Trajectory Planning for Linear Control Systems with Generalized Splines

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Abstract

When planning trajectories for linear control systems, a demand that arises naturally in, for instance, air traffic control, noise contaminated data interpolation, and planning for switched control systems, is that the curve interpolate through given points or intervals at given times. In this article, we address this by showing how standard optimal control techniques, together with mathematical programming, solve the problem and provide a theoretical framework for producing a set of curves called generalized smoothing splines.

1 Introduction

In this article, we look at the problem of finding the control that drives a given linear control system between predefined states, while interpolating through given points, or intervals. Our resulting curve will minimize a functional, and it turns out that the curve will be so called generalized spline, whose characteristics depend on the dynamics of the control system [5]. So, by establishing a framework for constructing optimal trajectories, based on control theory, we provide a systematic way to construct all of the classical splines, and even other types of curves that are potentially useful.

This type of interpolation problem needs to be solved for a number of different reasons. One is that when doing trajectory planning for switched control systems, we need to be able to generate curves that pass through predefined states at given times where the switching is supposed to take place. In other cases, such as air traffic control, the same problem needs to be solved, since in these cases, we need to be able to specify the position that the system will be in at a sequence of times. However, in most situations, it is not really crucial that we pass through these points exactly, but rather that we go reasonably close to them, while minimizing the cost functional. This is a desired property for two apparent reasons. First of all, a small deviation from the prespecified point can result in a significant decrease in the cost, and secondly, when the data that we work with is noise contaminated, it is not even desirable to interpolate through these points exactly. Inspired by [4], we can incorporate these aspects into our proposed method, and thus produce something that normally is called smoothing splines.

In Section 2, we start by discussing some facts about linear control systems, and show how they can be used to arrive at an optimal control problem, suitable for producing the curves discussed above. We then proceed to actually solving the problem of producing the generalized smoothing splines, using standard optimal control techniques together with mathematical programming, reducing the problem into a convex, easy-to-solve quadratic programming problem. We then, in Section 4, conclude by discussing some properties of the generated curves, such as differentiability and numerical efficiency.

2 Optimal Control for Linear Control Systems

In this article, we study linear single input, single output control systems on the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t) \\ y(t) &= c^T x(t), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$, and (A, b, c) are constant matrices of compatible dimensions. Now, the problem that we want to solve is the following - Given intervals and times $([a_i, b_i], t_i)$. How do we construct a curve that passes through each of these intervals $[a_i, b_i]$ at the specified time t_i, $0 \leq t_1 < t_2 < \ldots < t_m \leq T$, while minimizing

$$\min_u J = \min_u \frac{1}{2} \int_0^T u^2(t) dt. \tag{2}$$

It is a well known fact [1], [5] that the problem of interpolating the output of the system through given, distinct points, while minimizing (2), has a solution if the system (1) is both controllable and observable.

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Throughout this article, we will assume that this is the case, and we can now proceed to the interval interpolation conditions

$$a_i \leq y(t_i) \leq b_i \quad i = 1, \ldots, m, \tag{3}$$

where $y(t)$ is the linear output of the system. Solving (1) for y gives us

$$y(t) = c^T e^{At} x_0 + \int_0^t c^T e^{A(t-s)} bu(s) ds, \tag{4}$$

so if we let

$$\begin{aligned} \alpha_i &= a_i - c^T e^{At_i} x_0 \\ \beta_i &= b_i - c^T e^{At_i} x_0 \end{aligned} \tag{5}$$

and

$$z(t_i) = \int_0^{t_i} c^T e^{A(t_i - s)} bu(s) ds, \tag{6}$$

we can express the interpolation conditions (3) as

$$\alpha_i \leq z(t_i) \leq \beta_i, \quad i = 1, \ldots, m. \tag{7}$$

If we want to interpolate through points instead of intervals, we just shrink the intervals so that $\alpha_i = \beta_i$ for those i:s that are of interest.

We will assume that we are looking at all

$$u \in L^2[0, T], \tag{8}$$

although we will show, in Section 4, that the optimal solution actually is piecewise entire, and that it is differentiable of a degree k, to be defined later.

We first note that the linearity gives us the following lemma.

Lemma 1 *The set of controls that make $z(t)$ satisfy the constraints (7) is a closed and convex subset of $L^2[0, T]$.*

Proof : For each i ($\in \{1, 2, \ldots, m\}$), we have that

$$\sum_{k=1}^{N} \sigma_k \alpha_i \leq \sum_{k=1}^{N} \sigma_k z_k(t_i) \leq \sum_{k=1}^{N} \sigma_k \beta_i, \tag{9}$$

where

$$\begin{aligned} &\sum_{k=1}^{N} \sigma_k = 1, \\ &\sigma_k > 0, \quad k = 1, \ldots, N, \end{aligned} \tag{10}$$

and z_k is any solution that satisfies the constrains, given by (6), using the control u_k. Now, consider

$$\begin{aligned} &\sum_{k=1}^{N} \sigma_k z_k(t_i) = \sum_{k=1}^{N} \sigma_k \int_0^{t_i} c^T e^{A(t_i - s)} bu_k(s) ds \\ &= \int_0^{t_i} c^T e^{A(t_i - s)} b \sum_{k=1}^{N} \sigma_k u_k(s) ds. \end{aligned} \tag{11}$$

Thus the convex sum of controls make the curve satisfy the constraints if the individual controls, u_k, makes the resulting curve satisfy the constraints. That this convex set is non-empty follows from the assumption that (1) was both controllable and observable. On the other hand, assume that $\{u_k\}$ is a sequence of controls that each satisfy the constraints. Passing the limit through the integral in (6), because of the compactness of $[0, T]$, it follows that the limit also satisfies the constraints, and the lemma follows. ■

Now the existence and uniqueness of the optimal control problem follows from standard theorems. (See for example Luenberger [3].)

Theorem 1 *There exists a unique control function, $u(t)$, that satisfies the constraints, and minimizes the functional $J(u)$.*

Minimizing $J(u)$ under these constraints is, however, equivalent to an associated optimal control problem, given by the Lagrange Duality Theorem [3]

Theorem 2 *Minimizing $J(u)$ under these constraints, when the set of controls that satisfy the constraints is a closed and convex subset of $L^2[0, T]$, is equivalent to finding*

$$\begin{aligned} \mu_0 &= \max_{\lambda, \gamma \geq 0} \min_u \mu(u, \lambda, \gamma) \\ &= \max_{\lambda, \gamma \geq 0} \min_u \{ \frac{1}{2} \int_0^T u^2(t) dt \\ &+ \sum_{i=1}^{m} \lambda_i (\alpha_i - z(t_i)) + \sum_{i=1}^{m} \gamma_i (z(t_i) - \beta_i) \}. \end{aligned} \tag{12}$$

3 Generalized Smoothing Splines

3.1 Interval Interpolation

We now move on to actually solving the problem, using standard optimal control techniques together with some ideas from mathematical programming, and we start by defining a set of functions that will be of importance to us, when trying to find the optimal control explicitly. Let

$$g_i(t) = \begin{cases} c^T e^{A(t_i - t)} b & t \leq t_i \\ 0 & t > t_i, \end{cases} \tag{13}$$

which makes it possible to reformulate (12) as

$$\begin{aligned} &\mu_0 = \max_{\lambda, \gamma \geq 0} \min_u \{ \int_0^T (\tfrac{1}{2} u^2(t) + (\gamma^T - \lambda^T) g(t) u(t)) dt \\ &+ \lambda^T \alpha - \gamma^T \beta \}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} g^T(t) &= (g_1(t), g_2(t), \ldots, g_m(t))^T \\ \alpha^T &= (\alpha_1, \alpha_2, \ldots, \alpha_m)^T \\ \beta^T &= (\beta_1, \beta_2, \ldots, \beta_m)^T. \end{aligned} \tag{15}$$

We now minimize $\mu(u, \lambda, \gamma)$ over u, assuming that λ and γ are fixed, by finding the point where the Fréchet derivative of μ, with respect to u, is zero. After some calculations, this gives us our optimal control law

$$u^\star = (\lambda^T - \gamma^T)g(t). \tag{16}$$

We can now eliminate $u^\star$ from μ by substituting (16) into (14), which gives us

$$\mu_0 = \min_{\lambda,\gamma\geq 0} \frac{1}{2}(\lambda^T - \gamma^T)G(\lambda - \gamma) - \lambda^T\alpha + \gamma^T\beta, \tag{17}$$

where G is the Grammian

$$G = \int_0^T g(t)g^T(t)dt. \tag{18}$$

If we now define

$$\xi^T = (\lambda^T, \gamma^T), \tag{19}$$

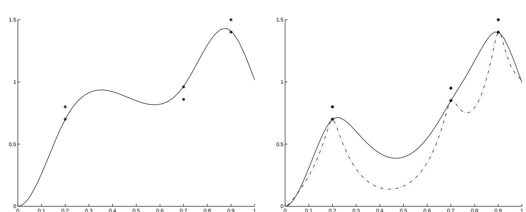
we have a quadratic programming problem

$$\mu_0 = \min_{\xi\geq 0} \frac{1}{2}\xi^T H\xi + F^T\xi, \tag{20}$$

where

$$\begin{aligned} H &= \begin{pmatrix} G & -G \\ -G & G \end{pmatrix} \\ F^T &= \left(-\alpha^T, \beta^T\right). \end{aligned} \tag{21}$$

Thus, in order to find the optimal $u^\star$, we just need to solve this convex, quadratic programming problem, for which there already exists efficient algorithms.



(a) Interpolation when the eigenvalues are both equal to -1.

(b) Here, the eigenvalues are both -10 (solid) or -100 (dotted).

Figure 1: The curves are solutions to the problem of interpolating through three intervals (with a fixed end point) for second order systems with different dynamics.

3.2 Penalized Deviations

In general, the control laws constructed, using the above proposed technique, are going to drive the system close to the end points of the intervals. A better control law is probably obtained by penalizing the control for making the system deviate from a given point in the interval, at the same time as we still require that we remain within the interval. We then get a new cost functional defined by

$$J(u) = \frac{1}{2}\left(\int_0^T \kappa u^2(t)dt + \sum_{i=1}^m (z(t_i) - \rho_i)^2\right), \tag{22}$$

where $\rho_i \in [\alpha_i, \beta_i]$. Here, the parameter $\kappa > 0$ controls the amount of smoothing. If it is small, then the spline gets really close to the desired interpolation points, the ρ_i:s, while a larger κ makes the spline more smooth. However, in this article, we simply set $\kappa = 1$, since it adds nothing new to the derivations, that are essentially the same in concept as before, but are a bit more complex in calculation.

Thus, our associated problem becomes

$$\begin{aligned} \mu_0 &= \max_{\lambda,\gamma\geq 0} \min_u \{\frac{1}{2}\int_0^T u^2(t)dt + \frac{1}{2}\sum_{i=1}^m (z(t_i) - \rho_i)^2 \\ &+ \sum_{i=1}^m \lambda_i(\alpha_i - z(t_i)) + \sum_{i=1}^m \gamma_i(z(t_i) - \beta_i)\}, \end{aligned} \tag{23}$$

and setting the Fréchet derivative of this expression, with respect to u, to zero gives that our optimal control, $u^\star$, has to satisfy

$$u + \sum_{i=1}^m g_i(z(t_i) - \rho_i) + (\gamma^T - \lambda^T)g = 0. \tag{24}$$

Thus we see that we must have the optimal u as a linear combination of the g_i:s. This has the effect of reducing the non-parametric problem to the problem of calculating parameters in a finite dimensional space.

So, if we assume that

$$u^\star = \tau^T g = g^T\tau, \tag{25}$$

and insert this into (24), we get

$$g(t)^T((I + G)\tau + \gamma^T - \lambda^T + \rho^T) = 0. \tag{26}$$

Solving this equation with respect to τ gives us

$$\tau = (I + G)^{-1}(\lambda + \rho - \gamma), \tag{27}$$

and inserting this into (23) gives after some calculations

$$\begin{aligned} \mu_0 &= \max_{\lambda,\gamma\geq 0} -\frac{1}{2}(\lambda^T - \gamma^T)(I + G)^{-1}G(\lambda - \gamma) \\ &- \rho^T(I + G)^{-1}G(\lambda - \gamma) + \alpha^T\lambda - \beta^T\gamma, \end{aligned} \tag{28}$$

which, once again, gives us a quadratic programming problem in the variables (λ, γ).

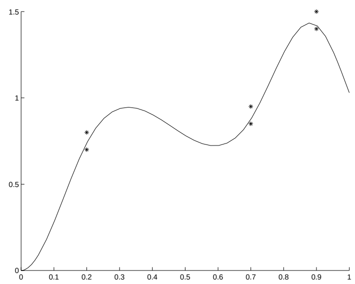


Figure 2: Here we interpolated through the intervals with penalized deviations (We placed the ρ_i:s in the middle of the intervals) for a second order system with both eigenvalues equal to -1.

4 Properties of the Solution

In this section, we will discuss some of the properties of the optimal curve, and we will begin by discussing the optimal control function, $u(t)$, before we consider the actual spline function $y(t)$.

First we note that the optimal control is piecewise entire since it has discontinuities of the function or the derivatives, only at the points t_i, and elsewhere it is of the form $c^T e^{A(t_i - t)} d$, for some choice of t_i and d. The degree of differentiability depends on the degree of differentiability of the functions $g_i(t)$. Now, recalling the definition of g_i in (13) gives that the kth derivative of g_i is

$$\frac{d^k}{dt^k} g_i(t) = (-1)^k c^T A^k e^{A(t_i - t)} b \text{ for } t < t_i. \qquad (29)$$

Thus the kth derivative is zero at t_i if and only if $c^T A^k b = 0$. We state this fact as a proposition.

Proposition 1 *The optimal control is differentiable of degree k provided $c^T A^i b = 0$ for $i = 0, \cdots, k$.*

Note that the optimal control may have a higher degree of differentiability at a given node if the coefficient of g_i is 0.

We now consider the spline function $y(t)$. Recall

$$y(t) = c^T e^{At} x_0 + \int_0^t c^t e^{A(t-s)} b u^*(s) ds.$$

Theorem 3 *The function*

$$y(t) = c^T e^{At} x_0 + \int_0^t c^t e^{A(t-s)} b u^*(s) ds.$$

is differentiable of degree $2k+2$ where k is the degree of differentiability of $u^*(t)$.

Proof: Let k be the maximal integer such that $cA^i b = 0$ for $i = 0, \cdots, k$ then we have that

$$\frac{d^{k+1}}{dt^{k+1}} \int_0^t c^T e^{A(t-s)} b u^*(s) ds = \int_0^t c^T A^{k+1} e^{A(t-s)} b u^*(s) ds,$$

since $c^T A^i b = 0$ for $i \leq k$. Now it follows that

$$\frac{d^{k+1+m}}{dt^{k+1+m}} \int_0^t c^T e^{A(t-s)} b u^*(s) ds = \sum_{i=0}^{m-1} c^T A^{k+1+i} b \frac{d^{m-1-i}}{dt^{m-1-i}} u^*(t) + \int_0^t c^T A^{k+1+m} e^{A(t-s)} b u^*(s) ds.$$

Thus we see that if $u^*(t)$ has k continuous derivatives, then $y(t)$ has a total of $2k+2$ continuous derivatives. The theorem then follows. ■

5 Conclusions

We have developed a method for constructing trajectories that minimizes some cost functional, depending on the control signal imposed on a given linear control system. This is done while the system is driven in such a way that it generates outputs that go through prespecified intervals at given times. Our approach has the nice property that it shows how these types of curves, called smoothing generalized splines, all can be generated within a coherent theoretical framework, based on ideas from both optimal control theory and mathematical programming. Our approach also has nice numerical features since it reduces the problem into a convex, quadratic programming problem that, after some transformations, can be solved easily.

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