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ON THE ROLE OF HOMOGENEITY WHEN CONTROLLING SINGLE-LEADER NETWORKS

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ABSTRACT

This paper presents an approach to controlling agent positions in single-leader networks to target points while explicitly taking agent homogeneity into account. When the capabilities of agents to accomplish tasks at each of the targets are identical, then the label of the target points may be permuted while still expressing the same intention. In single-leader networks which are not completely controllable, such a permutation of the target points may at times move a target closer to the system's reachable subspace, thereby allowing the network to surpass the limitations on controllability when homogeneity is not considered explicitly. To fully exploit this property in homogeneous networks, it is then necessary to find the permutation of a target point which brings it closest to the reachable subspace. However, finding this optimal permutation is shown to be in general a non-deterministic polynomial-time (NP)-hard problem. Specific network topologies are identified for when finding such an optimal permutation of a target point can be advantageous when controlling single-leader networks. Moreover, an alternate view of the problem of finding optimal permutations is presented in which clustering-based algorithms can be applied to find suboptimal solutions.

Key Words: Networked control, leader-follower networks, multi-agent systems, controllability, homogeneous agents.

I. INTRODUCTION

Research in multi-agent systems has focused mainly on designing decentralized controllers that allow for agents to autonomously achieve global goals, such as reaching consensus, *e.g.*, [4, 6, 13, 16, 19] or achieving formations, e.g., [3, 7, 17]. However, in many of the intended applications for multi-agent systems, such as search-and-rescue, it is more likely that agents will be working closely with humans as opposed to acting completely autonomously. The research problem that arises involves understanding how a human can control and direct a network of agents to perform a set of spatially-distributed tasks, without directly communicating with each individual agent.

In this paper, we will focus on the problem of controlling a network of agents to reach a set of desired target points. To do so without controlling each agent directly, a single-leader control paradigm will be used. Here, a human is assumed to take direct control of a single "leader" agent within the network, while all other agents play the roles of "followers" by executing a nearest neighbor averaging rule to maintain cohesion. The idea is that by directly controlling the state of the leader agent, the follower agents' states can be affected so as to reach a set of desired target points as is the case in [8, 11, 15, 18]. A known drawback to using the single-leader approach to control, however, is that the majority of network topologies actually yield single-leader networks which are not completely controllable. Therefore, a desired target point generally cannot be reached directly by follower agents in the network.

To help improve upon the performance limitations of single-leader networks, this paper will present a new method for controlling them that directly exploits homogeneity in the agents' capabilities. Traditionally, the controllability of single-leader networks is viewed as a point-to-point property in the state space, where it is desired to control each agent to its corresponding target. However, if there are identical tasks to be completed at each target point and the ability of each agent to carry out that task is the same, then the labels of the target points can be arbitrarily permuted while still expressing the same intention. In other words, when controlling a system of homogeneous agents to targets, it does not matter which agent goes where, as long as there is an agent assigned to each of the target locations. Using this new approach for controlling homogeneous single-leader networks, controllability is no longer a point-to-point property of the system, but instead becomes a point-to-set property, where the set consists of all permutations of the target point.

For single-leader networks that are not completely controllable, exploiting agent homogeneity and permuting the labels of the targets may, at times, move a target point closer to the system's reachable subspace than before. Therefore,

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directly incorporating agent homogeneity when controlling single-leader networks can allow for the network to possibly exceed previously established limitations on its reachability. The caveat to exploiting homogeneity when controlling such networks, however, is that finding the optimal permutation of a target point that brings it closest to the system's reachable subspace is, in general, a non-deteministic polynonial-time (NP)-hard problem. Specific network topologies will be identified as to when optimally permuting a target point guarantees a performance increase, compared to when homogeneity is not explicitly taken into account. Moreover, it will be shown that finding the optimal target point permutation is equivalent up to a polynomial-time transformation to a Euclidean minimum sum-of-squares clustering problem on the target points with equality constraints on the cluster sizes. Using this alternate view of the problem, it is demonstrated that clustering-based techniques can be used to arrive at suboptimal solutions.

The idea of agents in a network being controlled so as to disregard individual identities has been explored in some related literature. In particular, [10] modeled agents in an index-free manner by using indicator distribution representations that resulted in integro-differential dynamics. This Eulerian approach was shown to parallel the traditional graph-theoretic approaches of modeling multi-agent dynamics. With the model, stability properties could be proven for systems of agents and very simple controllers were produced for certain multi-agent problems. Moreover, [21] investigated the distributed formation control problem for agents where the assignment of agents to target points in the formation are not given *a priori*, but instead had to be determined dynamically. To do so, a local market-based protocol was used to determine the permutation of agents in the desired formation, and artificial potential fields were used to control the agents to that formation. This paper builds on previous results by the authors in [20] on controlling a network of homogeneous agents to a set of target points, where the labels of the targets may be permuted freely. The problem tackled in this paper differs from that of related work because the focus is for a human to take control of a single agent and direct a network of homogeneous agents to a set of desired target points, as opposed to having the agents act completely autonomously. Therefore, obstacles such as the lack of complete controllability arise, which greatly complicates the control task, and requires a new way to analyze the controllability of such systems if homogeneity is to be taken into account explicitly.

It should be noted that the methods presented in this paper are only meant to bring a target point closer to a homogeneous single-leader network's reachable subspace, but not necessarily always into the subspace. Therefore, the results discussed are most appropriate for mission-critical scenarios in which one would like to drive the network of agents as close as possible to a set of target points, even if they cannot be reached directly. For example, strategically positioning mobile sensors to cover an area in a search-and-rescue mission.

Although this paper focuses primarily on single-leader networks where all of the agents have homogeneous capabilities, the underlying concept can be expanded to heterogeneous single-leader networks as well. Any time some agents within a network have the same capabilities and there is a lack of complete controllability, the labels on their respective target points can be swapped in hopes of moving the target point closer to the system's reachable subspace. If multiple species of agents exist in the network, then a target point's label may be swapped with another target point's label if and only if their corresponding agents belong to the same species.

The outline of this paper is as follows: Section II will present the system dynamics for single-leader networks. Section III will review previous work on the controllability properties of single-leader networks, and present a new approach to analyzing controllability when homogeneity is present in the network. The problem of finding the optimal permutation of a target point to bring it closest to a system's reachable subspace will be discussed, and the computational complexity of solving the problem will be presented. Section IV will then give insight into the network topologies where exploiting agent homogeneity will allow for networks to be controlled beyond the traditionally-established limitations on reachability. Finally, Section V will demonstrate how taking an alternate approach to finding optimal target point permutations allows for clustering techniques to be used to find suboptimal solutions.

II. SYSTEM DYNAMICS

Consider a team of $N+1$ agents, numbered $1, \ldots, N+1$, with positions (states) $x_i \in \mathbb{R}^n$, for $i = 1, \ldots, N+1$ respectively. In this paper, it is assumed that each agent is only able to sense the relative displacement between itself and select other agents in the network as dictated by a network topology. In particular, agent sensing is bidirectional and so the information flow amongst agents in the network can be represented by a static undirected graph $G = (V, E)$, where $V = \{v_1, \ldots, v_{N+1}\}$ and $(v_i, v_j) \in E$ if and only if information flows between agents i and j. Moreover, let the neighbor set $N_i = \{j | (v_i, v_j) \in E\}$ represent the index set of all agents that share an edge with agent i in G.

In order to control the agent states without communicating with each agent directly, a single-leader control paradigm will be used where the state of a single leader agent will be controlled directly. The remaining agents will take on the role of follower agents and execute a nearest-neighbor averaging rule so as to maintain cohesion in the network. Without loss of generality, assume the N+1th agent is the leader while agents 1, . . . , N are the followers. The agents' dynamics are then given by:

$$\begin{cases} \dot{x}_i = -\sum_{j\in N_i}(x_i - x_j), \text{ for } i = 1,\dots,N, \\ x_{N+1} = u, \end{cases} \tag{1}$$

where $u \in \mathbb{R}^n$ is the control input.

The dynamics for the system of follower agent states can be written collectively as a linear time-invariant (LTI) system by using the graph Laplacian matrix. To do so, first let the adjacency matrix of G be the $(N + 1) \times (N + 1)$ symmetric matrix A where $A_{i,j}$, the element in the ith row and jth column, is given by

$$A_{i,j} = \begin{cases} 1 & \text{, if } (v_i, v_j) \in E, \\ 0 & \text{, otherwise.} \end{cases} \tag{2}$$

The degree matrix of the graph G is a $(N + 1) \times (N + 1)$ diagonal matrix Δ, where

$$\Delta_{i,j} = \begin{cases} |N_i| & \text{, if } i = j, \\ 0 & \text{, otherwise.} \end{cases} \tag{3}$$

Finally, the graph Laplacian matrix L is given by

$$L = \Delta - A. \tag{4}$$

Next, it is necessary to extract information on how the states of the follower agents evolve as a function of the control input. Let $x = \left[x_1^T, \dots, x_N^T\right]^T \in \mathbb{R}^{Nn}$ be the concatenated positions of all follower agents, where $x_j = [x_{j,1}, \ \dots, \ x_{j,n}]^T \in \mathbb{R}^n$, for $j = 1, \dots, N$. Moreover, define d_i: $\mathbb{R}^{Nn} \to \mathbb{R}^N$, for $i = 1, \dots, n$, as a function that returns the positions of the N follower agents along the ith dimension, *i.e.*, $d_i(x) = [x_{1,i}, \dots, x_{N,i}]^T$. The graph Laplacian matrix can be decomposed into the following blocks that isolate information on the inter-follower agent interactions, and the interaction between the leader and follower agents:

$$L = \left[\begin{array}{c|c} L_f & \ell \\ \hline \ell^T & \xi \end{array}\right], \tag{5}$$

where the dimension of L_f is $N \times N$, ℓ is $N \times 1$, and $\xi \in \mathbb{R}$.

Using this decomposition of the graph Laplacian matrix, the dynamics of the follower agents' positions along the ith dimension are given by the following LTI system:

$$d_i(\dot{x}) = -L_f d_i(x) - \ell u_i, \tag{6}$$

where u_i is the ith element of u. Since the dynamics along each dimension are decoupled, the dynamics of x can be written using the Kronecker product, as the LTI system

$$\dot{x} = -(L_f \otimes I_n)x - (\ell \otimes I_n)u, \tag{7}$$

where I_n is the $n \times n$ identity matrix, *e.g.*, [12]. Having established a dynamical model of the single-leader network, we will use it to determine how closely one can control a network of follower agents to a set of desired target points. However, in order to take agent homogeneity into account explicitly, a new way of analyzing the controllability properties of the system is required.

III. CONTROLLABILITY IN HOMOGENEOUS SINGLE-LEADER NETWORKS

The need to take agent homogeneity into account when controlling single-leader networks stems from the fact that the majority of network topologies yield systems which are not completely controllable. Therefore, by exploiting homogeneity and permuting the target point labels, one hopes to find a permutation of the target point that brings it closer to the system's reachable subspace. In this section, we will start by reviewing existing results on the controllability of single-leader networks to determine when such systems are not completely controllable. Once those networks have been highlighted, we will then present how one can potentially alleviate the lack of controllability in these networks by incorporating agent homogeneity.

3.1 Controllability of single-leader networks

In a single-leader network, the dynamics of the follower agents along each dimension are decoupled and given by the LTI system (6). Treating each dimension separately, the reachable subspace is given by the range space of the reachability matrix Γ, where

$$\Gamma = \left[-\ell \quad L_f\ell \quad \dots \quad (-L_f)^{N-1}(-\ell)\right]. \tag{8}$$

To better characterize the reachable subspace of such networks, the analysis tool of maximum leader-invariant externally equitable partitions will be used. In particular, it was shown in [11] that these partitions offer a useful network topological interpretation for the lack of complete controllability of single-leader networks. Before stating this result, however, we must first review some definitions from [5, 11, 12].

Definition III.1. Given a vertex set V, let $\Pi = \{C_1, \dots, C_M\}$ be a partition of V, where $C_i \subseteq V$ for $i = 1, \dots, M$, $C_1 \cup \dots \cup C_M = V$, and $C_i \cap C_j = \varnothing$ when $i \neq j$. We will call each C_i a cell.

Definition III.2. Given a vertex v and a cell C, the node-to-cell degree gives the number of vertices in cell C

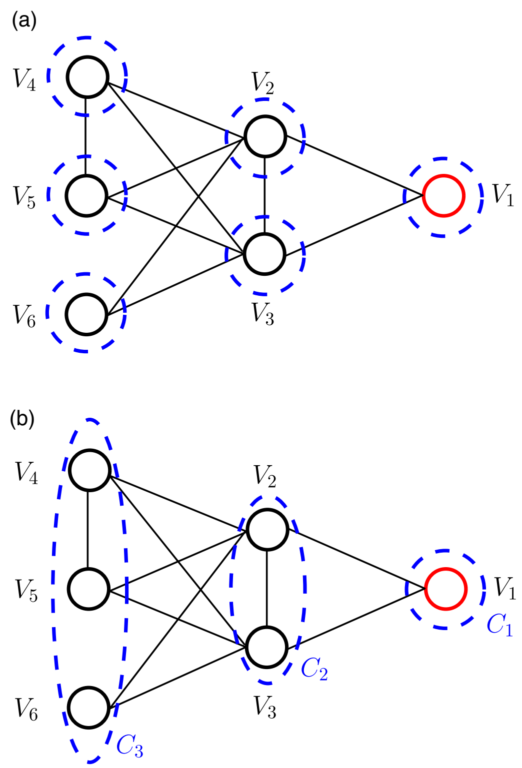

Fig. 1. Two examples of leader-invariant EEPs of a single-leader network, where V_1 is the vertex for the leader agent. (a) shows the trivial leader-invariant EEP, while (b) gives the maximal leader-invariant EEP. Since the two partitions are different, the network is not completely controllable.

that share an edge with v, and is given by $deg(v, C) = card(\{v' \in C | (v, v') \in E\})$.

For example, in Fig. 1b, C_1, C_2, C_3 are cells that partition the vertices in the network and $deg(v_2, C_3) = 3$.

Definition III.3. An external equitable partition (EEP) is a partition Π such that $\forall C \in \Pi$, $v \in C$ and $v' \in C \Rightarrow deg(v, C') = deg(v', C') \forall C' \in \Pi - \{C\}$.

Definition III.4. An EEP is leader-invariant if the vertex corresponding to the leader agent belongs to its own cell.

Definition III.5. A leader-invariant EEP is maximal if it has the fewest number of cells in any leader-invariant EEP.

For example, Fig. 1a is a leader-invariant EEP, while Fig. 1b is a maximal leader-invariant EEP. With these definitions in place, we can now state an important established result relating the controllability of a single-leader network to its maximal leader-invariant EEP which the analysis in the rest of this paper builds off of. It was shown in [11] that cells of the maximal leader-invariant EEP give information as to which groups of follower agents cannot be controlled independently of one another. That result is stated again below for easy reference.

Theorem III.1 [11]**.** Assume a single-leader network has a network topology with a maximal leader-invariant EEP of k^* cells, numbered $1, \ldots, k^*$, that do not contain the leader agent. The range space of the reachability matrix for (6), the follower agent dynamics along any dimension, is given by:

$$R(\Gamma) = span\{w_1, \ldots, w_k\} \tag{9}$$

where $1 \le k \le k^*$ and $w_i \in \mathbb{R}^N$. Moreover, letting $w_{i,j}$ represent the jth element of w_i, it must be that

1. $w_{i,j} \in \{0, 1\}$,
2. $w_{i,a} = 1 \Rightarrow w_{i,b} = 1$ for all v_b that belongs in the same cell as v_a,
3. $\sum_{i=1}^{k} w_i = \mathbb{1}$, where $\mathbb{1}$ is the vector of all 1's.

The theorem states that follower agents which are located within the same cell of the maximal leader-invariant EEP will asymptotically approach each other as they move according to the dynamics (7). Therefore, instead of being able to control each agent's position independently, only the centroid of agents within each cell can be controlled. However, sometimes even the centroids of certain cells cannot be controlled independently of one another either, resulting in all the agents of those cells asymptotically approaching each other. This is why $1 \le k \le k^*$. In order to determine how close agents in a single-leader network can get to a specific desired target point, it is necessary to restrict our attention to systems where the reachable subspace is known perfectly. Therefore, we make the following key assumption.

Assumption III.1. For general single-leader networks, maximal leader-invariant EEPs only give necessary conditions for controllability. However, in this paper, we restrict our attention to single-leader networks in which the reachable subspace $R(\Gamma)$ is completely determined by the agents' membership within the cells of the maximal leader-invariant EEP. In other words, we assume that $k = k^*$ in Theorem III.1, *i.e.*, the reachable subspace $R(\Gamma) = span\{w_1, \ldots, w_k\}$ is such that

$$w_{i,j} = \begin{cases} 1 & , if\ v_j \in cell\ i, \\ 0 & , otherwise. \end{cases} \tag{10}$$

In single-leader networks where the assumption does not hold, the results derived in this paper can be viewed as an upper bound on the limits of the system's controllability properties.

Although this may seem like a major assumption, it is not overly restrictive since, in many network topologies, having a trivial maximal leader-invariant EEP can act as both a necessary and sufficient condition for controllability. An example of a class of such networks where the assumption holds true is when the network topology also yields a distance regular graph as discussed in [22]. Under the assumption, a single-leader network is therefore characterized as being completely controllable and can reach any target point only when the maximal leader-invariant EEP is trivial, *i.e.*, each follower agent is contained within its own cell. Referring back to Fig. 1 in the example, we see, therefore, that the maximal leader-invariant EEP associated with the single-leader network is not trivial and hence the system is not completely controllable.

3.2 Optimally reachable target points

The set of reachable target points in a network is restricted by the choice of leader agent and network topology. In most scenarios, the resulting single-leader network is not completely controllable. However, a margin of error may be allowed between where the agents are located and where the user desires them to be. Thus, even if the targets cannot be reached exactly by the network, one would like to minimize this margin of error by finding a reachable target point that is as close as possible to the original target. To find this optimally reachable target point, it is necessary to first perform a reachability analysis of the system. Here, agents will be modeled as initially being at close proximity to one another by assuming zero initial conditions on the positions of the follower agents in the network.

Assumption III.2. All agent positions are initially zero, *i.e.*, agents which belong in the same maximal leader-invariant EEP cell start and stay together throughout the control task.

Although such an assumption is needed for the reachability analysis, it is not far from reality when controlling such single-leader networks since it is known that the states of agents belonging in the same cell will asymptotically converge over time due to the lack of controllability. Therefore, any multi-agent control task of considerable duration will have conditions that closely resemble those specified in the assumption.

With zero initial conditions on x, a target point of follower agents $x_T \in \mathbb{R}^{Nn}$ is reachable if and only if $d_i(x_T) \in R(\Gamma)$, for $i = 1, \ldots, n$. Depending on the network topology, the system of follower agents is not always completely controllable and so may not be able to reach a target point perfectly. Therefore, for a given x_T, the closest point along the reachable subspace that agents can reach will be referred to as $x^*(x_T)$, the optimally reachable target point which minimizes

$$J(x_T, x) = \|x_T - x\|^2 = \sum_{i=1}^{n} \|d_i(x_T) - d_i(x)\|^2, \tag{11}$$

such that $d_i(x) \in R(\Gamma)$, for $i = 1, \ldots, n$.

Proposition III.1. For a given x_T, the optimally reachable $x^*(x_T)$ that minimizes (11), where $d_i(x^*(x_T)) \in R(\Gamma)$ for $i = 1, \ldots, n$, is given by

$$x^*(x_T) = (WW^T \otimes I_n)x_T, \tag{12}$$

where

$$W = \left[\frac{w_1}{\|w_1\|} \cdots \frac{w_k}{\|w_k\|}\right], \tag{13}$$

and $w_1, \ldots, w_k$ are as given in (9).

Proof. Since $\|x_T - x\|^2 = \sum_{i=1}^{n} \|d_i(x_T) - d_i(x)\|^2$, minimizing $\|x_T - x\|^2$ is equivalent to minimizing $\|d_i(x_T) - d_i(x)\|^2$ individually for each i because the follower agent states along each dimension are controlled independently of one another. The Hilbert Projection Theorem says that the optimally reachable $d_i(x^*(x_T)) \in R(\Gamma)$ that minimizes $\|d_i(x_T) - d_i(x)\|$ is the projection of $d_i(x_T)$ onto the subspace $R(\Gamma)$. The reachable subspace $R(\Gamma)$ is spanned by vectors $w_1, \ldots, w_k$ as given in (9). Therefore, the optimal choice of $d_i(x)$ is given by

$$d_i(x^*(x_T)) = \sum_{j=1}^{k} \frac{w_j^T d_i(x_T)}{\|w_j\|^2} w_j. \tag{14}$$

For W as defined in (13), (14) can be rewritten as

$$d_i(x^*(x_T)) = WW^T d_i(x_T).$$

Since this holds for all i, $x^*(x_T)$ is written as (12). ■

The previous proposition presented a closed-form solution for the closest that agents in a single-leader network can reach to a desired target point. However, (14) in the proof can be used to come up with an alternate representation of the optimally reachable target point which offers an intuitive way to interpret the result visually, and opens the door to various clustering-based approaches when we start to incorporate agent homogeneity. To arrive at this alternate formulation of the result in Proposition III.1, first define g_i: $\mathbb{R}^{Nn} \to \mathbb{R}^n$, for $i = 1, \ldots, N$, as a function that returns the n dimensional coordinates of the ith agent, *i.e.*, $g_i(x) = x_i$. Furthermore, let m: $\{1, \ldots, N\} \to \{1, \ldots, k\}$ be a function that takes in the index of a follower agent and returns the index of the cell it belongs to in the maximal leader-invariant EEP. Let m^{-1} be the inverse image function that takes in a cell number and returns a set containing the indices of the follower agents that belong to that cell. With these definitions in place, we will now present an alternate representation of the result in Proposition III.1 for the optimally reachable target point.

Corollary III.1. For a given x_T and corresponding $x^*(x_T)$ that minimizes (11),

$$g_i(x^*(x_T)) = \frac{1}{|m^{-1}(m(i))|} \sum_{j \in m^{-1}(m(i))} g_j(x_T). \tag{15}$$

In other words, for each cell $j = 1, \ldots, k$ in the maximal leader-invariant EEP, agents in cell j of $x^*(x_T)$ are located at the centroid of the set of all target points associated with those agents.

Proof. From the definition of vectors w_j in (10), the expression for $d_i(x^*(x_T))$ in (14) can be interpreted. The numerator of each summand $w_j^T d_i(x_T)$ is the sum along the ith dimension of all target positions in cell j. That quantity is divided by $\|w_j\|^2$, which is the number of agents in cell j, so the result is the centroid along the ith dimension of all target positions in cell j of x_T. Finally, that value is multiplied to w_j, thereby assigning it to the ith dimensional component of all agents positions in cell j of $x^*(x_T)$. Since this holds for all dimensions along $i = 1, \ldots, n$, agents in cell j of $x^*(x_T)$ are all located at the centroid of the target positions in cell j of x_T. ■

With an understanding of $x^*(x_T)$ established, it is also possible now to compute the cost associated with the difference between any given x_T and its associated optimally reachable target point $x^*(x_T)$.

Corollary III.2. For a given x_T and corresponding $x^*(x_T)$, the minimum cost $J^*(x_T) = J(x_T, x^*(x_T))$ is given by

$$J^*(x_T) = x_T^T (I_{Nn} - WW^T \otimes I_n) x_T. \tag{16}$$

Proof. Substitution of (12) into the cost (11). ■

Having now computed the closest in which agents in a single-leader network can be controlled to a desired target point, we will now extend this analysis to determine what happens when agent homogeneity is taken into account explicitly and how the minimum cost associated with a target point is affected as a result.

3.3 Homogeneous networks

Equation (16) represents the cost associated with the closest that a particular single-leader network can reach to a target point x_T. Notice, however, that x_T represents the specific assignment of having each agent i be controlled to $g_i(x_T)$, for $i = 1, \ldots, N$. Nevertheless, in a network of agents with homogeneous capabilities, the roles of agents at each of the

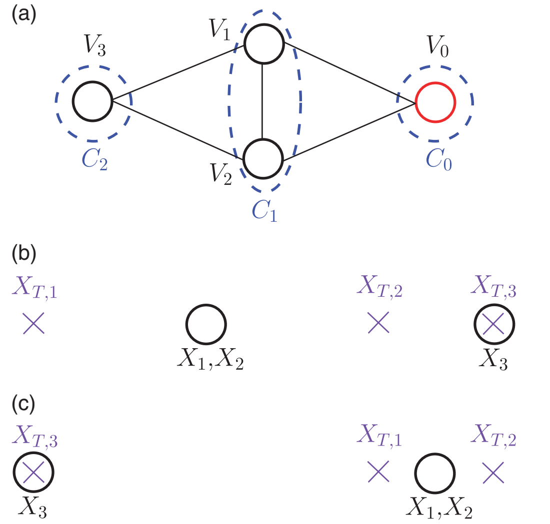

Fig. 2. The network topology of the single-leader network in Example III.1 is shown in (a), where V_0 is the vertex for the leader agent. (b) shows the closest the agents in the network can reach $x_T = [1\ 9\ 10]^T$, while (c) shows the closest the agents can reach $Px_T = [9\ 10\ 1]^T$. Notice that the Px_T results in a cost that is less than x_T.

target points are interchangeable and so it makes no difference if instead we ask agent i to go to $g_j(x_T)$ and agent j to go to $g_i(x_T)$. In fact, any permutation of the target point indices in x_T to some $(P \otimes I_n)x_T$, where P is a permutation matrix, ends up specifying the same intended target configuration if all we care about is the presence of an agent at each of the target positions. At times, however, this new permuted target point may be "more reachable" in the sense that it is closer to the system's reachable subspace, *i.e.*, $J^*((P \otimes I_n)x_T) < J^*(x_T)$.

Example III.1. Consider a single-leader network with scalar, *i.e.*, 1D, agent positions and $N = 3$ follower agents as illustrated in Fig. 2a, where agents 1 and 2 are in cell 1 and agent 3 is in cell 2 of the maximal leader-invariant EEP associated with the network. The range space of the reachability matrix is thus

$$R(\Gamma) = span\{w_1, w_2\} = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Let $x_T = [1 \quad 9 \quad 10]^T$ and

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

then $J^*(x_T) = 32$, while $J^*(Px_T) = 0.5$. This difference in performance when agent homogeneity is taken into account can be clearly seen in Fig. 2b and c, where the Xs represent target points and Os represent follower agent positions which are optimally reachable as described by Corollary III.2. Therefore, this example has shown that the permuted point Px_T is indeed more reachable than the original target point x_T.

As seen in the previous example, different permutations of a target point may be at different distances from the system's reachable subspace. In order to fully exploit agent homogeneity when controlling a single-leader network, it is then necessary to find the optimal permutation of the target point which brings it closest to the system's reachable subspace. Upon finding this permutation, controlling the network to reach that target point is trivial since (7) is a LTI system. Therefore, standard optimal control techniques for LTI systems with a fixed final state can be used to generate an appropriate control signal for the leader agent. However, the difficulty lies in actually finding an optimal permutation of the target point. In particular, fully exploiting agent homogeneity when controlling single-leader networks requires that one solve the following problem.

Problem III.1. Let $\mathcal{P}$ be the set of all $N \times N$ permutation matrices. Given a single-leader network and target point of follower agents x_T, find P^* such that

$$P^* = \arg\min_{P \in \mathcal{P}} J^*((P \otimes I_n)x_T). \tag{17}$$

Computing P^* can be viewed as finding the optimal specification of a target configuration for the follower agents to be controlled to. The next step is to determine the computational complexity of finding the closest reachable target point that homogeneous agents in a single-leader network can be controlled to, *i.e.*, solving for P^*. To approach this, an alternate representation Problem III.1 will first be formulated that uses Corollary III.2 to turn it into a variant of a standard clustering problem on the target points $g_1(x_T), \ldots, g_N(x_T)$. From there, the computational complexity can then be determined and appropriate clustering algorithms can be used to solve the problem. Before proceeding, however, it is first necessary to introduce some clustering-related definitions.

Definition III.6. A multiset is a collection of objects in which order is ignored, but where multiplicity is significant.

For example, $M_1 = \{1, 3, 4\}$, $M_2 = \{1, 3, 4, 4\}$, and $M_3 = \{1, 4, 3, 4\}$ are all multisets. $M_2 = M_3$, but $M_1 \neq M_2$ and $M_1 \neq M_3$. Also, $|M_1| = 3$, while $|M_2| = |M_3| = 4$.

Definition III.7. Given a multiset S, a clustering of S is a partitioning of the elements of S into multisets $c_1, \ldots, c_k$.

Now, let S be a multiset of agent positions. Within each cluster c_i, define the distortion measure of that cluster as

$$D(c_i) = \sum_{z \in c_i} \|z - \theta(c_i)\|^2, \tag{18}$$

where $\theta(c_i)$ is the centroid of all positions in c_i. Define the cost of a clustering as the total distortion measure, given by

$$H(c_1, \ldots, c_k) = \sum_{i=1}^{k} D(c_i). \tag{19}$$

These definitions together form the basis of the standard Euclidean minimum sum-of-squares clustering problem that is widely seen in machine learning and pattern recognition applications.

Problem III.2. The Euclidean minimum sum-of-squares clustering problem is to find a clustering $c_1^*, \ldots, c_k^*$, given a multiset of positions S, so as to minimize (19).

The following theorem will now show that finding an optimal permutation for a target point is equivalent to a variant of the standard Euclidean minimum sum-of-squares clustering problem on the set of target points $\{g_1(x_T), \ldots, g_N(x_T)\}$.

Theorem III.2. Suppose a single-leader network has a maximal leader-invariant EEP of exactly k cells containing follower agents, numbered $1, \ldots, k$. Finding the optimal permutation P^* for a target x_T in Problem III.1 is equivalent up to a polynomial-time transformation to solving Problem III.2 under Assumption III.1 for the multiset of target positions, $S = \{g_1(x_T), \ldots, g_N(x_T)\}$, with the constraint that $|c_i| = |m^{-1}(i)|$, the number of agents in cell i, for $i = 1, \ldots, k$.

Proof. Given a permutation matrix P, let p: $\{1, \ldots, N\} \rightarrow \{1, \ldots, N\}$ take in an agent index and returns the permuted index such that, for $j = 1, \ldots, N$, $g_j(x_T) = g_{p(j)}((P \otimes I_n)x_T)$. Let $c_i = \{g_j((P \otimes I_n)x_T) | m(j) = i|\}$, for $i = 1, \ldots, k$, be a clustering of $S = \{g_1(x_T), \ldots, g_N(x_T)\}$, where target positions in $(P \otimes I_n)x_T$ with indices in cell i are assigned to c_i.

Notice that $|c_i| = |m^{-1}(i)|$, the number of agents in each cell i, for $i = 1, \ldots, k$. Considering different permutations of agent indices for the target point x_T is equivalent to considering different cell assignments of the target positions, which is equivalent to considering clusterings $c_1, \ldots, c_k$ of S. The cost (16) associated with a chosen permutation P of target positions can be rewritten using (11) and Corollary III.1 as

$$\begin{aligned}
&J^*((P\otimes I_n)x_T)\\
&=\sum_{i=1}^{n}\|d_i((P\otimes I_n)x_T)-d_i(x^*((P\otimes I_n)x_T))\|^2\\
&=\sum_{i=1}^{N}\|g_i((P\otimes I_n)x_T)-g_i(x^*((P\otimes I_n)x_T))\|^2\\
&=\sum_{i=1}^{N}\|g_i((P\otimes I_n)x_T)-\theta(c_{m(i)})\|^2\\
&=\sum_{i=1}^{k}\sum_{j|m(j)=i}\|g_j((P\otimes I_n)x_T)-\theta(c_i)\|^2\\
&=\sum_{i=1}^{k}\sum_{z\in c_i}\|z-\theta(c_i)\|^2=H(c_1,\ldots,c_k),
\end{aligned}$$

which shows that the cost is equivalent to (19), *i.e.*, both problems are minimizing the same cost.

To show that the two problems are equivalent up to a polynomial-time transformation, algorithms will now be presented so as to transform the solution of one problem into the solution of the other in polynomial-time. First, given the P^* that solves Problem III.1, an optimal clustering $c_1^*, \ldots, c_k^*$ that solves Problem III.2 under the constraint that $|c_i| = |m^{-1}(i)|$, for $i = 1, \ldots, k$, can be computed by the polynomial-time algorithm:

Let $c_1^*,\ldots,c_k^*$ be empty multisets;
for $i=1,\ldots,N$ **do**
  Add $g_i((P^*\otimes I_n)x_T)$ to $c_{m(i)}^*$;
end

On the other hand, when given an optimal clustering $c_1^*, \ldots, c_k^*$, the matrix P^* can be computed using the polynomial-time algorithm:

Let $Q=R=\{1,\ldots,N\}$, and $P^*=0$ ($N\times N$ matrix);
for $i=1,\ldots,k$ **do**
  for *each* $z\in c_i^*$ **do**
    Find any $j\in Q$ such that $g_j(x_T)=z$;
    Remove j from Q;
    Find a $b\in m^{-1}(i)$ such that $b\in R$;
    Remove b from R;
    Set the element $P_{b,j}^*$ to 1;
  end
end

Thus, finding P^* in Problem III.1 under Assumption III.1 is equivalent up to a polynomial-time transformation to finding an optimhal clustering $c_1^*, \ldots, c_k^*$ for $S = \{g_1(x_T), \ldots, g_N(x_T)\}$, that minimizes (19) subject to $|c_i|$ equaling the number of agents in cell i, for $i = 1, \ldots, k$.

Viewing the problem of finding an optimally permuted target point as a size-constrained variant of the Euclidean minimum sum-of-squares clustering problem is useful because it allows us to find the computational complexity associated with solving the problem.

Theorem III.3. The problem of finding the optimal permutation matrix P^* in Problem III.1 under Assumption III.1 is NP-hard.

Proof. It was shown in [1] that the standard Euclidean minimum sum-of-squares clustering problem described in Problem III.2 is NP-hard by using a reduction from the DENSEST CUT problem for the case of $k = 2$ clusters. Using almost the same procedure, we will show that the optimization version of the MAX BISECTION problem, which was shown in [14] to be NP-hard, reduces to the size-constrained Euclidean minimum sum-of-squares problem in Theorem III.2 for $k = 2$ clusters. From there, leveraging the result from Theorem III.2 that the size-constrained Euclidean minimum sum-of-squares problem is equivalent, up to a polynomial-time transformation, to Problem III.1 under Assumption III.1 will prove the theorem.

Let $G = (V, E)$ be an undirected graph. Define B_1, B_2 as a partition of V such that $\|B_1\|=\|B_2\|=\frac{N}{2}$, where N is assumed to be even. The MAX BISECTION problem is to find B_1^* and B_2^* so as to maximize $|\mathcal{E}(B_1, B_2)|$, where $\mathcal{E}(B_1, B_2) = \{(v_i, v_j) \in E | v_i \in B_1 \text{ and } v_j \in B_2\}$.

Arbitrarily number and orient the edges in E as $e_1, \ldots, e_{|E|}$ so that each e_i is an ordered pair of vertices. Define the incidence matrix $\mathcal{I}$ as a $N\times|E|$ matrix such that for each $e_k = (v_i, v_j) \in E$, $\mathcal{I}_{i,k} = -1$ and $\mathcal{I}_{j,k} = 1$. Have $x_1, \ldots, x_N \in \mathbb{R}^{|E|}$ be such that x_i^T equals the ith row of $\mathcal{I}$. Define the multiset $S = \{x_1, \ldots, x_N\}$. Have c_1, c_2 be two clusters that partition S subject to the size constraint $|c_1|=|c_2|=\frac{N}{2}$. Let B_1 and B_2 be a partition of V, where $B_i = \{v_j | x_j \in c_i\}$, for $i = 1, 2$.

Let the function $\phi_j : \mathbb{R}^{|E|} \to \mathbb{R}$ take in a vector and return the jth element of its argument. Computing the total distortion of the cluster as in (19), we have

$$\begin{aligned}
H(c_1,c_2)&=\sum_{i=1}^{2}\sum_{z\in c_i}\|z-\theta(c_i)\|^2\\
&=\sum_{j=1}^{|E|}\sum_{i=1}^{2}\sum_{z\in c_i}(\phi_j(z)-\phi_j(\theta(c_i)))^2.
\end{aligned}$$

If $e_j \in \mathcal{E}(B_1, B_2)$, then either $\phi_j(z)$ equals 1 for exactly one $z \in c_1$ and equals -1 for exactly one $z \in c_2$ with all others equaling 0 and thus $\phi_j(\theta(c_1)) = \frac{2}{N}$ and $\phi_j(\theta(c_2)) = -\frac{2}{N}$, or the same statements above but with c_1 and c_2 switched. Furthermore, if $e_j \notin \mathcal{E}(B_1, B_2)$, then $\phi(\theta(c_1)) = \phi_j(\theta(c_2)) = 0$. Using these properties:

$$
\begin{aligned}
H(c_1, c_2) &= \sum_{e \in \mathcal{E}(B_1, B_2)} \sum_{i=1}^{2} \left(\left(\frac{N}{2} - 1 \right) \left(\frac{2}{N} \right)^2 \right. \\
&\quad \left. + \left(1 - \frac{2}{N} \right)^2 \right) + \sum_{e \notin \mathcal{E}(B_1, B_2)} 2 \\
&= 2 \left(1 - \frac{2}{N} \right) |\mathcal{E}(B_1, B_2| \\
&\quad + 2(|\mathcal{E}(B_1, B_1)| + |\mathcal{E}(B_2, B_2)|) \\
&= 2|E| - \frac{4}{N} |\mathcal{E}(B_1, B_2)|.
\end{aligned}
$$

Choice of B_1^* and B_2^*, or equivalently the choice of c_1^* and c_2^*, that minimizes $H(c_1, c_2)$ also maximizes $|\mathcal{E}(B_1, B_2)|$, since $|E|$ and N are constant. Therefore, the NP-hard MAX BISECTION problem reduces to size-constrained Euclidean minimum sum-of-squares, which is equivalent up to a polynomial-time transformation to finding P^* in Problem III.1 under Assumption III.1, so finding P^* is also NP-hard. ■

The above theorem describes the caveat associated with exploiting agent homogeneity when controlling single-leader networks to target points. Although considering permutations of target points can potentially bring a target point closer to a system's reachable subspace, determining how one can best permute a target point is, in general, a NP-hard problem. The remaining sections of this paper will discuss how to deal with this computational complexity.

IV. NETWORK TOPOLOGIES WHERE HOMOGENEITY CAN BE ADVANTAGEOUS

The previous section showed how, in order to fully exploit agent homogeneity when controlling a single-leader network to targets, it is necessary to solve the NP-hard Problem III.1 under Assumption III.1. Example III.1 showed one situation in which solving the NP-hard problem to find the optimal permutation of a target point could bring it closer to the reachable subspace of a single-leader network. However, is not clear yet how commonly such an increase in controllability can be achieved for general network topologies. The goal of this section is mainly to identify specific classes of network topologies where there exist optimally permuted target points which are closer to the system's reachable subspace than their un-permuted counterparts, thereby advocating the need to take homogeneity into account when controlling single-leader networks. We will start by looking at the case when agent positions are scalar, *i.e.*, $1D$, and then later consider the general case where agents have nD position coordinates.

4.1 Optimal permutations with *1D* agents

To identify network topologies where optimally permuting target points is advantageous for homogeneous single-leader networks with $1D$ agent positions, we will focus our analysis on bounding how far away an optimally permuted target point is from a system's reachable subspace. In particular, for a given reachability matrix Γ associated with the network topology of a system, the goal is to look at all possible target points with norm 1 and bound the worst case distance to the reachable subspace as posed in the problem below:

Problem IV.1. Given a single-leader network with $N \geq 2$ follower agents, $1D$ agent positions, and reachable subspace $R(\Gamma)$, find

$$M_1(\Gamma) = \max_{\|x\|=1} \left\{ \min_{P \in \mathcal{P}} \left\| \Pi_{R(\Gamma)^\perp}(Px) \right\|^2 \right\}, \tag{20}$$

where $\Pi_a(b)$ is the projection of vector b onto a.

Note that $M_1(\Gamma) \in [0, 1]$, where $M_1(\Gamma) = 0$ corresponds to the best case scenario where any target point can be permuted onto the system's reachable subspace. Meanwhile, $M_1(\Gamma) = 1$ corresponds to the worst case scenario when target points exist that remain orthogonal to the system's reachable subspace no matter how they are permuted.

In the case when $rank(\Gamma) = 1$, *i.e.*, when all follower agents cannot be moved independently of one another, the following theorem shows that such a worst case scenario occurs.

Theorem IV.1. The solution to Problem IV.1 for when $rank(\Gamma) = 1$ is given by $M_1(\Gamma) = 1$.

Proof. Since $rank(\Gamma) = 1$, then $R(\Gamma) = span\{\mathbb{1}\}$. Let $x \in \mathbb{R}^N$ be such that $\|x\| = 1$ and $x^T \mathbb{1} = 0$. Then for any $P \in \mathcal{P}$,

$$(Px)^T \mathbb{1} = x^T p^T \mathbb{1} = x^T (P^{-1} \mathbb{1}) = x^T \mathbb{1} = 0.$$

Thus, $\left\| \Pi_{R(\Gamma)^\perp}(Px) \right\|^2 = 1$ for any $P \in \mathcal{P}$ and so $M_1(\Gamma) = 1$. ■

Such a result should not be too surprising, since if $rank(\Gamma) = 1$, then all follower agents are confined to move together. Under such a scenario, the closest that a single-leader network can be to a target point is to have all agents go to the centroid of the targets, which is invariant to permutations. Fortunately, however, a situation where $rank(\Gamma) = 1$ only occurs for specific network topologies such as a star graph where the leader is the center node, or a complete graph. Note that neither of these network topologies are practical designs for multi-agent systems where the goal is to have each agent coordinate with only a subset of other agents while still be robust to connections breaking in the network topology.

Fortunately, however, the previous result does not hold true for single-leader networks where the reachable subspace has higher rank, *i.e.*, $1 < rank(\Gamma) \le N$, which accounts for the majority of network topologies. To show this, the following result must first be established.

Lemma IV.1. For a single-leader network as described in Problem IV.1 with $1 < rank(\Gamma) \le N$, $Px \perp R(\Gamma)$ for all $P \in \mathcal{P}$ $\Leftrightarrow x = 0$.

Proof. First, we prove the necessary condition ($\Leftarrow$). If $x = 0$, then $Px = 0$ for all $P \in \mathcal{P}$ and 0 is orthogonal to all possible subspaces.

To show the sufficient condition ($\Rightarrow$), two cases need to be considered. First, if $rank(\Gamma) = N$, then $R(\Gamma) = \mathbb{R}^N$. In this case, the only vector x which is orthogonal to $R(\Gamma)$ is $x = 0$. Now consider the case when $1 < rank(\Gamma) < N$, which can only occur when $N \ge 3$. Here, $R(\Gamma)$ must be spanned by a basis vector $v \in \mathbb{R}^N$ that consists of 0s and at least 2 but less than N 1s. If $Px \perp R(\Gamma)$ for all $P \in \mathcal{P}$, then certainly $v^T(Px) = 0$ for all $P \in \mathcal{P}$ as well. Given $i, j \in \{1, \ldots, N\}$, it is then possible to find permutation matrices $P_{ijA} \in$ and $P_{ijB} \in \mathcal{P}$ where

$$v^T(P_{ijA}x) = v^T(P_{ijB}x) = 0 \Rightarrow x_i = x_j.$$

Since this can be done for any $i, j \in \{1, \ldots, N\}$, it must be that $x = \alpha\mathbb{1}$ for some $\alpha \in \mathbb{R}$. However, since $v^T x = 0$, it must be that $\alpha = 0$ and thus $x = 0$. ■

With this result, an important bound on $M_1(\Gamma)$ for $1 < rank(\Gamma) \le N$ can be made.

Theorem IV.2. The solution to Problem IV.1 for when $1 < rank(\Gamma) \le N$ is bounded by $M_1(\Gamma) < 1$.

Proof. Since, when calculating $M_1(\Gamma)$, we only consider x such that $\|x\| = 1$, it is not the case that $x = 0$. Therefore, by Lemma IV.1, we see that for each target point x there exists a $P^*(x) \in \mathcal{P}$ such that $P^*(x)x \not\perp R(\Gamma)$ and so $\left\|\Pi_{R(\Gamma)^\perp}(P^*(x)x)\right\|^2 < 1$. Because this is true for all target points x such that $\|x\| = 1$, we conclude that $M_1(\Gamma) < 1$. ■

Theorem IV.1 and IV.2 give insight into when taking agent homogeneity into account can be advantageous when controlling single-leader networks. In particular, they say that with the exception of a small percentage of network topologies (*e.g.*, star graph with leader node as center and complete graph), solving for the optimal permutation of a target point can push the system beyond traditionally-established limitations on its controllability.

4.2 Optimal permutations with *nD* agents

Although agent positions can be controlled independently along each dimension in a single-leader network, not all results involving 1*D* agents can be naively applied to *nD* agents. One detail that must be taken into account in the *nD* case, which is not present in the 1*D* case, is the coupling amongst dimensions due to the same permutation matrix being applied to each dimensional component $d_i(x_T)$ of the target point. To see why, consider the following *nD* generalization of Problem IV.1 where one would like to bound how far any optimally permuted target point can be from the system's reachable subspace along any dimension.

Problem IV.2. Given a single-leader network with $N \ge 2$ follower agents, *nD* agent positions, and reachable subspace $R(\Gamma)$ along each dimension, find

$$M_n(\Gamma) = \max_{\substack{\|d_j(x)\|=1 \\ j=1,\ldots,n}} \left\{ \min_{P \in \mathcal{P}} \left\{ \max_{k=1,\ldots,n} \left\| \Pi_{R(\Gamma)^\perp}(Pd_k(x)) \right\|^2 \right\} \right\}. \tag{21}$$

What will be shown now is that when the dimensionality of agents is high compared to the number of agents (*i.e.*, $n \ge N!$), target points will exist which are always orthogonal to the system's reachable subspace along at least one dimension no matter how they are permuted. Even though most current multi-agent applications only involve agent states with low dimensionality (*i.e.*, 2*D* or 3*D* positions), the following result is instructive in that it demonstrates a previously unmentioned phenomenon that one should be aware of when generalizing from 1*D* to *nD* agents.

Theorem IV.3. The solution to Problem IV.2 for when $n \ge N!$ and $1 \le rank(\Gamma) < N$ is $M_n(\Gamma) = 1$.

Proof. To prove this theorem, we will construct a target point $x^* \in \mathbb{R}^{Nn}$ that will cause $M_n(\Gamma) = 1$. Assume that as stated in the theorem, $n \ge N!$ and $1 < rank(\Gamma) < N$. Since the system is not completely controllable, let x^* be such that $d_1(x^*) \perp R(\Gamma)$. Moreover, let $d_2(x^*), \ldots, d_{N!}(x^*)$ represent all $N! - 1$ other permutations of $d_1(x^*)$. In that case, no matter which $P \in \mathcal{P}$ is used, there is always a $k \in \{1, \ldots, N!\}$ such that $Pd_k(x^*) \perp R(\Gamma)$ and so $\left\|\Pi_{R(\Gamma)^\perp}(Pd_k(x^*))\right\|^2 = 1$. The ability to construct such a target point x^* means that $M_n(\Gamma) = 1$. ■

V. SUBOPTIMAL ALGORITHMS

The goal of this section is to bring to awareness of how the clustering representation discussed in Theorem III.2

opens the door to many suboptimal approaches based on clustering algorithms. This section will present some suboptimal clustering-based algorithms, but note that this is by no means an exhaustive list of approaches to solving the constrained clustering problem at hand.

5.1 Constrained k-means

The unconstrained version of the Euclidean minimum sum-of-squares clustering problem in Problem III.2 has been very well studied in machine learning and pattern recognition literature. In particular, a commonly used method for finding locally optimal solutions to the unconstrained clustering problem is the k-means algorithm, e.g., [9]. However, Theorem III.2 adds equality constraints on the size of individual clusters. To handle this additional layer of complexity, the constrained k-means clustering algorithm from [2] can be used to find locally optimal clusterings which minimize (19), while allowing for the minimum size of individual clusters to be specified. To use this algorithm for suboptimally solving the constrained clustering problem in Theorem III.2, equality constraints on the cluster sizes can be imposed by simply choosing the minimum cluster sizes so as to sum to N.

5.2 Compact clusters

Another approach to finding suboptimal solutions to the clustering problem in Theorem III.2 is to look for compact clusters, as defined by the following.

Definition V.1. In a clustering $c_1, \dots, c_k$ of a multiset S of $1D$ points, a cluster c_i is compact if $\nexists x_{i1}, x_{i2} \in c_i$ and $\nexists x_j \in c_j$ such that $x_{i1} < x_j < x_{i2}$, $\forall j \neq i$.

It will be shown now that in a $1D$ homogeneous single-leader network, the optimal clustering of target points for solving the NP-hard clustering problem in Theorem III.2 is always compact.

Lemma V.1. The optimal clustering $c_1^*, \dots, c_k^*$ of a multiset S of $1D$ points, which minimizes (19) fixed $|c_i|$ for $i = 1, \dots, k$, involves only compact clusters.

Proof. We start by showing that elements in every non-compact clustering can always be reassigned to decrease (19) without changing the cluster sizes. Assume $c_1, \dots, c_k$ are not all compact, then $\exists x_{a1}, x_{a2} \in c_a$ and $x_b \in c_b$ such that $x_{a1} < x_b < x_{a2}$, for some c_a and c_b where $a \neq b$. Furthermore, define

$$H_o(c_1,\dots,c_k,m_1,\dots,m_k) = \sum_{i=1}^{k}\sum_{z\in c_i}(z-m_i)^2,$$

where $H(c_1, \dots, c_k) \le H_o(c_1, \dots, c_k, m_1, \dots, m_k)$ with equality when $m_i = \theta(c_i)$, for $i = 1, \dots, k$. The total distortion of the clustering can be rewritten as

$$\begin{aligned} H(c_1,\dots,c_k) &= H_o(c_1,\dots,c_k,\theta(c_1),\dots,\theta(c_k)) \\ &= Q - 2R(c_a,c_b,\theta(c_a),\theta(c_b)), \end{aligned}$$

where

$$Q = \sum_{i=1}^{N} x_i^2 + \sum_{i=1}^{k}|c_i|\theta(c_i)^2 - 2\sum_{i=1, i\neq a,b}^{k}\theta(c_i)\sum_{z\in c_i} z,$$

and

$$R(c_a,c_b,m_a,m_b) = m_a\sum_{z\in c_a} z + m_b\sum_{z\in c_b} z.$$

If $\theta(c_a) \ge \theta(c_b)$, assign $\hat{c}_a$ the $|c_a|$ largest elements of $c_a \cup c_b$, while giving $\hat{c}_b$ the remaining elements. Otherwise, if $\theta(c_a) \ge \theta(c_b)$, then let $\hat{c}_a$ have the $|c_a|$ smallest elements of $c_a \cup c_b$, while $\hat{c}_b$ gets the rest. Furthermore, define $\hat{c}_i = c_i \forall i \neq a, b$. Notice that $|\hat{c}_i| = |c_i|$, for $i = 1, \dots, k$. Then, since after the reassignment, $\theta(\hat{c}_a) \neq \theta(c_a)$ and $\theta(\hat{c}_b) \neq \theta(c_b)$,

$$\begin{aligned} H(c_1,\dots,c_k) &= Q - 2R(c_a,c_b,\theta(c_a),\theta(c_b)) \\ &\ge Q - 2R(\hat{c}_a,\hat{c}_b,\theta(c_a),\theta(c_b)) \\ &= H_o(\hat{c}_1,\dots,\hat{c}_k,\theta(c_1),\dots,\theta(c_k)) \\ &> H(\hat{c}_1,\dots,\hat{c}_k). \end{aligned}$$

Whenever a clustering $c_1, \dots, c_k$ is not all compact, it is possible to obtain a new clustering $\hat{c}_1, \dots, \hat{c}_k$ with a lower total distortion. Since there are only a finite number of ways to cluster points in S, an optimal cluster must exist and it must involve only compact clusters. ■

As a consequence of Lemma V.1, the number of clusterings or permutations that need to be considered when searching for an optimal solution to Problem III.1 is decreased from at most $N!$ to now at most $k!$.

Theorem V.1. Finding $c_1^*, \dots, c_k^*$ to minimize (19) for a $1D$ single-leader network requires considering at most $k!$ clusterings.

Proof. For $1D$ points, only the ordering of the k compact clusterings matter in finding $c_1^*, \dots, c_k^*$. Thus, at most only $k!$ clusterings need to be considered. ■

Note that, although a nD generalization of Lemma V.1 may exist, any further investigation requires further analysis and would be tangential to the goal of this section, which was to simply demonstrate how the NP-hard clustering problem in

Theorem III.2 could be solved using a variety of suboptimal methods.

VI. CONCLUSIONS

This paper investigated the role of homogeneity when controlling agents in a single-leader network to a set of target points. When agents are all capable of the same tasks at each target, the target point can be permuted while still expressing the same intended goal for the system. However, in networks which are not completely controllable, permuting a target point could potentially bring it closer to the system's reachable subspace. To take this special property of homogeneous networks into account, the notion of controllability in a single-leader network was extended from a point-to-point property to a point-to-set property of the system. Here, the set corresponded to all permutations of a target point and one would like to find the permutation which brings the target closest to the reachable subspace. Finding this optimal permutation of a target point was shown to be a NP-hard problem. However, specific network topologies were identified in which exploiting agent homogeneity could allow for the system to surpass the limitations on its controllability. Moreover, clustering-based approaches were presented to solve for suboptimal target point permutations.

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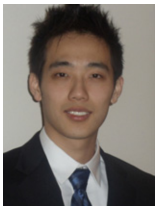

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