# Pathwise Observability Through Arithmetic Progressions and Non-Pathological Sampling

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***Abstract*— We consider the problem of establishing pathwise observability for a class of switched linear systems with constant, autonomous dynamics, but with switched measurement equations. Using van der Waerden's Theorem, a standard result in Ramsey Theory, we give a sufficient condition on the components of the system for it to be pathwise observable. This first result then enables us to extend the Kalman-Bertram criterion, which concerns the conservation of observability after the introduction of sampling, to switched linear systems. We then dualize these results to pathwise controllability.**

## I. Introduction

Consider the following class of switched linear systems:

$$\begin{aligned} x_{k+1} &= Ax_k \\ y_k &= C(\theta_k)x_k, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^p$, and where the matrices $A$ and $C(\cdot)$ are of compatible dimensions. The modes $\theta_k$ assume values in $\{1, \ldots, s\}$, so that $C(\theta_k)$ switches among $s$ different measurement matrices $C(1), \ldots, C(s)$. The system in (1) can be used for modeling switches between $s$ different sensory modes, as can occur, e.g., when sensors fail intermittently, or when the measurements $y_k$ are transmitted over a memoryless erasure channel [1], [2]. In [1], estimators were designed for the noisy counterpart of (1), and in [3], an asymptotic observer was proposed. In this paper, we are concerned with a particular aspect of the deterministic finite-time observability of the model, namely *pathwise observability* [4], which we define as follows:

Let a path $\theta$ of length $N$ be a string of length $N$, whose elements take values in $\{1, \ldots, s\}$, and let $|\theta| = N$ denote its length. Defining the observability matrix $\mathcal{O}(\theta)$ of a path $\theta$ as:

$$\mathcal{O}(\theta) \triangleq \begin{pmatrix} C(\theta_1) \\ \vdots \\ C(\theta_N)A^{N-1} \end{pmatrix}, \tag{2}$$

we say that $\theta$ is observable when its observability matrix is of full rank. If we let $\rho(M)$ denote the rank of a matrix $M$, this condition can thus be written as

$$\rho(\mathcal{O}(\theta)) = n. \tag{3}$$

We now arrive at the definition of *pathwise observability*, which we recall from [4]:

*Definition 1 (Pathwise Observability [4]):* The set of pairs $\{(A, C(1)), \ldots, (A, C(s))\}$ is pathwise observable if and only if there exists an integer $N$ such that all paths of length $N$ are observable. We refer to the smallest such integer as the index of pathwise observability. $\Diamond$

In [4], it was shown that pathwise observability is decidable. In fact, it was shown that the indexes of pathwise observability are bounded by numbers $\mathcal{N}(s, n)$ depending only on $s$ and $n$, which is an even stronger result, since it suggests a direct way of deciding whether or not a set of pairs is pathwise observable, in checking the rank of the observability matrix of every path of length $\mathcal{N}(s, n)$. In this paper, we will give sufficient conditions for pathwise observability that allow us to come up with a switched version of the Kalman-Bertram criterion for non-pathological sampling.

Note that pathwise observability plays a crucial role in the observability of switched linear systems: Assuming the mode sequences $\{\theta_k\}_{k=1}^{\infty}$ to be known and arbitrary, (1) would be completely observable for any $\{\theta_k\}_{k=1}^{\infty}$ if and only if its set of pairs $\{(A, C(1)), \ldots, (A, C(s))\}$ was pathwise observable. In fact, in [3], an asymptotic observer was introduced for (1) requiring pathwise observability to converge.

The contribution of this paper is twofold, and concerns the establishment of sufficient conditions for pathwise observability and controllability. In Section 2, we establish sufficient conditions based on structural properties of the individual pairs, which is an interesting result in that it dispenses from computing coupled observability matrices (2) (i.e. matrices involving multiple modes), enabling the study of classical observability matrices of standard dimensions. In Section 3, we use that result to extend a classical result from linear systems theory concerning the conservation of observability properties when sampling a continuous-time system. In Section 4, we dualize these results to the controllability case.

## II. Sufficient Conditions for Pathwise Observability

In this section, we establish sufficient conditions on the individual pairs $(A, C(i))$ for the set $\{(A, C(1)), \ldots, (A, C(s))\}$ to be pathwise observable. More precisely, the idea is that if a pair $(A^b, C(i))$ is observable, $b \in \mathbb{N}$, then whenever $\theta_{a+bk} = i$ for $k = 0, \ldots, n-1$ and for some integer $a$, i.e. whenever some mode $i$ occurs $n$ times in $\theta$ at constant interval $b$, then $\mathcal{O}(\theta)$ will contain the following matrix as a submatrix:

$$\begin{pmatrix} C(i) \\ C(i)A^b \\ \vdots \\ C(i)(A^b)^{n-1} \end{pmatrix} A^{a-1}, \tag{4}$$

which has rank $n$ if $A$ is invertible, and therefore ensures that $\rho(\mathcal{O}(\theta)) = n$. Note that this would not be the case if there were switching among different $A$-matrices as well. In that case, the matrix in (4) would, in general, still exhibit coupling with modes other than $i$. What we thus want to show is that whenever a pair $(A^l, C(i))$ is observable for all modes $i$ and for all $l$ smaller than a certain number, then the system is pathwise observable. This implies the possibility to assert that, in every path of at least a certain length $\mathcal{W}$, some mode $i$ has to occur $n$ equally separated times. It turns out that proving the existence of such $\mathcal{W}$ is a problem to which an answer is provided by a branch of combinatorial analysis, referred to as *Ramsey theory* [5]. Indeed, we wish to capitalize on the fact that any mode sequence has to exhibit certain regularity properties as long as it is long enough, which is a type of statement that falls precisely under the domain of Ramsey theory, whose main assertion is that complete disorder is an impossibility and that the appearance of disorder is really a matter of scale. As it turns out, our question finds its answer in van der Waerden's Theorem [6] (in its finite version), which is one of the central results of Ramsey theory:

*Theorem 1 (van der Waerden [6]):* For every positive integers $n$ and $s$, there exists a minimal constant $\mathcal{W}(n, s)$ such that if $N \geq \mathcal{W}(n, s)$, and $\{1, \ldots, N\} \subset C_1 \cup \ldots \cup C_s$, then some set $C_i$ contains an arithmetic progression of length $n$. $\Diamond$

Here, an arithmetic progression is simply a string of positive integers such that the difference between successive terms is constant. It is indeed easy to see how the solution to our problem follows from Theorem 1 by simply taking every $C_i$ to be the set of times at which mode $i$ occurs in $\theta$. In other words, if we ignore the trivial case $n = 1$ and assume $n \geq 2$, which will be done throughout the remainder of this paper, we have:

*Corollary 1:* Let $\theta$ be a path assuming values in $\{1, \ldots, s\}$. If $|\theta| \geq \mathcal{W}(n, s)$, then there exist an integer $i \in \{1, \ldots, s\}$ and two positive integers $a \in \{1, \ldots, |\theta|\}$ and $b < |\theta|/(n-1)$ such that $\theta_{a+bk} = i$ for every $k = 0, \ldots, n-1$. $\Diamond$

*Proof:* Let $C_i = \{k \in \{1, \ldots, |\theta|\} \mid \theta_k = i\}$ for all $i \in \{1, \ldots, s\}$. Clearly, $\{1, \ldots, |\theta|\} \subset C_1 \cup \ldots \cup C_s$. By Theorem 1, since $|\theta| \geq \mathcal{W}(n, s)$, some $C_i$ contains an arithmetic progression of length $n$. In other words, there exist two positive integers $a$ and $b$ such that $a + bk \in C_i$, and therefore $\theta_{a+bk} = i$, for $k = 0, \ldots, n-1$. Finally, $b < |\theta|/(n-1)$ because $b(n-1) < a + b(n-1) \leq |\theta|$. $\square$

Before establishing the main result of this section, which is a direct consequence of Corollary 1, we define, for $n \geq 2$,

$$\mathcal{W}'(n, s) \triangleq \left\lceil \frac{\mathcal{W}(n, s)}{n-1} \right\rceil - 1, \tag{5}$$

where $\lceil \cdot \rceil$ denotes the ceiling function (i.e. $\lceil \alpha \rceil = \min\{i \in \mathbb{N} \mid \alpha \leq i\}$).

*Theorem 2:* If $A$ is invertible, and if $(A^l, C(i))$ is an observable pair for all $i \in \{1, \ldots, s\}$ and all positive integers $l \leq \mathcal{W}'(n, s)$, then $\{(A, C(1)), \ldots, (A, C(s))\}$ is pathwise observable with an index no larger than $\mathcal{W}(n, s)$. $\Diamond$

*Proof:* Let $\theta$ be any path of length $\mathcal{W}(n, s)$. By Corollary 1, there exist an integer $i \in \{1, \ldots, s\}$ and two integers $a \in \{1, \ldots, |\theta|\}$ and $b < \mathcal{W}(n, s) \backslash (n-1)$ such that $\theta_{a+bk} = i$ for $k = 0, \ldots, n-1$. Therefore, the submatrix of $\mathcal{O}(\theta)$ consisting of the rows $a + bk$ of $\mathcal{O}(\theta)$, $k = 0, \ldots, n-1$, can be expressed as

$$\begin{pmatrix} C(i) \\ C(i)A^b \\ \vdots \\ C(i)(A^b)^{n-1} \end{pmatrix} A^{a-1}.$$

This matrix has rank $n$ since $A$ (and therefore $A^{a-1}$) is invertible, and because the pair $(A^b, C(i))$ is observable, since $b \leq \mathcal{W}'(n, s)$. Therefore $\mathcal{O}(\theta)$ has rank $n$, which completes the proof. $\square$

*Remarks:*

- These conditions are *not* necessary. For instance, the set of pairs $\{(A, C(1)), (A, C(2))\}$, where:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{cases} C(1) = (1 \ 0) \\ C(2) = (2 \ 0) \end{cases} \tag{6}$$

  is pathwise observable with index 2, but while $\mathcal{W}'(2, 2) = 2$, neither $(A^2, C(1))$ nor $(A^2, C(2))$ is an observable pair.
- The index of pathwise observability in Theorem 2 is not necessarily equal to $\mathcal{W}(n, s)$. For instance, the set of pairs $\{(A, C(1)), (A, C(2))\}$, where:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{cases} C(1) = (1 \ 1) \\ C(2) = (1 \ 2) \end{cases} \tag{7}$$

satisfies the assumptions of Theorem 2, but is pathwise observable with index 2, while $\mathcal{W}(2,2) = 3$. $\Diamond$

The numbers $\mathcal{W}(n,s)$ are referred to as the van der Waerden (vdW) numbers. Unfortunately, the only vdW numbers known exactly fit in Table I (for the sake of easy reference, we also give, in Table II, the known values of $\mathcal{W}'(n,s)$). Only upper bounds are known for the rest.

| s \ n | 2 | 3 | 4 | 5 | $\cdots$ | $n$ |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | $\cdots$ | $n$ |
| 2 | 3 | 9 | 35 | 178 | | |
| 3 | 4 | 27 | | | | |
| 4 | 5 | 76 | | | | |
| $\vdots$ | $\vdots$ | | | | | |
| $s$ | $s+1$ | | | | | |

TABLE I

KNOWN VALUES OF $\mathcal{W}(n,s)$

| s \ n | 2 | 3 | 4 | 5 | $\cdots$ | $n$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | $\cdots$ | 1 |
| 2 | 2 | 4 | 11 | 88 | | |
| 3 | 3 | 13 | | | | |
| 4 | 4 | 37 | | | | |
| $\vdots$ | $\vdots$ | | | | | |
| $s$ | $s$ | | | | | |

TABLE II

KNOWN VALUES OF $\mathcal{W}'(n,s)$

Those bounds grow at an enormous rate, which limits the applicability of Theorem 2. In fact, research is currently ongoing for finding tighter bounds, e.g. [7], [8]. However, Theorem 2 is fortunately all we need in order to show the more practical results of the next section concerning sampled systems.

## III. SAMPLED-DATA SYSTEMS

A problem of relevance to digital control is the study of properties of sampled-data systems since most modern, digital controllers are implemented in discrete-time. In particular, it is usually desirable for a discretized system to conserve some properties of the continuous-time system, especially observability and controllability. We start, without loss of generality, by considering the following autonomous, continuous-time system:

$$\begin{aligned} \dot{x}_t &= Ax_t \\ y_t &= Cx_t, \end{aligned} \tag{8}$$

and the discrete-time system obtained by sampling (8) at constant interval $T$, which is referred to as the sampling period (for any continuous-time quantity $z_t$, we let $\bar{z}_k \triangleq z_{kT}$):

$$\begin{aligned} \bar{x}_{k+1} &= e^{AT}\bar{x}_k \\ \bar{y}_k &= C\bar{x}_k. \end{aligned} \tag{9}$$

In 1963, the following result was proved in [9]:

*Theorem 3 (Kalman-Bertram Criterion):* Let $\sigma(A)$ denote the spectrum of $A$. If $(A, C)$ is an observable pair, then whenever the sampling period $T$ satisfies, for all $\{\lambda, \lambda'\} \in \sigma(A) \times \sigma(A)$,

$$\lambda \neq \lambda' + \frac{jk}{T} \quad \forall\, k \in \mathbb{Z}\backslash\{0\}, \tag{10}$$

then the discrete-time pair $(e^{AT}, C)$ is observable. $\Diamond$

A proof can be found in [9], but the result easily follows from the Popov-Belevitch-Hautus rank test (see, e.g., [10]). Further research on this subject has focused mainly on generalized hold functions [11], [12] (for controllability) and on robust sampling techniques [13].

Our aim in this section is to extend Theorem 3 to switched linear systems. In other words, we focus our attention on the continuous-time switched linear system:

$$\begin{aligned} \dot{x}_t &= Ax_t \\ y_t &= C(\theta_t)x_t, \end{aligned} \tag{11}$$

where $\theta_t$ is an arbitrary function of time assuming values in the set $\{1, \ldots, s\}$, and on its discretized counterpart:

$$\begin{aligned} \bar{x}_{k+1} &= e^{AT}\bar{x}_k \\ \bar{y}_k &= C(\bar{\theta}_k)\bar{x}_k. \end{aligned} \tag{12}$$

Note that, even though $\theta_t$ is arbitrary and may switch between samples, (12) can be characterized by a finite set of pairs $\{(e^{AT}, C(1)), \ldots, (e^{AT}, C(s))\}$, which cannot be the case when the dynamics (i.e. the $A$ matrix) switches as well (unless, e.g., $\theta_t$ switches only at the sampling times). What we wish to establish here is whether observability of every pair $(A, C(i))$ implies pathwise observability of the set of pairs $\{(e^{AT}, C(1)), \ldots, (e^{AT}, C(s))\}$. Fortunately, the following theorem follows almost directly from Theorems 2 and 3:

*Theorem 4:* Let $\sigma(A)$ denote the spectrum of $A$. If $(A, C(i))$ is an observable pair for all $i \in \{1, \ldots, s\}$, then whenever the sampling period $T$ satisfies, for all $\{\lambda, \lambda'\} \in \sigma(A) \times \sigma(A)$,

$$\lambda \neq \lambda' + \frac{jk}{lT} \quad \forall\, k \in \mathbb{Z}\backslash\{0\},\ \forall\, l \leq \mathcal{W}'(n,s), \tag{13}$$

the set of pairs $\{(e^{AT}, C(1)), \ldots, (e^{AT}, C(s))\}$ of the discretized system is pathwise observable with an index no larger than $\mathcal{W}(n,s)$. $\Diamond$

*Proof:* First, since $AT$ commutes with itself and $l$ is an integer, $e^{AlT} = (e^{AT})^l$. Therefore, by Theorem 3, (13) implies that the pair $((e^{AT})^l, C(i))$ is observable for all $i \in \{1, \ldots, s\}$ and all $l \leq \mathcal{W}'(n,s)$. Moreover, $e^{AT}$ being a matrix exponential, it is an invertible matrix. The result then follows from Theorem 2. $\square$

Now, even though some numbers $\mathcal{W}(n,s)$ may be unknown, they are finite, as discussed earlier. The following corollary follows:

*Corollary 2:* If $(A, C(i))$ is an observable pair for all $i \in \{1, \ldots, s\}$, then the set of pairs $\{(e^{AT}, C(1)), \ldots, (e^{AT}, C(s))\}$ of the discretized system is pathwise observable for all but a countable number of sampling periods $T$. $\Diamond$

*Proof:* If every eigenvalue of $A$ is real, then (13) always holds and the set is pathwise observable for all $T > 0$. Otherwise, defining the set $F$ of frequencies as

$$F \triangleq \{|\mathrm{Im}(\lambda_i) - \mathrm{Im}(\lambda_j)| \mid \lambda_i \neq \lambda_l \in \sigma(A),\ \mathrm{Re}(\lambda_i) = \mathrm{Re}(\lambda_j)\}, \quad (14)$$

we get that the set of pathological sampling periods is, by (13), a subset of

$$\left\{ \frac{k}{fl},\ k \in \mathbb{N}^*,\ f \in F,\ l \leq \mathcal{W}'(n,s) \right\}, \quad (15)$$

which is countable. Hence the result. $\square$

Finally, note that what needs to be avoided in Theorem 3 is the interaction between the natural frequencies of the linear system and the sampling frequency. It is therefore easily established that, under the same conditions, conservation of observability is guaranteed when the sampling period $T$ is small enough. The importance of this observation is actually further motivated by robust control problems, as pointed out in [13]. The following theorem extends this result to switched linear systems (11):

*Theorem 5:* If $(A, C(i))$ is an observable pair for all $i \in \{1, \ldots, s\}$, then there exists a positive real number $T$ such that whenever $0 < t < T$, the set of pairs $\{(e^{At}, C(1)), \ldots, (e^{At}, C(s))\}$ of the discretized system is pathwise observable with an index smaller than or equal to $\mathcal{W}(n,s)$. $\Diamond$

*Proof:* Clearly,

$$T = \frac{1}{\max(F)\mathcal{W}'(n,s)}, \quad (16)$$

which is the smallest element of the set in (15), works. $\square$

The most surprising fact about Theorems 4 and 5 is that there is inherently no coupling between the $s$ different modes in continuous-time, and yet pathwise observability is shown to be achieved for the sampled-data system. Moreover, note that we make absolutely no assumption on $\theta_t$, other than that it is a mapping from the continuous time line to $\{1, \ldots, s\}$. In particular, $T$ in Theorem 5 is an upper bound on the sampling period, and *not* a lower bound on the switching intervals (or minimum *dwell time*).

## IV. Pathwise Controllability

Notice that the first results of this paper naturally carry over, by duality, to the study of switched systems of the form:

$$x_{k+1} = Ax_k + B(\theta_k)u_k, \quad (17)$$

where the modes $\theta_k$ assume values in $\{1, \ldots, s\}$, so that $B(\theta_k)$ switches among $s$ different input matrices $\{B(1), \ldots, B(s)\}$, and where one may be concerned with *pathwise controllability*, defined as pathwise observability of the set of dual pairs $\{((A', B(1)'), \ldots, (A', B(s)')\}$ [4]. In fact, one gets, as a trivial extension of Theorem 2:

*Theorem 6:* If $A$ is invertible, and if $(A^l, B(i))$ is a controllable pair for all $i \in \{1, \ldots, s\}$ and all integers $l \leq \mathcal{W}'(n,s)$, then $\{(A, B(1)), \ldots, (A, B(s))\}$ is pathwise controllable with an index no larger than $\mathcal{W}(n,s)$. $\Diamond$

However, one should be careful when considering the sampling problem from the controllability point of view. Indeed, applying a *zero-order hold* to

$$\dot{x}_t = Ax_t + B(\theta_t)u_t, \quad (18)$$

i.e. letting $u_t \triangleq \bar{u}_k\ \forall\ t \in [kT, (k+1)T)$, yields

$$\bar{x}_{k+1} = e^{AT}\bar{x}_k + B_k\bar{u}_k, \quad (19)$$

where $B_k = \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}B(\theta_\tau)d\tau$. Once again, $B_k$ might switch among an infinite number of values, unless, e.g., the signal $\theta_t$ is constrained to switch at only the sampling times. In fact, the dual of our criterion (Theorem 4) involves the use of a Dirac impulse-based discretization as follows:

$$u_t = \bar{u}_k\delta(t - kT), \qquad kT \leq t < (k+1)T, \quad (20)$$

which allows us to rewrite (19) as

$$\bar{x}_{k+1} = e^{AT}\bar{x}_k + B(\theta_k)\bar{u}_k, \quad (21)$$

to which we can then apply the previous results. Now, even though (20) does not make any sense since perfect impulses cannot be produced in practice, we can state the following purely theoretical result:

*Theorem 7:* Let $\sigma(A)$ denote the spectrum of $A$. If $(A, B(i))$ is a controllable pair for all $i \in \{1, \ldots, s\}$, then whenever the sampling period $T$ satisfies, for all $\{\lambda, \lambda'\} \in \sigma(A) \times \sigma(A)$,

$$\lambda \neq \lambda' + \frac{jk}{lT} \quad \forall\ k \in \mathbb{Z}\backslash\{0\},\ \forall\ l \leq \mathcal{W}'(n,s), \quad (22)$$

the set of pairs $\{(e^{AT}, B(1)), \ldots, (e^{AT}, B(s))\}$ of the discretized system (21) obtained by applying the hold function (20) to (18) is pathwise controllable with an index no larger than $\mathcal{W}(n,s)$. $\Diamond$

Finally, note that Corollary 2 also extends to the controllability case.

## V. CONCLUSIONS

In this paper, we have introduced an application of Ramsey Theory to the study of a property of switched linear systems (i.e. *pathwise observability*). The result presented has enabled, for the first time, the study of the conservation of observability and controllability properties after the introduction of sampling in switched systems, which has resulted in a criterion very similar to the well-known Kalman-Bertram criterion.

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