Containment Control in Mobile Networks

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Abstract

In this paper, the problem of driving a collection of mobile robots to a given target destination is studied. In particular, we are interested in achieving this transfer in an orderly manner so as to ensure that the agents remain in the convex polytope spanned by the leader-agents, while the remaining agents, only employ local interaction rules. To this aim we exploit the theory of partial difference equations and propose hybrid control schemes based on stop-go rules for the leader-agents. Non-Zenoness, liveness and convergence of the resulting system are also analyzed.

I. INTRODUCTION

This paper investigates a particular subarea of multi-agent control, namely the so-called *containment* problem where a collection of autonomous, mobile agents are to be driven to a given target location while guaranteeing that their motion satisfies certain geometric constraints. These constraints are there to ensure that the agents are contained in a particular area during their transportation. Such issues arise for example when a collection of autonomous robots are to secure and then remove hazardous materials. This removal must be secure in the sense that the robots should not venture into populated areas or in other ways contaminate their surroundings.

We approach this problem from a leader-follower point-of-view [1], [2], [3]. In particular, we will let the agents move autonomously based on local, consensus-like interaction rules, commonly found in the literature under the banner of algebraic graph theory [4], [5], [6]. However, we will

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augment this control structure with the addition of leader-agents or anchor nodes [7]. These leaders are to define vertices in a convex polytope (the leader-polytope) and they are to move in such a way that the target area is reached while ensuring that the follower-agents stay in the convex polytope spanned by the leaders, up to a given tolerance. As such, the followers movements are calculated in a decentralized manner according to a fixed interaction topology, while the leaders are assumed to be able to detect if any of the followers violate the containment property.

For the leaders, we will use a hybrid Stop-Go policy [8], [9], in which the leaders move according to a decentralized formation control strategy until the containment property is about to be violated. At this point, they stop and let the followers settle back into the leader-polytope before they start moving again. For such a strategy to be successful, a number of results are needed, including a guarantee that the Laplacian-based follower-control will in fact drive the followers back into the leader-polytope. Moreover, we must also ensure that such a control strategy is feasible in the sense of non-Zeno, live in the sense of not staying in the Stop mode indefinitely, and convergent in the sense that the target area is in fact reached. This approach can also be generalized to hierarchial networks, as was illustrated by our preliminary work in [10].

II. Background and Mathematical Preliminaries

In this section we will present the basic mathematical framework and some enabling results in multi-agent control.

We start with basic notions of graph theory. For more details we refer the reader to [11]. An undirected graph G is defined by a set $\mathcal{N}_G = \{1, \ldots N\}$ of *nodes* and a set $\mathcal{E}_G \subset \mathcal{N}_G \times \mathcal{N}_G$ of *edges*. We will also use $|\mathcal{N}_G|$ for denoting the cardinality of $\mathcal{N}_G$. Two nodes x and y are *neighbors* if $(x, y) \in \mathcal{E}_G$. The neighboring relation is indicated with $x \sim y$ and $\mathcal{P}(x) = \{y \in \mathcal{N}_G : y \sim x\}$ collects all neighbors to the node x. A *path* $x_0 x_1 \ldots x_L$ is a finite sequence of nodes such that $x_{i-1} \sim x_i$, $i = 1, \ldots, L$. A graph G is *connected* if there is a path connecting every pair of distinct nodes.

Definition 1 *Let $S = (\mathcal{N}_S, \mathcal{E}_S)$ be an undirected host graph and $\mathcal{N}_{S'} \subset \mathcal{N}_S$. The subgraph S' associated with $\mathcal{N}_{S'}$ is the pair $(\mathcal{N}_{S'}, \mathcal{E}_{S'})$ where $\mathcal{E}_{S'} = \{(x, y) \in \mathcal{E}_S : x \in \mathcal{N}_{S'}, y \in \mathcal{N}_{S'}\}$*

Definition 1 allows basic operations in set theory to be extended to graphs. For instance, if S_1 and S_2 are two subgraphs of the graph S, then $S_1 \cup S_2$, $S_1 \cap S_2$, $S_1 \backslash S_2$ are the graphs associated with $\mathcal{N}_{S_1} \cup \mathcal{N}_{S_2}$, $\mathcal{N}_{S_1} \cap \mathcal{N}_{S_2}$, and $\mathcal{N}_{S_1} \backslash \mathcal{N}_{S_2}$, respectively. For our purposes, we will often use graphs with a boundary.

Definition 2 *Let S be a subgraph of G. The boundary of S is the subgraph $\partial S \subset G$ associated with $\mathcal{N}_{\partial S} \doteq \{y \in \mathcal{N}_G \setminus \mathcal{N}_S \ : \ \exists\, x \in \mathcal{N}_S \ : \ x \sim y\}$. The closure of S is $\bar{S} = \partial S \cup S$.*

Note that the definition of the boundary of a graph depends upon the host graph G. This implies that if one considers three graphs $S' \subset S \subset G$, the boundaries of S' in S and in G may differ.

In the context of multi-agent systems, the nodes of the host graph G represent agents and the edges are communication links. In particular, an agent x has access to the states of all its neighbors and can use this piece of information to compute its control law. Although a complete graph is not necessary for a distributed control algorithm, we always assume that the host graph is connected.

In order to model the collective behavior of the agents we will use functions $f : \mathcal{N}_G \mapsto \mathbb{R}^d$ defined over a graph G [12]. The *partial derivative* of f is defined as $\partial_y f(x) \doteq f(y) - f(x)$ and the Laplacian of f is given by

$$\Delta f(x) \doteq - \sum_{y \in \mathcal{N}_G, y \sim x} \partial_y^2 f(x) = + \sum_{y \in \mathcal{N}_G, y \sim x} \partial_y f(x), \tag{1}$$

where the last identity follows from the fact that $\partial_y^2 f(x) = -\partial_y f(x)$. The integral and the average of f are defined, respectively, as

$$\int_G f \ dx \doteq \sum_{x \in \mathcal{N}_G} f(x), \quad \langle f \rangle \doteq \frac{1}{|\mathcal{N}_G|} \int_G f \ dx. \tag{2}$$

Let $L^2(G|\mathbb{R}^d)$ be the Hilbert space composed by all functions $f : \mathcal{N}_G \mapsto \mathbb{R}^d$ endowed with the norm $\|f\|_{L^2}^2 = \int_G \|f\|^2$. We will use the shorthand notation L^2 when there is no ambiguity on the underlying domain and range of the functions.

Let S be a subgraph of G and ∂S be its boundary in G. We assume that $S \cup \partial S = G$. As in [12], we also consider the Hilbert space $H_0^1(S) = \{f \in L^2(G) \ : \ f_{|\partial S} = 0\}$ (see [12] for the definition of a suitable norm on $H_0^1(S)$). Note that a function $f \in H_0^1(S)$ is defined on $\bar{S}$ and possibly non null only on S.

The next theorem, proved in [12], characterize the eigenstructure of the Laplacian operator defined on $H_0^1(S)$.

Theorem 1 *Let G be a connected graph and S a proper subgraph of G. Then, the operator $\Delta : H_0^1(S|\mathbb{R}^d) \mapsto L^2(\bar{S}|\mathbb{R}^d)$ has $|\mathcal{N}_S|d$ strictly negative eigenvalues. Moreover, the corresponding eigenfunctions form a basis for $H_0^1(S|\mathbb{R}^d)$.*

III. Multiple Stationary Leaders

In this section we use PdEs for modelling and analyzing a group of agents with multiple leaders. A leader is just an agent that moves toward a predefined goal, and whose control policy is independent of the motion of all the followers. However, followers that are neighbors to the leader can use the leader state in order to compute their control inputs.

Let $r(x,t)$ be the position of the agent x at time $t \geq 0$, where[1] $r \in L^2$. The communication network is represented by the undirected and connected graph G. For distinguishing between leaders and followers, we consider two subgraphs S_F and S_L of G and assume that $S_L = \partial S_F$ and $S_F \cup S_L = G$, where the subscripts denote ”Leaders” and ”Followers” respectively. Note that we assume that all agents are either designated as leaders or followers.

As already mentioned in the introduction, we will assume that the followers obey the simple dynamics $\dot{r}(x,t) = u(x,t)$, where

$$u(x,t) \doteq \Delta r(x,t) \tag{3}$$

is the *Laplacian* control law. Let $\hat{r}(x,t)$, $x \in \mathcal{N}_{S_L}$ be the trajectory of the leaders. Then, the collective dynamics is represented by the model

$$\dot{r}(x,t) = \Delta r(x,t) \qquad x \in \mathcal{N}_{S_F} \tag{4a}$$

$$r(x,t) = \hat{r}(x,t) \qquad x \in \mathcal{N}_{S_L} \tag{4b}$$

endowed with the initial conditions $r(\cdot,0) = \tilde{r} \in L^2(S_F)$.

Model (4) is an example of a continuous-time Partial difference Equation (PdE) with non-homogeneous Dirichlet boundary conditions. We refer the reader to [12], [13], [14] for an introduction to PdEs.

[1]For sake of conciseness, for a function $f(x,t) : \mathcal{N}_G \times \mathbb{R}^+ \to \mathbb{R}^d$ we will often write $f \in L^2$ instead of $f(\cdot,t) \in L^2$.

The main results on Laplacian control available in the literature and specialized to model (4) are:

- in the leaderless case (i.e. $S_L = \emptyset$), the Laplacian control solves the rendezvous problem, i.e. $r(x,t) \to r^* \in \mathbb{R}^d,\ \forall x \in \mathcal{N}_G$ as $t \to +\infty$. Moreover, the agents converge exponentially to $r^* = \langle \tilde{r} \rangle$ thus achieving *average* consensus. These results have been established in [15], [16] through the joint use of tools in control theory and algebraic graph theory. A formal analysis of the PdE (4a) has been conducted in [17], [13], [14] showing a complete accordance with results available within the theory of the heat equation [18];
- in the case of a single leader (i.e. $\mathcal{N}_{S_L} = \{x_L\}$) with fixed position (i.e. $\hat{r}(x_L,t) = \bar{r} \in \mathbb{R}^d$), Laplacian control solves the rendezvous problem with $r^* = \bar{r}$ [15]. This property has also been shown in [13], [14] within the PdE framework, thus highlighting the profound links between model (4) and the heat equation with Dirichlet boundary conditions [18].

The first attempt of this paper is to characterize the asymptotic behavior of the followers in the presence of multiple leaders with fixed positions. To this end, for the remainder of this section, we will assume that $\hat{r}(x,t) = \bar{r}(x) \in L^2(S_L)$. The equilibria of (4) are then given by the solutions to the PdE

$$\Delta h(x) = 0 \qquad x \in \mathcal{N}_{S_F} \tag{5a}$$

$$h(x) = \bar{r}(x) \quad x \in \mathcal{N}_{S_L} \tag{5b}$$

and they have been studied in [12]. In particular, [12, Theorem 3.5] shows that if G is connected and $\mathcal{N}_{S_L} \neq \emptyset$ then, the PdE (5) has a unique solution[2] $h(x)$. By analogy with the jargon of Partial Differential Equations, h is termed the *harmonic extension of the boundary conditions* $\bar{r}$.

Our next aim is to verify that $r \to h$ as $t \to +\infty$. Let us consider the decomposition

$$r(x,t) = r_0(x,t) + h(x), \quad r_0 \in H^1_0(S_F) \tag{6}$$

Since h does not depend upon time and $\Delta h = 0$, $\forall x \in \mathcal{N}_{S_F}$, the PdE (4) is equivalent to the

[2][12, Theorem 3.5] assumes that the subgraph S is *induced* (see [12] for the definition of induced subgraphs). However, a careful examination of the proof, reveals that this assumption is unnecessary.

following one

$$\dot{r}_0(x,t) = \Delta r_0(x,t) \qquad x \in \mathcal{N}_{S_F} \tag{7a}$$

$$r_0(x,t) = 0 \qquad x \in \mathcal{N}_{S_L} \tag{7b}$$

From (6), it is apparent that the problem of checking if $r \to h$ as $t \to +\infty$ can be recast into the problem of studying the convergence to zero of the solutions to the PdE (7). The fact that $r_0 \to 0$ as $t \to +\infty$ follows from Theorem 1 and it can be shown by proceeding exactly as in the proof of [17, Theorem 5][3].

The next Theorem, proved in [19], highlights a key geometrical feature of $h(x)$. For a set X of points in $\mathbb{R}^d$, $\mathrm{Co}(X)$ will denote its convex hull. Moreover, the set Ω_L is the convex hull of leaders positions, i.e. $\Omega_L \doteq \mathrm{Co}(\{\bar{r}(y),\ y \in \mathcal{N}_{S_L}\})$.

Theorem 2 *Let S_1 be a nonempty connected subgraph of S_F and ∂S_1 be its boundary in G. Then, $\forall x \in \mathcal{N}_{S_1}$ it holds*

$$h(x) \in \mathrm{Co}(\{h(y),\ y \in \mathcal{N}_{\partial S_1}\}). \tag{8}$$

Moreover, one has that $h(x) \in \Omega_L$, i.e. that the position of each follower lies in the convex hull of the leaders positions. Finally, if Ω_L is full-dimensional[4], then $h(x) \in \Omega_L \backslash \partial\Omega_L$, $\forall x \in \mathcal{N}_{S_F}$.

Another geometrical feature which we need is the following:

Theorem 3 *Suppose that Ω_L is full-dimensional and that $r(x,t)$ is evolving according to* (4)*. Suppose that, at a given time $t = \bar{t}$, there is an agent $x \in \mathcal{N}_{S_F}$ such that $r(x,\bar{t}) \in \partial\Omega_L$ and $r(y,t) \in \Omega_L$, $\forall y \in \mathcal{P}(x)$. Then, two situations may occur:*

1) *there exists an (affine) hyperplane χ such that*

$$r(x,\bar{t}) \in \chi \cap \partial\Omega_L, \textit{and } r(y,\bar{t}) \in \chi \cap \partial\Omega_L \ \forall y \in \mathcal{P}(x).$$

Then:

$$\exists\, \alpha > 0 \ : \ r(x,\bar{t}) + \alpha\dot{r}(x,\bar{t}) \in \chi \cap \partial\Omega_L, \tag{9}$$

[3]Actually, [17, Theorem 5] proves a stronger property, namely that the origin of (7) is "exponentially stable on the space $H_0^1(S)$". The definition of stability of equilibria on subspaces is provided in [17].

[4]The set $\Omega_L \subset \mathbb{R}^d$ is full-dimensional if the dimension of the affine hull generated by Ω_L is d (see [20]).

2) *otherwise,*

$$\exists\, \alpha > 0 \ :\ r(x,\bar{t}) + \alpha \dot{r}(x,\bar{t}) \in \Omega_L \setminus \partial\Omega_L. \tag{10}$$

Note that (9) means that the velocity of x will be along the hyperplane χ (in other words, the agent may slide on the boundary $\partial\Omega_L$), whereas (10) means that the velocity of x is pointing inside the polytope Ω_L. While Theorem 2 and the fact that $r \to h$ as $t \to +\infty$ guarantee that followers asymptotically enter Ω_L, Theorem 3 ensures that once all followers are in Ω_L they cannot exit from this set and therefore containment will be never violated.

Proof: (Theorem 3)

Since $r(x,t)$ obeys to (4), by rearranging terms we obtain:

$$\dot{r}(x,\bar{t}) = -|\mathcal{P}(x)| r(x,\bar{t}) + \sum_{y \in \mathcal{P}(x)} r(y,\bar{t}).$$

Then, setting $\alpha = |\mathcal{P}(x)|^{-1}$, it holds:

$$r(x,\bar{t}) + \alpha \dot{r}(x,\bar{t}) = |\mathcal{P}(x)|^{-1} \sum_{y \in \mathcal{P}(x)} r(y,\bar{t}),$$

i.e., $r(x,\bar{t}) + \alpha\dot{r}(x,\bar{t})$ is the barycenter $b(\mathcal{Y}_x)$ of the polytope $\mathcal{Y}_x \doteq \mathrm{Co}(\{r(y,\bar{t}),\ y \in \mathcal{P}(x)\})$. Note that, if $r(y,\bar{t}) \in \Omega_L$, $\forall y \in \mathcal{P}(x)$ one has $\mathcal{Y}_x \in \Omega_L$. Moreover, thanks to convexity, the barycenter of $\mathcal{Y}_x$ lies in the relative interior of $\mathcal{Y}_x$. Thus, if all $y \in \mathcal{P}(x)$ verify that $r(y,\bar{t}) \in \chi \cap \partial\Omega_L$ then $\mathcal{Y}_x \subset \chi \cap \partial\Omega_L$ and so does $b(\mathcal{Y}_x)$, i.e. $b(\mathcal{Y}_x) \in \chi \cap \partial\Omega_L$; otherwise $b(\mathcal{Y}_x) \in \Omega_L \setminus \partial\Omega_L$. ■

IV. Leader-Follower Containment Control

Containment of all the followers is achieved in the case of static leaders in the last section. However, if the leaders are moving, this property might be violated. In order to prevent the followers from leaving the polytope spanned by the leaders, appropriate control strategies need to be designed for the leaders to guarantee the containment. In what follows, we propose a hybrid strategy for this purpose and analyze liveness and reachability of the resluting closed-loop system.

A. Hybrid Control Strategy

For the sake of containment, we define two distinctly different control modes for the evolution of the leaders. The first of the two control modes is the $STOP$ mode that corresponds to the leaders halting their movements altogether in order to prohibit a break in the containment:

$$STOP : \text{(4a), (4b) and } \dot{\hat{r}}(x,t) = 0,\ x \in \mathcal{N}_{S_L} \tag{11}$$

It is clear that in order to execute this mode, no information is needed for the leaders whatsoever.

The second control mode under consideration is the GO mode, in which the leaders move toward a given target formation. A number of different control laws can be defined for this, but, for the sake of conceptual unification, we let the GO mode be given by a Laplacian-based control strategy as well.

$$GO : \text{(4a), (4b) and } \dot{\hat{r}}(x,t) = \Delta_{S_L}(\hat{r}(x,t) - r_T(x)),\ x \in \mathcal{N}_{S_L} \tag{12}$$

where $r_T(x),\ x \in \mathcal{N}_{S_L}$ denotes the desired target position of leader x and Δ_{S_L} denotes the Laplacian operator defined solely over the subgraph S_L, i.e.

$$\Delta_{S_L} f(x) \doteq - \sum_{y \sim x,\ y \in \mathcal{N}_{S_L}} \partial_y^2 f(x).$$

Under the assumption that S_L is connected, and by exactly the same reasoning as for the standard rendezvous problem, under the influence of the GO mode alone the leaders will converge exponentially to $r_L(x) = \langle \hat{r}(\cdot,0) - r_T(\cdot)\rangle + r_T(x)$, i.e. $\exists k > 0, \eta > 0$ such that $\|\hat{r}(\cdot,t) - r_L(x)\|_{L^2} \leq k e^{-\eta t}\|\hat{r}(\cdot,0) - r_L(x)\|_{L^2}$. In other words, no convergence to a predefined point is achieved. Rather, this control law ensures that the leaders arrive at a translationally invariant target formation.

Note that the details of the leaders' motion is not crucial and this particular choice is but one of many possibilities. However, this choice is appealing in that it makes the information flow explicit, and the leaders only need access to the positions (and target locations) of their neighboring leaders in order to compute their motion. As such the decentralized character of the algorithm is maintained.

In order to fully specify the hybrid Stop-Go leader policy transition rules are needed as well. As before, let Ω_L denote the leader-polytope and let $d(\mu, \Omega_L)$ denote the signed distance

$$d(\mu, \Omega_L) \doteq \zeta_{\Omega_L}(\mu) \min_{x \in \partial\Omega_L} \|\mu - x\|_2, \tag{13}$$

where $\|\cdot\|_2$ denotes the Euclidean 2-norm, and where $\zeta_{\Omega_L}(\mu) = -1$ if $\mu \in \Omega_L$ and +1 otherwise. Using this distance measure we let the two guards, i.e. transition conditions, be given by

$$GO2STOP : \exists y \in \mathcal{N}_{S_F} \mid d(r(y,t), \Omega_L) \geq 0? \tag{14a}$$

$$STOP2GO : d(r(y,t), \Omega_L) < -\epsilon \; \forall y \in \mathcal{N}_{S_F}? \tag{14b}$$

where a transition from GO to $STOP$ triggers when the conditions in $GO2STOP$ are met, and similarly for $STOPT2GO$, and where $\epsilon > 0$ is a threshold.

Note that the guard $STOP2GO$ is crossed only if the following assumptions are verified:

Assumption 1 *Let* $\hat{h}(\cdot, t)$ *be the solution to* (5) *for* $\bar{r}(\cdot) = \hat{r}(\cdot, t)$, $\forall t \geq 0$ *and consider the set* $\Omega_L^\epsilon(t) = \{y \in \Omega_L(t) : d(y, \partial\Omega_L(t)) < -\epsilon\}$. *Then*

1) $\Omega_L^\epsilon(t)$ *is nonempty,* $\forall t \geq 0$*;*
2) $\mathrm{Co}(\{\hat{h}(x,t), x \in \mathcal{N}_{S_F}\}) \subset \Omega_L^\epsilon(t)$.

Note that, for a given time $t \geq 0$, the uniqueness of $\hat{h}(\cdot, t)$ follows from the uniqueness of the solution to (5). In particular, Assumption 1 implies that Ω_L must be full-dimensional at all times and "sufficiently fat" along every direction (see condition 1). Conditions relating property 2 of Assumption 1 to the graph topology are currently under investigation. A few comments must be made about the computation and communication requirements that these guards give rise to. If two leaders are located at the end-points of the same face of Ω_L, then they must be able to determine if any of the followers are in fact on this face. This can be achieved through a number of range sensing devices, such as ultrasonic, infra-red, or laser-based range-sensors. Moreover, in order for all leaders to transition between modes in unison, they must communicate between them, which means that either S_L is a complete graph, or that multi-hop strategies are needed. In either way, a minimal requirement for these mode transitions to be able to occur synchronously, without having to rely on information flow across follower-agents, is that S_L must be connected.

The hysteresis threshold $\epsilon > 0$ in the $STOP2GO$ guard and the next assumption are needed in order to avoid Zeno behaviors. Let ρ_{Ω_L} denote the supremum of the diameter of Ω_L during an execution.

Assumption 2 $\exists M < \infty$ *such that* $\rho_{\Omega_L} \leq M$.

It is easy to check that Assumption (2) is verified when Laplacian control governs the leaders' motion in the GO mode as in (12). Indeed, the exponential convergence of $\hat{r}(x,t)$ to $r_L(x) = \langle \hat{r}(\cdot,0) - r_T(\cdot)\rangle + r_T(x)$ implies that $\hat{r}(x,t)$ is bounded at all times. However, Laplacian control is but one of many possible control strategies and can be replaced by other control schemes (e.g. plan-based leader control laws) without generating Zeno executions as long as Assumption 2 is verified.

Theorem 4 *Under Assumptions 2 and 1, the hybrid automaton defined by* (11), (12) *and* (14) *is non-Zeno.*

Proof: Let the system be in the $STOP$ mode. Under Assumption 2 we have

$$\|\dot{r}(x,t)\| = \|\Delta r(x,t)\| \leq \sum_{y\sim x} \|\partial_y r(x)\| \leq \sum_{y\sim x} \rho_{\Omega_L} \leq N\rho_{\Omega_L}, \quad \forall x \in \mathcal{N}_{S_F}. \tag{15}$$

From Assumption 1, in order for the system to leave the $STOP$ mode, at least one follower agent must have travelled at least a distance ϵ, which in turn implies that the system will always stay for a time greater than or equal to $\epsilon/N\rho_{\Omega_L}$ in the $STOP$ mode. In order for the system to exhibit Zeno executions, a necessary condition is that the difference between the transition times must approach zero [21]. Since this is not the case here, the non-Zeno property is established. ■

B. Liveness and Reachability

As already mentioned, the proposed solution is non-Zeno. However, as it is currently defined, the Stop-Go policy may be blocking in the sense that the system never leaves the $STOP$ mode. One remedy to this problem is to allow the containment to be slightly less tight. In other words, we can select different guards, e.g.

$$GO2STOP : \exists y \in \mathcal{N}_{S_F} \mid d(r(t,y),\Omega_L) > 2\delta? \tag{16a}$$

$$STOP2GO : d(r(t,y),\Omega_L) \leq \delta \ \forall y \in \mathcal{N}_{S_F}? \tag{16b}$$

where $\delta > 0$. What this means is that we do not enter the $STOP$ mode until a follower is 2δ outside Ω_L. Let us define

$$\Omega_{L,\delta} \doteq \{y \in \mathbb{R}^d : d(y,\Omega_L) \leq \delta\}$$

Note that, one has $\Omega_L \subset \Omega_{L,\delta}$. The next Theorem summarizes the main properties of the resulting hybrid automaton. A remarkable feature of the guards (16) is that Assumption 1 is no longer needed in order to guarantee liveness.

Theorem 5 *Under Assumption 2, the hybrid automaton by* (11), (12) *and* (16) *is non-Zeno, live, in the sense of always leaving the* $STOP$ *mode eventually, and convergent in the sense that* $\hat{r}(x,t) \to \langle \hat{r}(\cdot,0) - r_T(\cdot)\rangle + r_T(x)$.

Proof: We first prove liveness. Assume that the system is in the $STOP$ mode. From Theorem 2 we have that $h \in \Omega_L$. Since $\forall x \in S_F$, $r(x,t) \to h$, and $\Omega_L \subset \Omega_{L,\delta}$, every follower will eventually get back in $\Omega_{L,\delta}$ in finite time (recall that the leaders are stationary in the $STOP$ mode) hence triggering a transition to the GO mode.

Under Assumption 2, it holds $\|\dot{r}(x,t)\| \leq N(\rho_{\Omega_L} + 2\delta)$ and we can repeat the non-Zeno argument in the proof of Theorem 4 in order to see that the system always stays in the GO mode for a time greater than or equal to $\delta/(N(\rho_{\Omega_L} + 2\delta))$.

As a result, in a non-blocking system the leaders will be given infinitely many opportunities to move during a finite (bounded away from zero) time horizon, which implies convergence to the target location as long as the leaders would in fact end up at the target location under the influence of the GO mode alone. ■

V. CONCLUSIONS

In this paper we presented a hybrid Stop-Go control policy for the leaders in a multi-agent containment scenario. In particular, the control strategy allows us to transport a collection of follower-agents to a target area while ensuring that they stay in the convex polytope spanned by the leaders. The enabling results needed in order to achieve this is that, for stationary leaders, the followers in a connected interaction graph will always converge to locations in the leader-polytope. Extensions to the proposed control strategy are moreover given in order to ensure certain liveness properties.

Acknowledgment

The work by Giancarlo Ferrari-Trecate was partially supported by the European Commission under the Network of Excellence HYCON, contract number FP6-IST-511368. The work by Magnus Egerstedt and Meng Ji was supported by the U.S. Army Research Office through Grant #99838.

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