On-Line Optimization of Switched-Mode Systems: Algorithms and Convergence Properties

Y. Wardi, X.C. Ding, M. Egerstedt, and S. Azuma

***Abstract*— This paper concerns the problem of on-line (real-time) computation of solutions to the optimal switching time problem in hybrid systems. The systems under consideration are autonomous, and the performance measure to be optimized has the form of a cost functional defined on the state trajectory. The state variable cannot, however, be measured directly and it has to be estimated by a suitable observer. In this paper, we propose an on-line optimization algorithm based on the state observer, and derive bounds on its convergence rate.**

I. Introduction

In recent years there has been a mounting interest in optimal control of switched-mode hybrid dynamical systems, where the control parameter includes the schedule of the system's modes and the performance metric consists of a cost functional defined on the system's state (see [4], [6], [8], [9], [16], [17], [18], [20]). A general optimal-control framework for such systems was established in [4], and variants of the maximum principle were derived in [16], [17], [18]. References [2], [3], [8], [14] considered piecewise-linear or affine systems, while nonlinear autonomous systems were considered in [8], [10], [22], [23]. Gradient-descent and second-order algorithms for nonlinear systems were developed in [22], [23]. This paper also considers nonlinear, autonomous systems, where the absence of the external term u allows us to focus on the salient features inherent in hybrid systems. Moreover, the absence of a control term is inherently natural in a number of problems where different modes are defined through different feedback laws, thus already incorporating the control terms in the dynamics.

The dynamics that we focus on in this paper have the following form,

$$\dot{x} \;\in\; \{f_\alpha(x)\}_{\alpha\in\mathcal{A}}, \qquad t\in[0,T], \tag{1}$$

where $x(t)\in R^n$ is the state variable, $\mathcal{A}$ is a finite index-set, and $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is a collection of functions $f_\alpha : R^n \rightarrow R^n$. The final time $T>0$ and the initial state $x(0):=x_0$ are assumed to be fixed. Let $L : R^n \rightarrow R$ be a cost function, and define the cost functional J by

$$J \;:=\; \int_0^T L(x)dt. \tag{2}$$

J is a function of the schedule of the functions f_α in the Right-Hand Side (RHS) of (1), and the objective of the optimal control problem is to find (compute) the schedule (mode sequence as well as switching times) that minimizes the cost functional J.

Such optimal control problems arise in a variety of applications, including situations where a control module has to switch its attention among a number of subsystems [11], [15], [19], or collect data sequentially from a number of sensory sources [5], [10].

As already mentioned, a schedule typically consists of two (sub)variables: the sequencing variable and the timing variable. The sequencing variable consists of the sequence of the functions f_α in the RHS of (1), corresponding to the sequence of modes of the system. The timing variable is comprised of the vector of switching times between successive functions in a given sequence $\{f_\alpha\}_{\alpha\in\mathcal{A}}$. Optimization with respect to the timing variable (for a given sequence) is an easier problem than optimization over the space of sequences, and most existing algorithms pertain to it. Lately, however, a number of approaches to the optimal sequencing problem have emerged as well [7], [17]. This paper considers the timing problem for a given sequence of functions, and its principal contribution is in an on-line (real-time) optimization algorithm, but it also points out possible extensions to the sequencing problem as a topic for future research. It should be noted that initial work (without proofs) was carried out in the brief manuscript [21], and this paper provides an in-depth analysis of the problem.

Nonlinear optimal control problems generally admit open-loop solutions which depend on the initial state, except under special circumstances. However, the initial state variable is not always known and it has to estimated by a suitable observer. In this case, an algorithm for computing the optimal control must rely on the trajectory of the state estimator and hence be implemented on line. This served to motivate the development of the algorithm in [1], which tracks the solution of the optimal cost-to-go along the trajectory of the state estimator. However, that algorithm has no apparent way of getting back on track once it departs from the trajectory of the optimal cost-to-go. The algorithm in this paper gets around this difficulty since it does not have to track the minimum of the optimal cost-to-go. Moreover, it is more general that the algorithm in [1] while including it as a special case for a particular choice of the initial condition and step size. Under a specific step-selection rule it resembles Newton's method, and it has a quadratic convergence rate in a suitable sense defined below.

The main convergence result of the paper is that the on-line, output-driven estimate of the optimal switching-times

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vector at time t, denoted by $\bar{\tau}(t)$, satisfies the following inequality,

$$||\bar{\tau}(t+\Delta t)-\bar{\tau}|| \leq K\Big(||\bar{\tau}(t)-\bar{\tau}||^2+\Delta t||\bar{\tau}(t)-\bar{\tau}||+||e(t+\Delta t)||\Big),$$

where Δt is the numerical step length of the algorithm, $\bar{\tau}$ is the stationary solution obtained through complete state information, K is an appropriate (dynamics-dependent) constant, and e is the estimation error. This equation is noteworthy in that it tells us something about convergence rates and explicitly incorporates terms stemming from the numerical accuracy, the rate at which the state estimate is converging, and the quadratic convergence rate of Newton's method. These issues will be made explicit in a later part of the paper, which is organized as follows. Section II formulates the problem and recounts relevant existing results, and Section III presents the algorithm, analyzes its convergence rate, and discusses some implementation-related issues. Finally, Section IV concludes the paper.

II. PROBLEM FORMULATION AND PREVIOUS RESULTS

The switched-mode hybrid dynamical system that concerns this paper has the following form. Given a time horizon $[0,T]$, let τ_i, $i=1,2,\ldots,N$, be a monotone-nondecreasing finite sequence of time-points such that $0\leq\tau_1,\ldots,\tau_N\leq T$. Given $N+1$ differentiable functions $f_i:R^n\rightarrow R^n$, consider the differential equation

$$\dot{x} \;=\; \begin{cases} f_1(x), & \text{for all } t\in[0,\tau_1), \\ f_i(x), & \text{for all } t\in[\tau_{i-1},\tau_i), \quad i=2,\ldots,N, \\ f_{N+1}(x), & \text{for all } t\in[\tau_N,T), \end{cases} \tag{3}$$

with a given initial condition $x_0:=x(0)$. The variable $x(t)$ is the state of the system, the functions f_i, $i=1,\ldots,N+1$, are called the *modal functions*, and the times τ_i, $i=1,\ldots,N$, are called the *switching times*. To simplify the notation in (3) we define $\tau_0:=0$ and $\tau_{N+1}:=T$, and we further define

$$F(x,t) \;:=\; f_i(x) \quad \text{for all } t\in[\tau_{i-1},\tau_i), \quad \text{for every } i=1,\ldots,N+1, \tag{4}$$

so that Eq (3) has the following form,

$$\dot{x} \;=\; F(x,t). \tag{5}$$

Let $L:R^n\rightarrow R$ be a continuously-differentiable function, and consider the *cost functional* J, defined by

$$J \;:=\; \int_0^T L(x)dt. \tag{6}$$

Let us denote the vector of switching times by $\bar{\tau}:=(\tau_1,\ldots,\tau_N)^T$. Assuming that the sequence of modal functions, $\{f_i\}_{i=1}^{N+1}$ is fixed, we can view J as a function of $\bar{\tau}$ via Eq. (3). This gives rise to the *timing optimization problem* of minimizing $J(\bar{\tau})$ subject to the constraints

$$0=\tau_0\leq\tau_1\leq\ldots,\leq\tau_N\leq\tau_{N+1}=T. \tag{7}$$

As mentioned in the introduction, a number of algorithms have been proposed to solve this problem, and most of them are based on gradient descent. The partial derivatives $\frac{\partial J}{\partial\tau_i}$, $i=1,\ldots,N$, comprising the gradient can be computed by costate techniques as shown in [7].

When the sequencing of the modal functions becomes a part of the optimization variable, the problem becomes more difficult. Reference [7] proposed an approach consisting of the insertion of a modal function to a given schedule, at a given point in time, for a brief period, and computing the derivative of the cost functional with respect to the insertion's duration. This allows us to overlay the inherently-discrete parameter space by a continuous structure and use a gradient-descent technique for computing a suboptimal schedule. Although this paper concerns the timing optimization problem, we mention the possibility of modifying the mode-sequence because it will become relevant later in the sequel.

Recently there has been a mounting interest in on-line algorithms for the timing optimization problem, and a particular approach was proposed in [1], [21]. Consider the problem of minimizing J as defined in (6), subject to the constraints defined by (5) and (7), for given sequence of modal functions $\{f_i\}_{i=1}^{N+1}$, initial state $x(0)$, and final time $T>0$. Suppose that the state variable $x(t)$ cannot be measured or observed, and it is estimated by the output $\hat{x}(t)$ of a suitable state observer. We are not concerned in this paper with the details of the observer subsystem, but mention that it can be the Luenberger observer in the case where the system is linear, or the Moraal-Grizzle observer [12] in the case where the system is nonlinear. We must point out that finding a simultaneously stable observer for multiple systems (and hence for switched-mode systems) is an issue that we do not consider here since it must be addressed individually for each particular system. Instead, we assume that such a stable observer exists.

The cost-to-go function at time $t\in[0,T]$ is defined as follows. Given a switching-time vector $\bar{\tau}$, consider the equation

$$\dot{y}(\xi) \;=\; F(y,\xi) \tag{8}$$

on the time-interval $\xi\in[t,T]$ with the boundary condition $y(t)=\hat{x}(t)$, where F is defined by (4). The cost-to-go, denoted by $J(t,\hat{x}(t),\bar{\tau})$ to emphasize its dependence on t and the state estimator $\hat{x}(t)$, is defined by the following equation,

$$J(t,\hat{x}(t),\bar{\tau}) \;:=\; \int_t^T L(y(\xi))d\xi. \tag{9}$$

Observe that the dynamic equation for y (Equation (8)) is identical to the dynamic equation for the state x (Equation (5)) from time t onward, but the boundary conditions at time t are different: the state estimator $\hat{x}(t)$ for the former, and the actual state $x(t)$ for the latter. Next, for a given $\bar{\tau}=(\tau_1,\ldots,\tau_N)^T$ we define the integer $i(t,\bar{\tau})$ by

$$i(t,\bar{\tau}) \;=\; \max\{j=0,1,\ldots,N \mid \tau_j\leq t\}, \tag{10}$$

i.e., $i(t,\bar{\tau})$ is the index (j) of the last switching time that has occurred prior to or at time t. It follows from (10) that

$$\tau_{i(t,\bar{\tau})} \;\leq\; t \;<\; \tau_{i(t,\bar{\tau})+1}. \tag{11}$$

Moreover, assuming that $\hat{x}(t)$ is available at time t, and by the boundary condition $y(t) = \hat{x}(t)$ for Eq. (8), it is readily seen that the switching times that occurred before time t have no effect on $y(\xi)$ for any $\xi \geq t$. Consequently, and by (9), the cost-to-go $J(t, \hat{x}(t), \bar{\tau})$ is a function only of the future switching times τ_j, $j = i(t, \bar{\tau}) + 1, \ldots, N$, and not of the past or present switching times τ_j, $j = 1, \ldots, i(t, \bar{\tau})$. This certainly corresponds to the intuitive notion that in an on-line algorithm, timing variables that occurred in the past cannot be changed. Thus, the actual variable of the cost-to-go function is a sub-vector of $\bar{\tau}$ consisting of the future switching times, namely $(\tau_{i(t,\bar{\tau})+1}, \ldots, \tau_N)^T \in R^{N-i(t,\bar{\tau})}$. Nonetheless we will use the notation $\bar{\tau}$ in (9) and its derivatives if no confusion arises. The problem of optimizing $J(t, \hat{x}(t), \bar{\tau})$ with respect to $(\tau_{i(t,\bar{\tau})+1}, \ldots, \tau_N)^T$ henceforth will be denoted by $\pi_{t,\hat{x}(t)}$.

Let $\bar{\tau}^\star(t)$ denote a solution point of the cost-to-go at time t. The algorithm in [1] for computing this point has two phases: an initial, off-line phase whereby it computes $\bar{\tau}^\star(0)$ by the initial time $t = 0$ and with the initial state estimator $\hat{x}(0)$, and the on-line phase where it tracks $\bar{\tau}^\star(t)$ in time. That algorithm has the following three shortcomings: (1). Once it departs from the optimal trajectory $\{\bar{\tau}^\star(t)\}$ it has no natural way of returning to it. (2). It cannot easily modify the sequence of modal functions since that would involve recomputing the initial phase (which is time-consuming) on-line. (3). Once a switching time is surpassed by the real time, it becomes a part of the past and hence cannot be modified. If this happens at an early stage of the algorithm when the state estimation error is large, the result can be a far-from-optimal switching time that cannot be changed. Our algorithm circumvents these shortcomings by eliminating the initial, off-line phase and by not being required to track the optimal trajectory $\{\bar{\tau}^\star(t)\}$. This allows it to append the sequence of modal functions without interrupting its on-line execution, whenever it is felt that an early switching time was surpassed by the real time t prematurely.

The next section describes our algorithm and derives a quantitative result indicating a balance between convergence, and compliance with constraints on its ability to perform in real time. With a suitable descent direction, the algorithm resembles the Newton-Raphson method. It will be shown moreover that once it reaches an optimum point of the cost-to-go then it becomes identical to the algorithm in [1]. This demonstrates that the algorithm in [1] is essentially a Newton-Raphson method, which is not clear from its presentation and is not true once that algorithm does not run on the optimal trajectory $\{\bar{\tau}^\star(t)\}$.

III. ON-LINE ALGORITHM

Let us fix a sequence of modal functions $\{f_1, \ldots, f_{N+1}\}$, and consider the timing optimization problem of computing $\bar{\tau}$ that minimizes the cost functional J as defined in (6) and subject to the inequality constraints in (7). The algorithm presented below is underscored by three processes: the time process $\{t\}_{t=0}^T$, the state-estimator process $\{\hat{x}(t)\}_{t=0}^T$, and the optimization process which we denote by $\{\bar{\tau}(t)\}_{t=0}^T$. The first process, $\{t\}$, acts as a temporal platform for computation, that is, at any time t a certain computation may be performed. The state-estimator process consists of the output of the observer subsystem, and the optimization process acts to minimize the cost-to-go at each time t based on relevant information available at that time. More specifically, suppose that $\hat{x}(t)$ and $\bar{\tau}(t)$ are given at time t, then at time $t+dt$, first $\hat{x}(t+dt)$ becomes available from the observer subsystem, and then $\bar{\tau}(t+dt)$ is computed by taking a single step of an optimization algorithm that aims at solving $\pi_{t+dt,\hat{x}(t+dt)}$. Formally, for a suitable vector $h(t) \in R^{N-i(t+dt,\bar{\tau}(t))}$,

$$\bar{\tau}(t+dt) \;=\; \bar{\tau}(t) + h(t). \tag{12}$$

We point out that the process $\{\bar{\tau}(t)\}$ is not meant to be continuous, rather it expresses the fact the at time $t + dt$, one step of an optimization algorithm is computed via (12). We assume for the moment that this computation is instantaneous, which allows us to analyze the process $\{\bar{\tau}(t)\}$ as it evolves in infinitesimal time-quanta of duration dt. That assumption will be dispensed with later, when we replace dt by a positive time-step $\Delta t > 0$ which must bound from above the time required to compute $\bar{\tau}(t + \Delta t)$.

There are two natural choices of $h(t)$ that come to mind: one is based on gradient descent, and the other is based on the Newton-Raphson method. The algorithm described below follows the latter approach. It requires certain smoothness properties of the function J which are guaranteed by the following assumption.

Assumption 3.1: The functions f_i, $i = 1, \ldots, N+1$, and L, are three-times continuously differentiable.

The following result is an immediate corollary which can be proved by standard arguments from the theory of differential equations (see, e.g., the appendix of [13] for such arguments), and hence its proof is omitted.

Corollary 3.1: The function $J(t, \hat{x}, \bar{\tau})$ is twice continuously differentiable and its second derivatives are locally Lipschitz continuous.

Now we define $h(t)$ in (12) in the following way. Suppose that the vector $\bar{\tau}(t)$ is feasible for $\pi_{t+dt,\hat{x}(t+dt)}$. This means, by (7), (10), and (11), that

$$\begin{aligned} &\tau_{i(t+dt,\bar{\tau}(t))}(t) \;\leq\; t + dt \\ &< \; \tau_{i(t+dt,\bar{\tau}(t))+1}(t) \;\leq\; \ldots \;\leq\; \tau_N(t) \;\leq\; T. \end{aligned} \tag{13}$$

Let us define the $(N - i(t+dt)) \times (N - i(t+dt))$ matrix $H(t)$ by

$$H(t) \;=\; \frac{\partial^2 J}{\partial \tau^2}\big(t, \hat{x}(t), \bar{\tau}(t)\big), \tag{14}$$

and suppose that $H(t)$ is positive definite (otherwise it makes no sense to consider Newton-like algorithms). Furthermore, let $\tilde{h}(t)$ be the projection of $-H(t)^{-1}\frac{\partial J}{\partial \tau}(t + dt, \hat{x}(t + dt), \bar{\tau}(t))$ onto the feasible set as defined by (13), and for a given $c > 0$, define the step size $\gamma(t)$ by $\gamma(t) := \max\{c \leq 1 \mid \bar{\tau}(t) + c\tilde{h}(t) \text{ is feasible}\}$. Finally, we define $h(t)$ by $h(t) = \gamma(t)\tilde{h}(t)$. Note that $\tilde{h}(t)$ is a feasible direction from $\bar{\tau}(t)$, and the choice of $\gamma(t)$ ensures that (12) does not overshoot the feasible set and hence $\bar{\tau}(t + dt)$ is feasible.

We next show that under special circumstances, (12) with the above-defined $h(t)$ preserve local optimality and, in fact, yield the algorithm in [1]. These circumstances arise when $\bar{\tau}$ is a local minimum for the problem $\pi_{t,\hat{x}(t)}$, it lies in the interior of the feasible set for that problem, and $\gamma(t)=1$.

Proposition 3.1: Suppose that Assumption 3.1 is in force; and for every $t\in[0,T)$, $H(t)$ is positive definite; $-\frac{\partial J}{\partial\tau}(t+dt,\hat{x}(t+dt),\bar{\tau}(t))$ lies in the interior of the feasible set at $\bar{\tau}(t)$; $\gamma(t)=1$ and hence

$$\bar{\tau}(t+dt) \;=\; \bar{\tau}(t)-H(t)^{-1}\frac{\partial J}{\partial\tau}(t+dt,\hat{x}(t+dt),\bar{\tau}(t)); \quad (15)$$

and $\bar{\tau}(t+dt)$ lies in the interior of the feasible set of $\pi_{t+dt,\hat{x}(t+dt)}$. Then, if $\bar{\tau}(0)$ is a stationary point, then for every $t\in[0,T]$, $\bar{\tau}(t)$ is a stationary point for $\pi_{t,\hat{x}(t)}$.

Proof: Fix $t\in[0,T)$, and suppose that $\bar{\tau}(t)$ is stationary for $\pi_{t,\hat{x}(t)}$, and hence

$$\frac{\partial J}{\partial\tau}(t,\hat{x}(t),\bar{\tau}(t)) \;=\; 0. \quad (16)$$

Therefore, a first-order approximation of the last term in the RHS of (15) implies that

$$\bar{\tau}(t+dt) \;=\; \bar{\tau}(t)-H(t)^{-1}\times \Big(\frac{\partial^2 J}{\partial\tau\partial t}(t,\hat{x}(t),\bar{\tau}(t))dt+\frac{\partial^2 J}{\partial\tau\partial x}(t,\hat{x}(t),\bar{\tau}(t))\dot{\hat{x}}(t)dt\Big), \quad (17)$$

and consequently

$$\dot{\bar{\tau}}(t) \;=\; -H(t)^{-1}\times \Big(\frac{\partial^2 J}{\partial\tau\partial t}(t,\hat{x}(t),\bar{\tau}(t))+\frac{\partial^2 J}{\partial\tau\partial x}(t,\hat{x}(t),\bar{\tau}(t))\dot{\hat{x}}(t)\Big). \quad (18)$$

Next, taking the total derivative of $\frac{\partial J}{\partial\tau}(t,\hat{x}(t),\bar{\tau}(t))$ with respect to t yields,

$$\frac{d}{dt}\frac{\partial J}{\partial\tau}(t,\hat{x}(t),\bar{\tau}(t)) \;=\; \frac{\partial^2 J}{\partial\tau\partial t}(t,\hat{x}(t),\bar{\tau}(t)) +\frac{\partial^2 J}{\partial\tau\partial x}(t,\hat{x}(t),\bar{\tau}(t))\dot{\hat{x}}(t)+\frac{\partial^2 J}{\partial\tau^2}(t,\hat{x}(t),\bar{\tau}(t))\dot{\bar{\tau}}(t). \quad (19)$$

Plugging the RHS of (18) for $\dot{\bar{\tau}}(t)$ in (19) and noting that (by definition) $H(t)=\frac{\partial^2 J}{\partial\tau^2}(t,\hat{x}(t),\bar{\tau}(t))$, it follows that

$$\frac{d}{dt}\frac{\partial J}{\partial\tau}(t,\hat{x}(t),\bar{\tau}(t)) \;=\; 0. \quad (20)$$

Consequently, the gradient term $\frac{\partial J}{\partial\tau}(t,\hat{x}(t),\bar{\tau}(t))$ has a constant value as a function of t. Since by assumption $\frac{\partial J}{\partial\tau}(0,\hat{x}(0),\bar{\tau}(0))=0$, it follows that $\frac{\partial J}{\partial\tau}(t,\hat{x}(t),\bar{\tau}(t))=0$ for every $t\in[0,T]$, which completes the proof. ■

We remark that Eq. (18) defines the process $\{\bar{\tau}(t)\}$ in [1] which is shown there to preserve optimality. On the other hand, if $\bar{\tau}(t)$ is not stationary for $\pi_{t,\hat{x}(t)}$ then (17) is not identical to (15) and the algorithms here and in [1] are different.

In order to turn Eq. (12) into an algorithm we have to replace the infinitesimal time-step dt by a finite-length time-step $\Delta t>0$. This results in the following procedure for computing $\bar{\tau}(t+\Delta t)$ from $\bar{\tau}(t)$.

Algorithm 3.1:
Step 1. Compute $\tilde{h}(t)$, defined as the projection of $-H(t)^{-1}\frac{\partial J}{\partial\tau}(t+\Delta t,\hat{x}(t+\Delta t),\bar{\tau}(t))$ onto the feasible set at $\bar{\tau}(t)$.
Step 2. Compute $\gamma(t) := \max\{c\leq 1 \mid \bar{\tau}(t)+c\tilde{h}(t)$ is feasible$\}$.
Step 3. Define $h(t):=\gamma(t)\tilde{h}(t)$, and set

$$\bar{\tau}(t+\Delta t) \;:=\; \bar{\tau}(t)+h(t). \quad (21)$$

To have this algorithm implementable on line, we have to specify the time at which its steps would be computed. We can suppose that $\hat{x}(t)$ is available from the observer subsystem at time t, and likewise that $\hat{x}(t+\Delta t)$ is available at time $t+\Delta t$. Let us define the following quantities:

- $\lambda_1(t)$ - the time it takes to compute $\frac{\partial J}{\partial\tau}(t+\Delta t,\hat{x}(t+\Delta t),\bar{\tau}(t))$ from $\hat{x}(t+\Delta t)$ and $\bar{\tau}(t)$ (this is needed by Step 1).
- $\lambda_2(t)$ - the time it takes to compute $H(t)^{-1}$ from $\hat{x}(t)$ and $\bar{\tau}(t)$ (see (14)).
- $\lambda_3(t)$ - the time it takes to run through one cycle of the algorithm and compute $\bar{\tau}(t+\Delta t)$ from $H(t)^{-1}$ and $\frac{\partial J}{\partial\tau}(t+\Delta t,\hat{x}(t+\Delta t),\bar{\tau}(t))$.

Define $\lambda_j := \sup\{\lambda_j(t):t\in[0,T]\}$, $j=1,2,3$, and suppose that $\lambda_1+\lambda_2+\lambda_3<\Delta t$.

The following result points out a feasible computation schedule.

Proposition 3.2: Suppose that $\sum_{j=1}^{3}\lambda_j<\Delta t$. For every $t\in\{0,\Delta t,2\Delta t,\ldots\}$, it is possible to have $\bar{\tau}(t)$ available at time $t+\lambda_1+\lambda_3$.

Proof: The proof is by induction. Suppose that $\bar{\tau}(t)$ is available at time $t+\lambda_1+\lambda_3$ for some $k=0,1,\ldots$. Recall that $\hat{x}(t)$ was available at time t. Therefore, $H(t)^{-1}$ is available at time $t+\lambda_1+\lambda_3+\lambda_2$, and since by assumption $\sum_{j=1}^{3}\lambda_j<\Delta t$, $H(t)^{-1}$ is available at time $t+\Delta t$. Next, $\hat{x}(t+\Delta t)$ becomes available at time $t+\Delta t$, and therefore $\frac{\partial J}{\partial\tau}(t+\Delta t,\hat{x}(t+\Delta t),\bar{\tau}(t))$ becomes available at time $t+\Delta t+\lambda_1$. Consequently, $\bar{\tau}(t+\Delta t)$ can be available at time $t+\Delta t+\lambda_1+\lambda_3$, which complete the induction's hypothesis and hence the proof. ■

We next address a convergence result of this algorithm. Asymptotic convergence is a central concept in nonlinear programming and a minimal requirement of an algorithm. Typically it means that every accumulation point of an iteration sequence computed by an algorithm satisfies a suitable optimality condition, like stationarity or more generally, the Kuhn-Tucker condition. However, it is hard to speak of asymptotic properties of Algorithm 3.1, because it is designed to run only for a finite number of iterations due to the finiteness of the horizon interval $[0,T]$ and the time-step Δt. Moreover, once a switching time $\tau_i(t)$ is surpassed by the time t it retains its value thereafter, and this is another factor which limits the applicability of asymptotic concepts to the algorithm. For these reasons, we characterize convergence of the algorithm not in an asymptotic sense, but rather in the sense of convergence rate, as will be explained in the next paragraph.

Consider the Newton-Raphson algorithm for minimizing a twice-continuously differentiable function $f : R^n \to R$:

$$x_{k+1} = x_k - H(x_k)^{-1}\nabla f(x_k), \qquad k = 1, 2, \ldots, \quad (22)$$

where $H(x) := \frac{df^2}{dx}(x)$ is the Hessian. Then, if $\lim_{k\to\infty} x_k = x^*$ for some $x^* \in R^n$, and if $H(x^*)$ is positive definite, then x^* is a local minimum for f, and there exist $\delta > 0$ and $K > 0$ such that, if $||x_k - x^*|| < \delta$, then $||x_{k+1} - x^*|| \leq K||x_k - x^*||^2$ (see [13]). This means that the algorithm has a quadratic convergence rate to stationary points.

Our analysis of Algorithm 3.1 involves a similar concept, but it accounts for approximations associated with the facts that the computations are carried out every Δt seconds, and the state is estimated rather than measured. To set the stage, let us fix a time-point $t \in [0, T)$, and consider a vector $\bar{\tau} = (\tau_1, \ldots, \tau_N)^T$ in the interior of the feasible set, namely

$$\tau_{i(t,\bar{\tau})} \leq t < \tau_{i(t,\bar{\tau})+1} < \ldots < \tau_N < T; \quad (23)$$

to keep the discussion simple we do not consider here the case where $\bar{\tau}$ lies on the boundary of the feasible set, but an extension to that case appears to be conceptually straightforward by using arguments from the theory of constrained optimization. Suppose also that $\bar{\tau}$ is a stationary point for the problem $\pi_{t,x(t)}$, namely the problem of minimizing the cost-to-go at time t where the initial condition for (8) is the actual state $x(t)$. Then, we have that

$$\frac{\partial J}{\partial \tau}(t, x(t), \bar{\tau}) = 0. \quad (24)$$

Such a stationary point $\bar{\tau}$ is the target of our algorithm. Let us denote the Hessian $\frac{\partial^2 J}{\partial \tau^2}(t, x(t), \bar{\tau})$ by H, and define the error $e(t)$ by $e(t) := \hat{x}(t) - x(t)$ for all $t \in [0, T]$.

The following result was stated in [21] without a proof.

Proposition 3.3: Suppose that $\bar{\tau}$ is a stationary point for $\pi_{t,x(t)}$ lying in the interior of the feasible set, and that H is positive definite. Then there exist constants $\delta > 0$ and $K > 0$ such that, if $||\bar{\tau}(t) - \bar{\tau}|| < \delta$; $\Delta t < \delta$; $||e(t)|| < \delta$ and $||e(t + \Delta t)|| < \delta$; $\gamma(t) = 1$ and hence Eq. (15) is satisfied; and

$$i(t, \bar{\tau}(t)) = i(t + \Delta t, \bar{\tau}(t + \Delta t)) = i(t, \bar{\tau}) = i(t + \Delta t, \bar{\tau}), \quad (25)$$

then

$$||\bar{\tau}(t + \Delta t) - \bar{\tau}|| \leq K\Big(||\bar{\tau}(t) - \bar{\tau}||^2 + \Delta t||\bar{\tau}(t) - \bar{\tau}|| + ||e(t + \Delta t)||\Big). \quad (26)$$

Proof: By Eq. (25), the dimensions of $\bar{\tau}$, $\bar{\tau}(t)$, and $\bar{\tau}(t + \Delta t)$ are all the same, and the sequences of modal functions at times t and $t + \Delta t$ are identical. By Eq. (15), we have that

$$\bar{\tau}(t + \Delta t) - \bar{\tau} = \bar{\tau}(t) - \bar{\tau} - H(t)^{-1}\frac{\partial J}{\partial \tau}\big(t + \Delta t, \hat{x}(t + \Delta t), \bar{\tau}(t)\big). \quad (27)$$

By assumption $\bar{\tau}$ is a local minimum for $J(t, x(t), \cdot)$. Therefore, and by Bellman's optimality principle, $\bar{\tau}$ is a local minimum for $J(t + \Delta t, x(t + \Delta t), \cdot)$ as well, and hence

$$\frac{\partial J}{\partial \tau}\big(t + \Delta t, x(t + \Delta t), \bar{\tau}\big) = 0. \quad (28)$$

Consequently, and by (27),

$$\begin{aligned}\bar{\tau}(t + \Delta t) - \bar{\tau} &= \bar{\tau}(t) - \bar{\tau} - H(t)^{-1}\Big(\frac{\partial J}{\partial \tau}\big(t + \Delta t, \hat{x}(t + \Delta t), \bar{\tau}(t)\big) \\ &\quad - \frac{\partial J}{\partial \tau}\big(t + \Delta t, x(t + \Delta t), \bar{\tau}\big)\Big). \quad (29)\end{aligned}$$

By Corollary 3.1, the last difference term in the RHS of (29) can be written as

$$\begin{aligned}&\frac{\partial J}{\partial \tau}\big(t + \Delta t, \hat{x}(t + \Delta t), \bar{\tau}(t)\big) - \frac{\partial J}{\partial \tau}\big(t + \Delta t, x(t + \Delta t), \bar{\tau}\big) \\ &= \frac{\partial^2 J}{\partial \tau \partial x}\big(t + \Delta t, x(t + \Delta t), \bar{\tau}\big)e(t + \Delta t) + O\big(||e(t + \Delta t)||^2\big) \\ &\quad + \frac{\partial^2 J}{\partial \tau^2}\big(t + \Delta t, x(t + \Delta t), \bar{\tau}\big)(\bar{\tau}(t) - \bar{\tau}) + O\big(||\bar{\tau}(t) - \bar{\tau}||^2\big), \quad (30)\end{aligned}$$

where $\limsup_{\eta\to 0} \frac{O(\eta^2)}{\eta^2} < \infty$.

Next, recall (14) that $H(t) = \frac{\partial^2 J}{\partial \tau^2}(t, \hat{x}(t), \bar{\tau}(t))$, and that $H := \frac{\partial^2 J}{\partial \tau^2}(t, x(t), \bar{\tau})$ is positive definite by assumption. Therefore, and by Corollary 3.1, there exists $\delta_1 > 0$ and $K_1 > 0$ such that, if $\Delta t < \delta_1$, $||e(t)|| < \delta_1$, and $||\bar{\tau}(t) - \bar{\tau}|| < \delta_1$, then

$$\begin{aligned}&||H(t)^{-1} - \Big(\frac{\partial^2 J}{\partial \tau^2}\big(t + \Delta t, x(t + \Delta t), \bar{\tau}\big)\Big)^{-1}|| \\ &\leq K_1\Big(\Delta t + ||e(t + \Delta t)|| + ||\bar{\tau}(t) - \bar{\tau}||\Big). \quad (31)\end{aligned}$$

Now plug the RHS of (30) in (29) to obtain,

$$\begin{aligned}\bar{\tau}(t + \Delta t) - \bar{\tau} &= \bar{\tau}(t) - \bar{\tau} \\ &- H(t)^{-1}\Big(\frac{\partial^2 J}{\partial \tau \partial x}\big(t + \Delta t, x(t + \Delta t), \bar{\tau}\big)e(t + \Delta t) + O\big(||e(t + \Delta t)||^2\big) \\ &+ \frac{\partial^2 J}{\partial \tau^2}\big(t + \Delta t, x(t + \Delta t), \bar{\tau}\big)(\bar{\tau}(t) - \bar{\tau}) + O\big(||\bar{\tau}(t) - \bar{\tau}||^2\big)\Big). \quad (32)\end{aligned}$$

Subtracting and adding the term $\Big(\frac{\partial^2 J}{\partial \tau^2}(t + \Delta t, x(t + \Delta t)\bar{\tau})\Big)^{-1}$ from $H(t)^{-1}$ in (32) we obtain, after some algebra, that

$$\begin{aligned}
& \bar{\tau}(t+\Delta t) - \bar{\tau} \\
= & -\Big(H(t)^{-1} - \big(\frac{\partial^2 J}{\partial \tau^2}(t+\Delta t, x(t+\Delta t), \bar{\tau})\big)^{-1}\Big) \\
& \times\Big(\frac{\partial^2 J}{\partial \tau \partial x}\big(t+\Delta t, x(t+\Delta t), \bar{\tau}\big)e(t+\Delta t) \\
& +O\big(||e(t+\Delta t)||^2\big) \\
& + \frac{\partial^2 J}{\partial \tau^2}\big(t+\Delta t, x(t+\Delta t), \bar{\tau}\big)\big(\bar{\tau}(t)-\bar{\tau}\big) \\
& +O\big(||\bar{\tau}(t)-\bar{\tau}||^2\big)\Big) \\
& -\big(\frac{\partial^2 J}{\partial \tau^2}(t+\Delta t, x(t+\Delta t), \bar{\tau})\big)^{-1} \\
& \times\Big(\frac{\partial^2 J}{\partial \tau \partial x}\big(t+\Delta t, x(t+\Delta t), \bar{\tau}\big)e(t+\Delta t) \\
& +O\big(||e(t+\Delta t)||^2\big) + O\big(||\bar{\tau}(t)-\bar{\tau}||^2\big)\Big). \qquad (33)
\end{aligned}$$

Consequently, and by (31) and Corollary 3.1, there exist $K_2 > 0$ and $\delta_2 > 0$ such that, if $\Delta t < \delta_2$, $||e(t)|| < \delta_2$, $||e(t+\Delta t)|| < \delta_2$, and $||\bar{\tau}(t) - \bar{\tau}|| < \delta_2$, then

$$\begin{aligned}
& ||\bar{\tau}(t+\Delta t) - \bar{\tau}|| \\
< & \; K_2\Big(\Delta t + ||e(t+\Delta t)|| + ||\bar{\tau}(t)-\bar{\tau}||\Big) \times \\
& \Big(||e(t+\Delta t)|| + ||\bar{\tau}(t)-\bar{\tau}||\Big) \\
& + K_2\Big(||e(t+\Delta t)|| + ||\bar{\tau}(t)-\bar{\tau}||^2\Big). \qquad (34)
\end{aligned}$$

Finally, by reducing K_2 and δ_2 if necessary, there exist $K > 0$ and $\delta > 0$ such that, if the assumptions of the lemma are satisfied, then Eq. (26) is in force. ■

A few remarks are in order. The upper bound in the RHS of (26) consists of three parts (scaled by K): $||\bar{\tau}(t) - \bar{\tau}||^2$, $\Delta t||\bar{\tau}(t) - \bar{\tau}||$, and $||e(t+\Delta t)||$. In the first part, the quadratic term is due to the fact that the algorithm has the form of a Newton-Raphson method. For the same reason, Δt in the second part does not appear alone, but only as multiplying $||\bar{\tau}(t) - \bar{\tau}||$. This term, Δt, ought be as small as possible but large enough so that it bounds from above the computing times of the algorithm (see Proposition 3.2), and thus its value comprises a balance between accuracy and on-line computational requirements. Finally, the third part, $||e(t+\Delta t)||$, appears alone, and to guarantee its smallness an aggressive state observer should be put in place.

IV. Conclusions

In this paper, the problem of computing optimal solutions on-line for switched-mode hybrid dynamical systems was addressed. In particular, we considered autonomous systems with performance measures in the form of a cost functional over the state trajectory. The state variable could not, however, be measured directly, and instead it was approximated through an appropriate observer. The contribution of the paper consists of a novel framework for on-line optimal control, and the derivation of upper bounds on its convergence rate. Applications of this framework and its algorithms to several problems are currently under investigation.

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