3 establishes a parametrization of the convex-valued inclusion (5) by an equivalent control system, namely, by system (5). Clearly, every trajectory of (5) is also a trajectory of (6). Evidently, that the vector-function $\alpha(\cdot)$ introduced above takes its values from the simplex of the space $\mathbb{R}^{n+1}$. It is also possible to prove that every $\alpha^j(\cdot)$, $j = 1, ..., n+1$ takes only two values, namely 0 and 1 and can be interpreted as a characteristic function of a measurable subset of $[0, t_f]$ (see (11) for details). Note that a pair $v(\cdot) := \left((\alpha^1(\cdot), ..., \alpha^{n+1}(\cdot))^T, (u^1(\cdot), ..., u^{n+1}(\cdot))^T\right)$ from the Cartesian product $\Lambda(n+1) \times \mathcal{U}^{n+1}$ can be considered as a control vector for the relaxed system (6).

Finally, let us prove the following abstract result.

Theorem 4. Let $\mathcal{X}$ be a separable Banach space and $\{S, \Sigma, \mu\}$ be a measurable space with a probability measure μ. Let $C \subset \mathcal{X}$ be closed and convex. If $h : S \to C$ is a μ-measurable function, then $\int h(\tau)\mu(d\tau) \in C$.

Proof: Assume $\beta := \int h(\tau)\mu(d\tau) \notin C$ and let

$$V(\beta, R) := \{\xi \in \mathcal{X} \mid ||\xi - \beta||_{\mathcal{X}} < R\}$$

be the ball around β with radius R. Evidently, there is a radius R such that $C \cap V(\beta, R) = \emptyset$. Using a Separating Theorem from convex analysis (see e.g., (19)), we obtain a nontrivial $\mathcal{L} \in \mathcal{X}^*$ with $\mathcal{L}\xi \leq \mathcal{L}\psi$ for all variables $\xi \in V(\beta, R)$, $\psi \in C$. By $\mathcal{X}^*$ we have denoted the (topological) dual space to $\mathcal{X}$. Thereby we have the inequality

$$\sup_{||\xi||_{\mathcal{X}} \leq 1} (\mathcal{L}(\beta + R\xi)) = \mathcal{L}\beta + R|\mathcal{L}| \leq \mathcal{L}(h(\tau)),\ \tau \in S.$$

and - by integration with respect to μ - the corresponding inequality

$$\mathcal{L}\beta + R|\mathcal{L}| \leq \int \mathcal{L}(h(\tau))\mu(d\tau). \qquad (7)$$

Because of $\int \mathcal{L}(h(\tau))\mu(d\tau) = \mathcal{L}\int h(\tau)\mu(d\tau) = \mathcal{L}\beta$, (7) leads to $\mathcal{L}\beta + R|\mathcal{L}| \leq \mathcal{L}\beta$ contradicting the fact that $\mathcal{L}$ is nontrivial. Therefore $\beta = \int h(\tau)\mu(d\tau)$ belongs to C. The proof is finished. □

We will use Theorem 4 in the next section and derive an over estimation of the state variables of NACSs under consideration.

3. CONTINUITY AND APPROXIMABILITY OF THE CONVENTIONAL AFFINE CONTROL SYSTEMS

This section is devoted to some fundamental continuity properties of NACSs introduced above. The original NACS (1) is studied over the set of measurable control functions (the elements of $\mathbb{L}^1_m(0, t_f)$). In parallel with this main control space we also consider its (topologically) dual, namely, the space $\mathbb{L}^\infty_m(0, t_f)$ of all essentially bounded m-valued functions. For $v(\cdot) \in \mathbb{L}^1_m(0, t_f)$, $\psi(\cdot) \in \mathbb{L}^\infty_m(0, t_f)$ the corresponding duality pairing is denoted by $\langle v(\cdot), \psi(\cdot)\rangle_{1,\infty}$. Recall that a sequence $\{u^k(\cdot)\}$ from $\mathbb{L}^1_m(0, t_f)$ converges weakly to an element $u(\cdot) \in \mathbb{L}^1_m(0, t_f)$ if

$$\lim_{k\to\infty} \langle u^k(\cdot), v(\cdot)\rangle_{1,\infty} = \langle u(\cdot), v(\cdot)\rangle_{1,\infty}$$

for all $v(\cdot) \in \mathbb{L}^\infty_m(0, t_f)$. We us use the notation $\mathbb{C}_n(0, t_f)$ for the Banach space of all n-valued continuous functions on $[0, t_f]$ equipped with the usual sup-norm $|| \cdot ||_{\mathbb{C}_n(0,t_f)}$. We now are able to present our main theoretical result. This result establishes a fundamental continuity property of the NACSs under consideration.

Theorem 5. Consider the initial value problem (1), an associated sequence $\{u^k(\cdot)\}$, $k \in \mathbb{N}$ of control functions $u^k(\cdot) \in \mathbb{L}^1_m(0, t_f)$ and a sequence of vectors $\{x^k_0\}$ from $\mathbb{R}^n$ such that $\{u^k(\cdot)\}$ converges weakly to an element $u(\cdot) \in \mathbb{L}^1_m(0, t_f)$ and $\lim_{k\to\infty} x^k_0 = x_0$. Then, for all $k \in \mathbb{N}$ the following problem

[1] Note that an initial value problem for a non-stationary differential equation $\dot{z}(t) = \tilde{g}(t, \tilde{z}(t))$ can be reduced to the above problem (4) by introducing the additional equation $\dot{t} = 1$, the corresponding variable $z := (\tilde{z}^T, t)^T$ and the right-hand side $g := (\tilde{g}^T, 1)^T$.

$$\dot{x}(t) = a(t,x(t)) + b(t,x(t))u^k(t) \text{ a.e. on } [0,t_f], \quad x(0) = x_0^k, \tag{8}$$

has a unique (absolutely continuous) solution $x^k(\cdot)$ on the time interval $[0,t_f]$ and moreover,

$$\lim_{k\to\infty} ||x(\cdot) - x^k(\cdot)||_{\mathbb{C}_n(0,t_f)} = 0,$$

where $x(\cdot)$ is the solution of (1) corresponding to $u(\cdot)$.

Proof: Evidently, the graph

$$\text{Graph}(x(\cdot)) := \{(t,x(t)) \mid t \in (0,t_f)\}$$

of the absolutely continuous solution $x(\cdot)$ to 1 is a compact subset of $(0,t_f) \times \mathcal{R}$. For a small $\epsilon > 0$ a closed ϵ-neighborhood $\mathcal{V}_\epsilon$ of Graph$(x(\cdot))$ also belongs to $(0,t_f)\times\mathcal{R}$. Without loss of generality we can assume that functions a and b are also uniformly bounded on $\mathcal{V}_\epsilon$, measurable with respect to variable t and uniformly Lipschitz continuous in x (for all $(t,x) \in \mathcal{V}_\epsilon$). From the Caratheodory existence theorem it follows that for all $k \in \mathbb{N}$ there exists a $\delta_k > 0$ such that the initial value problem (8) has a unique solution on the interval $[0,\delta_k]$. This solution can be extended to a time interval $I_k := [0,\Delta_k]$, where

$$\Delta_k \le \min\{\tau \in (0,t_f) \mid x(\tau) \in \partial\text{Pr}[\mathcal{V}_\epsilon]\}$$

and $\partial\text{Pr}[\mathcal{V}_\epsilon]$ is the boundary of the projection of $\mathcal{V}_\epsilon$ on the space $\mathbb{R}^n$. Define $\tilde{x}^k(\cdot) := x^k(\cdot) - x(\cdot)$, $\tilde{u}^k(\cdot) := u^k(\cdot) - u(\cdot)$ and $\tilde{x}_0^k := x_0^k - x_0$. For a.e. t from I_k, $k \in \mathbb{N}$ we have the following differential equation

$$\begin{aligned}&\dot{\tilde{x}}^k(\cdot) = \dot{x}^k(t) - \dot{x}(t) = a(t,x(t)+\tilde{x}^k(t)) + b(t,x(t)+\\ &\tilde{x}^k(t))[u(t)+\tilde{u}^k(t)] - [a(t,x(t)) + b(t,x(t))u(t)] =\\ &[a(t,x(t)+\tilde{x}^k(t)) - a(t,x(t))] + [b(t,x(t)+\tilde{x}^k(t)) -\\ &b(t,x(t))]u(t) + b(t,x(t)+\tilde{x}^k(t))\tilde{u}^k(t) = [a(t,x(t)+\\ &\tilde{x}^k(t)) - a(t,x(t))] + [b(t,x(t)+\tilde{x}^k(t)) -\\ &b(t,x(t))]u^k(t) + b(t,x(t))\tilde{u}^k(t),\end{aligned} \tag{9}$$

where $\tilde{x}^k(0) = \tilde{x}_0^k$. Let L be a common Lipschitz constant for the above functions a and b. Then for $||\tilde{x}||_{\mathbb{R}^n} \le \epsilon$ we obtain the estimates

$$\begin{aligned}&||a(t,x+\tilde{x}) - a(t,x)|| \le L||\tilde{x}||,\\ &||b(t,x+\tilde{x}) - b(t,x)|| \le L||\tilde{x}|| \quad \forall x \in \text{Pr}[\mathcal{V}_\epsilon].\end{aligned} \tag{10}$$

Since $\{u^k(\cdot)\}$ converges weakly to $u(\cdot)$, we have the weak convergence of the sequence $\{\tilde{u}^k(\cdot)\}$ to the zero element $0_{\mathbb{L}^1_m(0,t_f)}$ of the space $\mathbb{L}^1_m(0,t_f)$. Using the Dunford-Pettis Theorem (see e.g., (6)), we conclude that all the functions from $\{\tilde{u}^k(\cdot)\}$ have a common modulus of absolute continuity of the Lebesgue integral, i.e., there exists a nondecreasing continuous function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{l\to 0+} \nu(l) = 0$ and

$$\int_E ||\tilde{u}^k(t)||dt \le \nu(l) \tag{11}$$

for every (Lebesgue) measurable subset E of $[0,t_f]$ with measure mes$\{E\} \le l$. Let $K > 0$ be a boundness constant for the function b. Since $\{\tilde{u}^k(\cdot)\}$ converges weakly to $0_{\mathbb{L}^1_m(0,t_f)}$, we conclude that

$$\lim_{k\to\infty} \int_0^t b(\tau,x(\tau))\tilde{u}^k(\tau)d\tau = 0 \quad \forall t \in [0,t_f].$$

The obtained pointwise convergence of the above integral implies the uniform convergence

$$\lim_{k\to\infty} ||\int_0^\cdot b(\tau,x(\tau))\tilde{u}^k(\tau)d\tau||_{\mathbb{C}_n(0,t_f)} = 0. \tag{12}$$

Moreover, from (11) it follows that

$$\begin{aligned}&||\int_0^{t^1} b(\tau,x(\tau))\tilde{u}^k(\tau)d\tau - \int_0^{t^2} b(\tau,x(\tau))\tilde{u}^k(\tau)d\tau|| \le\\ &K\int_{t^2}^{t^1} ||\tilde{u}^k(\tau)||d\tau \le K\nu(t^1 - t^2)\end{aligned}$$

for all $t^1, t^2 \in [0,t_f]$. Therefor, the function $K\nu(l)$ is a common modulus of absolute continuity for all functions $\int_0^\cdot b(\tau,x(\tau))\tilde{u}^k(\tau)d\tau$ on the time interval $[0,t_f]$. We now rewrite the resulting differential equation from (9) in an integral form:

$$\begin{aligned}&\tilde{x}^k(t) = \tilde{x}_0^k + \int_0^t [a(\tau,x(\tau)+\tilde{x}^k(\tau)) - a(\tau,x(\tau))]d\tau +\\ &\int_0^t [b(\tau,x(\tau)+\tilde{x}^k(\tau)) - b(\tau,x(\tau))]u^k(\tau)d\tau +\\ &\int_0^t b(\tau,x(\tau))\tilde{u}^k(\tau)d\tau.\end{aligned}$$

Using (10) and (11), we derive the following estimate for the sup-norm of solution $\tilde{x}^k(\cdot)$ on the time interval $I_k = [0,\Delta_k]$

$$\begin{aligned}&||\tilde{x}^k(\cdot)||_{\mathbb{C}_n(0,\Delta_k)} \le L\Delta_k||\tilde{x}^k(\cdot)||_{\mathbb{C}_n(0,\Delta_k)} +\\ &2\nu(\Delta_k)L||\tilde{x}^k(\cdot)||_{\mathbb{C}_n(0,\Delta_k)} + ||\tilde{x}_0^k|| +\\ &||\int_0^\cdot b(\tau,x(\tau))\tilde{u}^k(\tau)d\tau||_{\mathbb{C}_n(0,\Delta_k)},\end{aligned}$$

The last inequality implies that

$$\begin{aligned}&||\tilde{x}^k(\cdot)||_{\mathbb{C}_n(0,\Delta_k)}[1 - L\Delta_k - 2\nu(\Delta_k)L] \le ||\tilde{x}_0^k|| +\\ &||\int_0^\cdot b(\tau,x(\tau))\tilde{u}^k(\tau)d\tau||_{\mathbb{C}_n(0,\Delta_k)}.\end{aligned} \tag{13}$$

For a suitable constant $0 < c < 1$ one can put $\Delta_k = \Delta$ for all sufficiently large $k \in \mathbb{N}$, where Δ is a solution of the following equation $L\Delta + 2\nu(\Delta)L = c$. Using the properties of the function $\nu(\cdot)$ introduced above, we conclude that the last equation has a unique solution. From (12), (13) and from the given convergence of the initial points $\lim_{k\to\infty} ||\tilde{x}_0^k|| = 0$, we finally deduce that

$$||\tilde{x}^k(\cdot)||_{\mathbb{C}_n(0,\Delta_k)} \le ||\tilde{x}_0^k|| + ||\int_0^\cdot b(\tau,x(\tau))\tilde{u}^k(\tau)d\tau||_{\mathbb{C}_n(0,\Delta_k)}$$

and $\lim_{k\to\infty} ||\tilde{x}^k(\cdot)||_{\mathbb{C}_n(0,\Delta_k)} = 0$ on a common fixed time interval $\tilde{I}_k := [0,\Delta]$. We now define a new initial time instant $\tilde{t}^1 := \Delta$ and consider the next interval $[\tilde{t}^1, \tilde{t}^1 + \Delta]$ of the length Δ. Since all the above inequalities/estimates are still valid on this new time interval, the resulting differential equation from (9) has a solution also on the interval $[\tilde{t}^1, \tilde{t}^1 + \Delta]$ which converges to zero. Further, consider the initial point $\tilde{t}^j := j\Delta$ for $j = 2, \ldots$. In a finite number of j-steps we will cover the full time interval $[0,t_f]$. Therefore, the initial value problem for the resulting differential equation from (9) with the corresponding initial value $\tilde{x}_0^k$ and the original initial value problem (8) have unique solutions. These solutions are uniquely prolongable on the full time interval $[0,t_f]$. Moreover, we have $\lim_{k\to\infty} ||\tilde{x}^k(\cdot)||_{\mathbb{C}_n(0,\Delta_k)} = 0$ on $[0,t_f]$. This implies that $\lim_{k\to\infty} ||x(\cdot) - x^k(\cdot)||_{\mathbb{C}_n(0,t_f)} = 0$. □

Note that Theorem 5 can be interpreted as follows: for the NACS (1) the weak convergence of controls and the convergence of the initial conditions cause the uniform convergence of the corresponding state variables. Clearly, this result can also be interpreted as a kind of "robustness" of NACS (1) with respect to perturbations of controls and initial state variables.

We now apply the abstract convexity result from Section 2, namely, Theorem 4 and derive an over estimation for solutions of the initial value problem (1).

Theorem 6. Consider the initial value problem (1) under conditions of Section 1. Let $a(t,x), b(t,x) \in C$ for all $(t,x) \in \mathbb{R}^{n+1}$, where C is a compact convex subset of $\mathbb{R}^n$, and let $U \subseteq \mathbb{R}^m$ be a convex compact set containing 0. Then $x(t) \in x_0 + tC[1 + \text{diam}(U)]$, where $\text{diam}(U)$ is a diameter of the set U.

Proof: Introduce the following Lebesgue probability measure $\mu = \tau/t_f$ on $[0,t] \subseteq [0,t_f]$. Then we apply Theorem 4 to our affine control system and compute the state of (1) at a time $t \in [0,t_f]$

$$x(t) = x_0 + \int_0^t [a(\tau, x(\tau)) + b(\tau, x(\tau))u(\tau)]d\tau =$$

$$x_0 + t\Big[\int_0^t a(\tau, x(\tau))\mu(d\tau) + \int_0^t b(\tau, x(\tau))u(\tau)\mu(d\tau)\Big] \in$$

$$x_0 + tC + tC\text{diam}(U)$$

The proof is completed. □

The presented result makes it possible to estimate the state vector of a NACS of the type (1) under some general assumptions. In particular, from Theorem 6 follows that the reachable sets of systems (1) on $[0,t_f]$ and belongs to the compact convex set $x_0 + t_f C[1 + \text{diam}(U)]$.

4. SLIDING MODE DYNAMICS AND NONLINEAR AFFINE HYBRID SYSTEMS

Let us now apply the theoretical results form the previous section to sliding mode dynamics and to the NAHSs mentioned in Introduction. Returning to control systems of type (1) with the feedback control strategy (2), we consider the corresponding closed-loop system

$$\begin{aligned} &\dot{x}(t) = a(t,x(t)) + b(t,x(t))w(t,x(t)) \text{ a.e. on } [0,t_f], \\ &x(0) = x_0. \end{aligned} \tag{14}$$

We now investigate the continuity/approximability properties of the closed-loop system from (14) under perturbations of controls and initial states. Evidently, some "stable" properties of (14) in the above sense can be interpreted as a kind of robustness for the corresponding control systems with sliding mode regimes.

Generally, the state equation from the resulting system (14) is a differential equation with discontinuous right hand side. The discontinuity here is determined by the specific form of the chosen feedback control strategy. The function (2) usually guarantees some strong stability properties of the generic sliding mode regimes. For example, the asymptotic Lyapunov-type or the finite-time stability with respect to the (smooth) sliding surface $\sigma(t,x) = 0$ (see (9; 15)). Since the closed-loop system from (14) is a special case of (4), we apply the Filippov approach from Section 2 and consider the associated differential inclusion

$$\dot{\tilde{x}}(t) \in \mathcal{K}[a,b](\tilde{x}(t)) \text{ a.e. on } [0,t_f],\ \tilde{x}(0) = (x_0, 0), \tag{15}$$

where $\tilde{x} := (x^T, t)^T$ is the extended state vector,

$$\mathcal{K}[a,b](\tilde{x}) := \overline{\text{co}}\{\lim_{j\to\infty}[a(\tilde{x}_j) + b(\tilde{x}(t))w(\tilde{x}(t))] \mid \tilde{x}_j \to \tilde{x},\ \tilde{x}_j \notin \mathcal{S}\bigcup\mathcal{W}\}$$

and $\mathcal{S}, \mathcal{W} \subset \mathbb{R}^n$ are some sets of measure zero. Note that the initial value problem (15) not obligatory has a unique solution. Let $\mathbb{L}^p_m(0,t_f,\mathbb{R}^n)$, $p \in \mathbb{N}$ be a Lebesgue space of the p-measurable functions given on the product $[0,t_f] \times \mathbb{R}^n$ and let $||\cdot||_{\mathbb{L}^p_m(0,t_f,\mathbb{R}^n)}$ be the corresponding norm. We now specify the general Theorem 5 in the case of the closed-loop system (14).

Theorem 7. Consider the initial value problem (1) under conditions of Section 1 and the corresponding closed-loop system (14). Let $\{w^k(\cdot,\cdot)\}$, $k \in \mathbb{N}$ be a sequence of some bounded measurable $\mathbb{L}^p_m(0,t_f,\mathbb{R}^n)$-convergent feedback controls $w^k : [0,t_f] \times \mathbb{R}^n \to \mathbb{R}^m$ with

$$\lim_{k\to\infty} ||w^k(\cdot,\cdot) - w(\cdot,\cdot)||_{\mathbb{L}^p_m(0,t_f,\mathbb{R}^n)} = 0,$$

Let also $\lim_{k\to\infty} x_0^k = x_0$. Then, for all $k \in \mathbb{N}$ the approximating initial value problem

$$\begin{aligned} &\dot{x}(t) = a(t,x(t)) + b(t,x(t))w^k(t,x(t)) \text{ a.e. on } [0,t_f], \\ &x(0) = x_0^k, \end{aligned}$$

has an absolutely continuous solution $x^k(\cdot)$ on the time interval $[0,t_f]$ and $\lim_{k\to\infty} ||x(\cdot) - x^k(\cdot)||_{\mathbb{C}_n(0,t_f)} = 0$, where $x(\cdot)$ is a solution generated by the closed-loop system (14).

Proof: For every $k \in \mathbb{N}$ we define

$$\mathcal{K}^k[a,b](\tilde{x}) := \overline{\text{co}}\{\lim_{j\to\infty}[a(\tilde{x}_j) + b(\tilde{x}(t))w^k(\tilde{x}(t))] \mid \tilde{x}_j \to \tilde{x},\ \tilde{x}_j \notin \mathcal{S}\bigcup\mathcal{W}\}$$

and consider the corresponding sequence of the approximating initial value problems

$$\dot{\tilde{x}}^k(t) \in \mathcal{K}^k[a,b](\tilde{x}(t)),\ \tilde{x}^k(0) = (x_0^k, 0). \tag{16}$$

Since for every $k \in \mathbb{N}$ the set-valued map $\mathcal{K}^k[a,b]$ is upper semi-continuous with nonempty, convex and compact values, and it is also locally bounded it follows that Filippov solution $x^k(\cdot)$ to an approximating initial value problem exists (see (9)). On the given trajectories of the closed-loop systems under consideration we now determine the following functions

$$u_w(t) := w(t,x(t)),\ u_w^k(t) := w^k(t,x^k(t)).$$

Since the measurable functions w and w^k are bounded and $x(\cdot)$, $x^k(\cdot)$ are absolutely continuous function, the introduced functions u_w and u_w^k are also measurable and bounded. Moreover, there exists a compact convex set $U \subseteq \mathbb{R}^m$, $0 \in U$ such that $u_w(t), u_w^k(t) \in U$ for almost all $t \in [0,t_f]$. For this control set U we introduce the following differential inclusions

$$\begin{aligned} &\dot{z}(t) \in \mathcal{F}_U(z(t)) := a(t,z(t)) + b(t,z(t))U, \\ &\dot{z}^k(t) \in \mathcal{F}_U(z^k(t)) := a(t,z^k(t)) + b(t,z^k(t))U. \end{aligned} \tag{17}$$

Since inclusions from (17) have a convex right hand side, these inclusions are equivalent to the corresponding Gamkrelidze system (see Proposition 3). For some multipliers $\alpha \in \Lambda(n+1)$ we define

$$v(\cdot) := \big((\alpha^1(\cdot), ..., \alpha^{n+1}(\cdot))^T, (u_w(\cdot), ..., u^{n+1}(\cdot))^T\big)$$

and $v^k(\cdot) := \big((\alpha^1(\cdot), ..., \alpha^{n+1}(\cdot))^T, (u_w^k(\cdot), ..., u^{n+1}(\cdot))^T\big)$ for every $k \in \mathbb{N}$. Evidently, the convergence property

$\lim_{k\to\infty} ||w^k(\cdot,\cdot) - w(\cdot,\cdot)||_{\mathbb{L}^p_m(0,t_f,\mathbb{R}^n)} = 0$ implies the week convergence of the sequence $\{u^k_w(t)\}$ to the control function $u_w(\cdot)$ and also implies the week convergence of $\{v^k(\cdot)\}$ to $v(\cdot)$. Therefore, from Theorem 5 it follows that $\lim_{k\to\infty} ||\eta(\cdot)-\eta^k(\cdot)||_{\mathbb{C}_n(0,t_f)} = 0$, where $\eta(\cdot)$ and $\eta^k(\cdot)$ are solutions to the corresponding initial value problems for Gamkrelidze systems with $(x_0, v(\cdot))$ and $(x^k_0, v^k(\cdot))$. From the definitions of the right-hand sides of all the inclusions under consideration we deduce that

$$\mathcal{K}[a,b](\tilde{x}) \subseteq \mathcal{F}_U(z), \quad \mathcal{K}^k[a,b](\tilde{x}) \subseteq \mathcal{F}_U(z^k)$$

for all $x \in \mathbb{R}^n$. This means that the set of solutions to (17) contains all the solutions to the initial value problems (15) and (16). In the particular case with

$$v(\cdot) := \big((1,...,0)^T, (u_w(\cdot),...,0)^T\big),$$
$$v^k(\cdot) := \big((1,...,0)^T, (u^k_w(\cdot),...,0)^T\big)$$

we obtain $\lim_{k\to\infty} ||x(\cdot)-x^k(\cdot)||_{\mathbb{C}_n(0,t_f)} = 0$ and the proof is complete. □

It is well-known that some applied control strategies are caused by problems with incomplete information. The practically implementable sliding mode control schemes are in fact some observer-based schemes (see e.g., (4)). This fact motivates the following example.

Example 8. Under the assumptions of possible model uncertainties an adequate feedback control low of the type (2) contains the corresponding errors. This situation can be described, for example, by some bounded additive errors $\delta^k_p(\cdot)$, $p = 1,...,(l-1)$ in the state/derivatives estimations. The above-mentioned feedback control strategy can now be presented in the following form

$$\begin{aligned} &w^k(t,x(t)) := \tilde{w}(\sigma(t,x(t)+\delta^k_1(t)),\\ &\dot{\sigma}(t,x(t)+\delta^k_2(t)),...,\sigma^{(l-1)}(t,x(t)+\delta^k_{l-1}(t))), \end{aligned} \tag{18}$$

where $\{w^k\}$ is a sequence of the feedback control lows generated by a current k-estimation of the state and sliding surface. Assume that all the functions $\delta^k_p(\cdot)$ are elements of the space $\mathbb{L}^1_s(0,t_f)$ (recall that $\sigma(t,x)$ from (2) is a s-dimensional vector). Moreover, let us assume that $\lim_{k\to\infty} ||\delta^k_p||_{\mathbb{L}^1_s(0,t_f)} = 0$ for all $p = 1,...,(l-1)$. Evidently, a bounded sequence $\{w^k\}$ from (18) satisfies the conditions of Theorem 7. Then the trajectories generated by the feedback control strategies $\{w^k\}$ with uncertainties δ^k_p converge uniformly to a trajectory generated by the error-free control (2).

Let us now consider hybrid systems, namely, the NAHSs introduced in Section 1 (Definition 2 and Definition 2). It is well known that some important structural properties of the hybrid and switched dynamical systems can not be expressed by the same properties of active subsystems (locations). For example, the Lyapunov stability of a switched system is generally not a consequence of the corresponding stability properties of all the subsystems (see e.g., (16)). The same is also true, for example, with respect to optimality (see (2; 3; 7) for details). Otherwise, the dynamical structure of the NACSs make it possible to extend the fundamental continuity/approximability result from the main Theorem 5 to hybrid control systems under consideration.

Theorem 9. For a sequence of locations $\{q_i\} \subset \mathcal{Q}$, where $i = 1,...r$, consider the initial value problem (3), and associated sequences of controls

$$\{u^k_i(\cdot)\}, \; k \in \mathbb{N}, \; u^k_i(\cdot) \in \mathbb{L}^1_m(0,t_f)$$

such that every $\{u^k_i(\cdot)\}$ converges weakly to an element $u_i(\cdot) \in \mathbb{L}^1_m(0,t_f)$. Moreover, let $\{x^k_0\}$ from $\mathbb{R}^n$ be a convergent sequence of the initial vectors with $\lim_{k\to\infty} x^k_0 = x_0$. Then, for all $k \in \mathbb{N}$ the following initial value problem

$$\begin{aligned} \dot{x}(t) = &\sum_{i=1}^{r} \chi_{[t_{i-1},t_i)}(t)\big(a_{q_i}(t,x_i(t)) + \\ &b_{q_i}(t,x_i(t))u_i(t)\big) \;\text{ a.e. on } [0,t_f], \; x(0) = x^k_0, \end{aligned} \tag{19}$$

has a unique (absolutely continuous) solutions $x^k(\cdot)$ on the time interval $[0,t_f]$ and $\lim_{k\to\infty} ||x(\cdot)-x^k(\cdot)||_{\mathbb{C}_n(0,t_f)} = 0$, where $x(\cdot)$ is the solution of (3) corresponding to $u(\cdot)$.

Note that Theorem 9 is an immediate consequence of our mail Theorem 5. The corresponding proof contains two additional facts. Firstly, for a given sequence of locations the structure of the initial value problem (3) is similar to the structure of a NACS (1). Moreover, every control $u_i(\cdot)$ associated with the location q_i is considered as a reduction of an admissible general control function $u(\cdot)$ (see Definition 2). Therefore, the week convergence of all sequences $\{u^k_i(\cdot)\}$ from Theorem 9 also implies the week convergence of the function $u_i(\cdot)$ to an admissible control function $u(\cdot)$.

As we can see in contrast to the above-mentioned stability and optimality properties for general hybrid systems the established continuity property of the NAHS under consideration follows from the same continuity properties of the given subsystems. Evidently, Theorem 9 also determines a robustness-like behavior of the NAHS with respect to possible disturbances of control functions.

5. CONCLUDING REMARKS

In this contribution, we study general affine control systems and establish some fundamental continuity and approximability properties of the corresponding dynamical models. These properties are direct consequences of the given system structure and can also be interpreted as a kind of the robustness-like behavior with respect to possible disturbances of the admissible control functions. Using the obtained theoretical facts for affine systems, we next examine systems with sliding mode regimes and specify the above general result for affine control systems with a specific feedback control strategy. This strategy provides a usual modelling framework for the conventional sliding mode dynamics. Our main theoretic approach is also considered for a class of hybrid systems with autonomous location transitions. We introduce a class of nonlinear affine hybrid control systems and obtain the corresponding continuity property for these systems. Finally, note that the basic theoretical result presented in this paper can be extended to some other classes of hybrid and switched systems. It seem also be possible to study the presented continuity/approximability concept introduced in this paper in the context of the feedback control strategies for the above family of affine hybrid dynamical systems.

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