Controllability Decompositions of Networked Systems Through Quotient Graphs

Simone Martini, Magnus Egerstedt, and Antonio Bicchi

***Abstract*— In this paper we study decentralized, networked systems whose interaction dynamics are given by a nearest-neighbor averaging rule. By letting one node in the network take on the role of a leader in the sense that this node provides the control input to the entire system, we can ask questions concerning the controllability. In particular, we show that the controllable subspaces associated with such systems have a direct, graph theoretic interpretation in terms of so-called quotient graphs, providing us with a smaller, approximate bisimulation of the original network.**

***Index Terms*— Networked control systems, Network analysis and control, Communication networks**

I. Introduction

The emergence of decentralized, mobile multi-agent networks, such as distributed robots, mobile sensor networks, or mobile ad-hoc communications networks, has imposed new challenges when designing control algorithms. These challenges are due to the fact that the individual agents have limited computational, communications, sensing, and mobility resources. In particular, the information flow between nodes of the network must be taken into account explicitly already at the design phase, and a number of approaches have been proposed for addressing this problem, e.g. [1], [2], [3], [4], [5], [6], [7], [8].

Regardless of whether the information flow is generated over communication channels or through sensory inputs, the underlying geometry is playing an important role. For example, if an agent is equipped with omnidirectional range sensors, it can only detect neighboring agents if they are located in a disk around the agent. Similarly, if the sensor is a camera, the area becomes a wedge rather than a disk. But, to make the interaction geometry explicit when designing control laws is not an easy task, and an alternative view is to treat interactions as purely combinatorial. In other words, all that matters is whether or not an interaction exists between agents, and under certain assumptions on the global interaction topology, one can derive remarkably strong and elegant results. (For a representative sample, see [1], [6], [7].) What then remains to be shown is that the actual geometry in fact satisfies the combinatorial assumptions.

In this paper, we continue down this path, by investigating controllability from a graph-theoretic point-of-view, which was first proposed in [9], and later investigated in [10]. In [10], necessary conditions for controllability were given entirely in terms of the graph topology and, as such, it provides a starting point for the undertakings in this paper. In particular, we show that when the network is not completely controllable, the controllable subspace can be given a graph-theoretic interpretation. What this means is that it is possible to construct a smaller, completely controllable network (the so-called *controllable quotient graph*) that is equivalent to the original network in terms of controllable subspaces. This design allows the control designer to focus directly on a smaller network when producing control laws.

Moreover, it is shown that the dynamics associated with the uncontrollable part of the network is asymptotically stable for all connected networks. As such, the controllable quotient graph is an *approximate bisimulation* of the original network, in the sense of [11].

The outline of this paper is as follows: In Section II, we briefly review the basic premises behind leader-follower networks and recall some definitions from algebraic graph theory. In Section III, we review some results from [10], [9], allowing us to study controllability of single-leader networks from a graph-theoretic vantage-point. Quotient graphs, obtained through so-called *equitable partitions* of the graph, are the topic of Section IV, while the uncontrollable part of network is discussed in Section V. The main results of this paper are given in Section VI, where the Theorem 1 formalizes that the *quotient graph* represents a controllable model reduction of the original system. Finally, in Section VII simulations are shown to emphasize the relevance of the main Theorem.

II. Leader Follower Consensus Networks

In this section we start with some basic notions in graph theory. In multi-agents systems, it is common to let the nodes of a graph represent the agents, and to let the arcs in the graph represent the inter-agent communication links.

Let the undirected graph G be given by the pair $(V, \mathcal{E})$, where $V = \{1, \ldots, n\}$ is a set of n vertices, and $\mathcal{E}$ is a set of edges. We can associate the *adjacency matrix* $\mathcal{H} \in \mathcal{R}^{n\times n}$ with G, whose entries satisfy $h_{kj} = 1$ if $(j,k) \in \mathcal{E}$. Two nodes j and k are neighbors if $(j,k) \in \mathcal{E}$, and the set of the neighbors of the node j is defined as $N_j = \{k \,|\, [\mathcal{H}]_{jk} = 1\}$. The degree of a node is given by the number of its neighbors, and a graph G is *connected* if there is a path between any pair of distinct nodes, where a path $i_0 i_1 \ldots i_S$ is a finite sequence of nodes such that $i_{k-1} \in N_k$ with $k = 2, 3 \ldots S$.

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In this paper we let the state of each node, x_i, be scalar. (This does not affect the generality of the derived results.) The standard, consensus algorithm is the update law

$$\dot{x}_i(t) = \sum_{j \in N_i} (x_j(t) - x_i(t)), \tag{1}$$

or equivalently $\dot{x}(t) = -\mathcal{L}x(t)$, where $x(t)$ is the vector with the states of all nodes at time t, and $\mathcal{L}$ is the graph Laplacian. Let $\mathcal{D} \in \mathbb{R}^{n\times n}$ be the diagonal matrix of the degrees of the nodes, it is easy to verify that $\mathcal{L} = \mathcal{D} - \mathcal{H}$.

Under some connectivity conditions, the consensus algorithm is guaranteed to converge, i.e. $\lim_{t\to+\infty} x_i(t) = g$, $i \in \{1,\ldots,n\}$, where g is a constant depending on $\mathcal{L}$, and on the initial conditions $x_0 = x(0)$. (See for example [1], [12], [13].)

As in [7], [10], [14], we imagine that a subset of the agents have superior sensing, computation, or communication abilities. We thus partition the node set V into a leader set L of cardinality n_l, and a follower set F of cardinality n_f, so that $L \cap F = \emptyset$ and $L \cup F = V$. Leaders differ in their state update law in that they can arbitrarily update their positions, while the followers execute the agreement procedure (1), and are therefore controlled by the leaders.

Under the assumption that the first n_f agents are followers, and the last $n_l = n - n_f$ are leaders, the introduction of leaders in the network induces a partition in the the graph Laplacian $\mathcal{L}$ that becomes

$$\mathcal{L} = \left[\begin{array}{c|c} \mathcal{L}_f & \mathcal{L}_{fl} \\ \hline \mathcal{L}_{fl}^T & \mathcal{L}_l \end{array}\right],$$

with $\mathcal{L}_f \in \mathbb{R}^{n_f\times n_f}$, $\mathcal{L}_l =\in \mathbb{R}^{n_l\times n_l}$ and $\mathcal{L}_{fl} \in \mathbb{R}^{n_f\times n_l}$. Note that the subscripts f and l denote respectively the affiliation with the leaders and followers set.

The control system we now consider is the controlled agreement dynamics (or leader-follower system), in which followers evolve through the Laplacian-based dynamics

$$\begin{array}{l} \dot{x}_f(t) = -\mathcal{L}_f x_f(t) - \mathcal{L}_{fl} x_l(t) \\ x_l(t) = u(t), \end{array} \tag{2}$$

where x_f and x_l are respectively the state vectors of the followers and the leaders, and $u(t)$ denotes the exogenous control signal dictated by the leaders.

III. Controllability of Single-Leader Networks

In this section we recall some previous results of relevance to the developments in this paper. To conform to standard notation, we denote with $n = n_f$ the number of followers, we identify matrices A and B with $-\mathcal{L}_f \in \mathbb{R}^{n\times n}$ and $-\mathcal{L}_{fl} \in \mathbb{R}^{n\times 1}$ respectively, and we will equate x_f and x_l with x and u. Thus the system (2) becomes

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3}$$

with controllability matrix

$$C = \left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array}\right]. \tag{4}$$

As A is symmetric it can be written on the form $U\Lambda U^T$, where Λ is the diagonal matrix of eigenvalues of A and U is the unitary matrix comprised of its pairwise orthogonal unit eigenvectors. Since $B = UU^TB$, by factoring the matrix U from the left in (4), C assumes the form

$$C = U\left[\begin{array}{cccc} U^TB & \Lambda U^TB & \cdots & \Lambda^{n-1}U^TB \end{array}\right].$$

If one of the columns of U is perpendicular to all the columns of B, then C will have a row equal to zero and hence be rank deficient. On the other hand, in the case of one leader, if any two eigenvalues of A are equal, then C will have two linearly dependent rows, and again, the controllability matrix becomes rank deficient, as shown in [9]. Moreover, if the system is *leader symmetric* then there is a non-identity permutation J (matrix defined over the follower nodes F) such that $JA = AJ$ and in that case the system (3) is uncontrollable because one of the eigenvectors of A is also orthogonal to all columns of B, as shown in [15].

In this paper, we will focus on networks that are leader symmetric, restricted to the case when $n_l = 1$, i.e. when there only is one leader present. This is related to the necessity to use the most simple framework to control a set of followers. By moving only one super-node, the leader, we are able to control all the agents belonging to the network.

Definition 1 (LS^{2}L Network): A network is said to be LS^{2}L (Leader-Symmetric, Single Leader) if it leader-symmetric with a single leader.

In the following we give a graph-theoretic interpretation of the controllable part of a LS^{2}L network starting from the analysis of the controllable subspace.

IV. Equitable Partitions and Quotient Graphs

To obtain the controllable quotient graphs, the notion of an *equitable partition* is needed. (We refer the readers to [16] for more details about this subject.)

Definition 2 (Equitable Partition): A partition π of V, with cells $C_1, C_2, \ldots, C_r$ is said to be *equitable* if each node in C_i has the same number of neighbors in C_j $\forall i, j \in \{1,\ldots,r\}, i \neq j$, with $r = |\pi|$, which denotes the cardinality of the partition.

The directed graph with the r cells of π as its vertices and b_{ij} edges from the ith to the jth cells of π is called the *quotient graph*, and it is denoted by G/π. Moreover, a partition π with at least one cell with more than one node is said to be a *Nontrivial Equitable Partition* (NEP) and the adjacency matrix of this quotient is given by $\mathcal{H}(G/\pi)_{ij} = b_{ij}$.

Definition 3 (Characteristic Vector): A *characteristic vector* $p_i \in \mathbb{R}^{n\times 1}$ of a nontrivial cell C_i is defined as:

$$[p_i]_j = \begin{cases} 1 & \text{if } j \in C_i \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4 (Characteristic Matrix): A *characteristic matrix* $P \in \mathcal{R}^{n\times r}$ of a partition π of $V(G)$ is a matrix with the characteristic vectors of the cell as its columns.

Definition 5 (Leader-Invariant Equitable Partition (LEP)): By the leader-invariant equitable partition (LEP), we understand the maximal equitable partition $\pi_M = \pi_F \bigcup \pi_L$, where $\pi_F = \{C_1^M, C_2^M, \ldots, C_s^M\}$ is the maximal equitable partition of followers such that the cardinality of π_F is minimal (i.e. has the fewest cells), and the leader L belongs to the singleton cell $C_{s+1}^M = \{L\}$ of the partition $\pi_L = \{C_{s+1}^M\}$.

Remark 1: As shown in [15], if the network is LS^{2}L , then the LEP is nontrivial, i.e. not all cells are singletons.

V. Controllability Decomposition

We first recall the concepts of the Kalman decomposition for controllability. Considering the system (3) of a LS^{2}L network, we construct the controllability matrix (4) and, as previously discussed, we know that it is rank deficient. The controllability subspace, is equal to the range space of C, $(\mathcal{R}(C))$, and $rank(C)$ defines the dimension of this subspace.

Consider now any basis for this subspace. Let $d = rank(C)$ and let $(p_1, p_2, \dots, p_d)$ be the orthogonal, unit length vectors of this basis. We can now use these vectors to obtain the first d columns of the transformation matrix $\mathcal{T} = \left[\begin{array}{c|c|c|c|c} p_1 & p_2 & \dots & p_d & \dots \end{array}\right]$.
As $\mathcal{T}$ must be an $n \times n$ square matrix, we use then $n - d$ orthogonal, unit length vectors of the basis belonging to subspace $\mathcal{R}^\perp(C)$ to produce $\mathcal{T}$. Let $(p_{d+1}, p_{d+2}, \dots, p_n)$ be these vectors. $\mathcal{T}$ is non singular and produces the following system:

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \qquad \bar{x} = \mathcal{T}^{-1}x = \begin{bmatrix} \bar{x}_c \\ \bar{x}_{uc} \end{bmatrix}, \tag{5}$$

where

$$\bar{A} = \mathcal{T}^{-1}A\mathcal{T} = \begin{bmatrix} \bar{A}_c & 0 \\ 0 & \bar{A}_{uc} \end{bmatrix} \quad \bar{B} = \mathcal{T}^{-1}B = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}. \tag{6}$$

Here the subscripts c and uc refer to the controllable and uncontrollable parts respectively.

The reason why $\bar{A}$ in the decomposition (6) takes on this form, i.e. that $\bar{A}$ is block diagonal, follows directly from the fact that $A = A^T$ and $\mathcal{T}$ is orthonormal, i.e. $\bar{A}^T = (\mathcal{T}^{-1}\ A\ \mathcal{T})^T = \mathcal{T}^{-1}A^T\mathcal{T} = \mathcal{T}^{-1}A\mathcal{T} = \bar{A}$.

As a result, we can decouple the system into two different subsystems, namely

$$\dot{\bar{x}}_c = \bar{A}_c\bar{x}_c + \bar{B}_c u \tag{7}$$

for the controllable part of the network, and

$$\dot{\bar{x}}_{uc} = \bar{A}_{uc}\bar{x}_{uc} \tag{8}$$

for the uncontrollable part.

Proposition 1: Let G be a single leader network with dynamics described by (3). Its uncontrollable subsystem (8) is always asymptotically stable, i.e

$$\lim_{t\to\infty} \bar{x}_{uc}(t) = 0.$$

Proof: Since we apply a similarity transformation $\mathcal{T}$ to A, this doesn't change its eigenvalues. So we need to prove that A is negative definite, which follows from the fact that $\mathcal{L}_f$ is positive definite, as shown in [14]. ■

Proposition 2 (Range space of C): Let G be a LS^{2}L network with dynamics described by (3), and let π_M be its LEP. The range space of C corresponds to the spanning set of the characteristic vectors of π_F, i.e.

$$\mathcal{R}(C) = \text{span}\left\{ \begin{bmatrix} E_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ E_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_s \end{bmatrix} \right\}$$

where E_i is a column vector of ones, with $r_i = |C_i^M|$ components.

Proof: We denote by r_i the cardinality of each set C_i^M of the partition π_F of G, and we now consider the graph G' in which the first r_1 vertices belong to C_1^M, the second r_2 vertices belong to C_2^M, and so on. Let $\mathcal{L}'$ be the graph Laplacian of G' and $P(G/\pi_F) \in \mathbb{R}^{n\times r}$ be the characteristic matrix of G/π_F. Recalling Definition 4, in this case we have

$$P(G/\pi_F) = \begin{bmatrix} E_1 & 0 & & 0 \\ 0 & E_2 & & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & E_s \end{bmatrix} \tag{9}$$

where $E_i \in \mathbb{R}^{r_i\times 1}$ is a vector with ones in each position. Now, since $A' = -\mathcal{L}'_f$ is symmetric it can be rewritten as a block matrix:

$$A' = \begin{bmatrix} A'_{11} & A'_{12} & \cdots & A'_{1,s} \\ A'_{21} & A'_{22} & \cdots & A'_{2,s} \\ \vdots & \ddots & & \vdots \\ A'_{s,1} & \dots & \dots & A'_{s,s} \end{bmatrix}, \tag{10}$$

where each diagonal submatrix $A'_{ii} \in \mathbb{R}^{r_i\times r_i}$ represents the set C_i^M, and each other submatrix $A_{ij} \in \mathbb{R}^{r_i\times r_j}$ represents the connections between nodes belonging to set C_i^M and C_j^M. From Definition 2, for each submatrix A'_{ij}, we have

$$\sum_{k=1}^{r_j} a_{i^\star k} = \sum_{k=1}^{r_j} a_{j^\star k} \qquad \forall\ i^\star, j^\star \in C_i^M. \tag{11}$$

Moreover, $B' = -\mathcal{L}'_{fl}$ has always the form

$$B' = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E \end{bmatrix} \tag{12}$$

with E column vectors of ones with l elements, where l denotes the number of the neighbors of the leader. The controllability matrix can thus be recursively calculated as

$$C = \begin{bmatrix} B' & A'\cdot B' & A'\cdot A'B' & \cdots & A'\cdot A'^{(n-2)}B' \end{bmatrix},$$

and, since A' is as in (10) with condition (11) and B' is as in (12), it becomes

$$C = \begin{bmatrix} 0 & \cdots & \widehat{C}_{1n} \\ 0 & \ddots & \vdots \\ E & \cdots & \widehat{C}_{sn} \end{bmatrix} \qquad \begin{array}{l} \widehat{C}_{ij} = f_{ij}E_i \\ f_{ij} \in \mathbb{R} \end{array} \tag{13}$$

So the range space of C (13) is such that

$$\mathcal{R}(C) = \text{span}\left\{ \begin{bmatrix} E_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ E_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_s \end{bmatrix} \right\}$$

which corresponds to the spanning set of the characteristic vectors of π_F, which proves the proposition. ■

Corollary 1: The dimension of the controllable subspace of the network G is equal to the cardinality of π_F of its LEP.

Proof: The range space of C is equal to the spanning set of the characteristic vectors of π_F. It follows that $dim(\mathcal{R}(C))$, is equal to the number of the columns of its characteristic matrix, i.e. the number of sets of π_F. ■

Corollary 2: Agents of the network belonging to each set C_i^M of π_F starting from the same point will move together, i.e. $\forall\ t > 0$,

$$\left.\begin{matrix} x_1(0) = \cdots = x_{r_1}(0) \\ \vdots \\ x_{n-r_s}(0) = \cdots = x_n(0) \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x_1(t) = \cdots = x_{r_1}(t) \\ \vdots \\ x_{n-r_s}(t) = \cdots = x_n(t) \end{matrix}\right.$$

Proof: A possibile choice for $\mathcal{R}(C)^\perp$ is to take vectors with column sums to zero and with blocks $P_i \in \mathbb{R}^{r_i \times (r_i - 1)}$ in the position associate to each block E_i of $\mathcal{R}(C)$, such that

$$P_i = \begin{bmatrix} I_{r_i - 1} \\ -\mathbf{1}^T \end{bmatrix}_{r_i \times (r_i - 1)}.$$

In other words we have that $\mathcal{R}(C)^\perp = \bigcup_{i=1}^s R_i$ where

$$R_1 = \text{span}\left\{ \begin{bmatrix} E_{11}^\perp \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} E_{12}^\perp \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} E_{1r_1-1}^\perp \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

$$R_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ E_{21}^\perp \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ E_{22}^\perp \\ \vdots \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 \\ E_{1r_2-1}^\perp \\ \vdots \\ 0 \end{bmatrix} \right\}$$

and so on, where

$$E_{i1}^\perp = \begin{bmatrix} P \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad E_{i2}^\perp = \begin{bmatrix} 0 \\ P \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots$$

$$\text{with} \quad P \in \mathbb{R}^{2\times 1} \quad \text{s.t.} \quad P = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

It follows that for every block E_i, i.e. for every set C_i^M, we have, $\forall t > 0$

$$\left.\begin{matrix} x_1(0) = \cdots = x_{r_1}(0) \\ \vdots \\ x_{n-r_s}(0) = \cdots = x_n(0) \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x_1(t) = \cdots = x_{r_1}(t) \\ \vdots \\ x_{n-r_s}(t) = \cdots = x_n(t) \end{matrix}\right.$$

which proves the corollary. ■

VI. Approximate Bisimulation Through Equitable Partition Graph

A common theme in the theory of distributed processes and in systems and control theory is to characterize systems which are "externally equivalent". The intuitive idea is that we only want to distinguish between two systems if the distinction can be detected by an external system interacting with these systems. This is a fundamental notion in design, allowing us to switch between externally equivalent representations of the same system and to reduce subsystems to externally equivalent but simpler subsystems.

A crucial notion in this sense is the concept of bisimulation. The notion of bisimulation, introduced in [17], and which has been further developed for example in [18], [19], [20], is one such formal notion of abstraction that has been used for reducing the complexity of finite state systems and expresses when a subprocess can be considered to be externally equivalent to another (hopefully simpler) process.

Bisimulation is a concept of equivalence that has become a useful tool in the analysis of concurrent processes. It also reflects classical notions in systems and control theory such that state - space equivalence of dynamical systems, and especially the reduction of a dynamical system to an equivalent system with minimal state - space dimension.

In the following we apply concepts of approximate bisimulations to multi agent systems. We aim to find a subgraph of the original graph that we can use to move all the agents belonging to the network, and we aim to give a graphic and immediate interpretation to this one using equitable partitions. Indeed, since the uncontrollable part of the system is always asymptotically stable, we can simplify the original network with one which corresponds exactly to the controllable part of the network. In order to move all the agents of the network, it is possible to control this smaller entity and ignoring the uncontrollable part. Moreover, we will prove that this controllable subgraph can be found by investigating the network through equitable partitions.

Consider the controllability decomposition (5,6) with

$$\mathcal{T} = \left[\ \mathcal{T}_c \ \middle|\ \mathcal{T}_{uc}\ \right] = \left[\ \mathcal{T}_c^1 \quad \mathcal{T}_c^2 \quad \cdots \quad \mathcal{T}_c^s \ \middle|\ \mathcal{T}_{uc}\ \right], \tag{14}$$

$$\mathcal{T}^{-1} = \left[\begin{array}{c} \mathcal{T}\text{inv}_c \\ \hline \mathcal{T}\text{inv}_{uc} \end{array}\right] = \left[\begin{array}{c} \mathcal{T}^1\text{inv}_c \\ \vdots \\ \mathcal{T}^s\text{inv}_c \\ \hline \mathcal{T}\text{inv}_{uc} \end{array}\right], \tag{15}$$

where $\mathcal{T}_c$ denote the first $s = dim(\mathcal{R}(C))$ columns of $\mathcal{T}$, and $\mathcal{T}\text{inv}_c$ the first s rows of $\mathcal{T}^{-1}$.
Therefore

$$\bar{A}_c = \mathcal{T}\text{inv}_c\ A\ \mathcal{T}_c, \qquad \bar{B}_c = \mathcal{T}\text{inv}_c\ B, \tag{16}$$

which allows us to state the following lemma.

Lemma 1: Let G be a LS^2L network, with dynamics described by (3), and let π_M be its LEP. $\mathcal{T}\text{inv}_c$ (15) is such that

$$\mathcal{T}^i\text{inv}_c = \frac{(\mathcal{T}_c^i)^T}{|C_i^M|}. \tag{17}$$

Proof: $\mathcal{T}^{-1} = \mathcal{T}\text{inv} = (\mathcal{T}^T\ \mathcal{T})^{-1}\ \mathcal{T}^T$ with $\mathcal{T} = \left[\ \mathcal{T}_c \mid \mathcal{T}_{uc}\ \right]$ and, as we proved in Proposition 2, $\mathcal{T}_c$ correspond to the characteristic matrix of π_F. Since $\mathcal{T}_c$ and $\mathcal{T}_{uc}$ are orthonormal, the matrix $\mathcal{T}^\star = (\mathcal{T}^T\ \mathcal{T})$ is such that:

$$\mathcal{T}^\star = \left[\begin{array}{c|c} \mathcal{T}_c^T\ \mathcal{T}_c & 0 \\ \hline 0 & \mathcal{T}_{uc}^T\ \mathcal{T}_{uc} \end{array}\right] = \left[\begin{array}{c|c} \mathcal{T}_{11}^\star & 0 \\ \hline 0 & \mathcal{T}_{22}^\star \end{array}\right],$$

where $\mathcal{T}_{11}^\star \in \mathbb{R}^{s\times s}$ is a diagonal matrix s.t. $[\mathcal{T}_{11}^\star]_{ii} = |C_i^M|$ as shown in [16], and $\mathcal{T}_{22}^\star \in \mathbb{R}^{(n-s)\times(n-s)}$.
$\mathcal{T}^\star$ is a diagonal block matrix and, its inverse can be easily evaluated:

$$(\mathcal{T}^\star)^{-1} = \left[\begin{array}{c|c} (\mathcal{T}_c^T\ \mathcal{T}_c)^{-1} & 0 \\ \hline 0 & (\mathcal{T}_{uc}^T\ \mathcal{T}_{uc})^{-1} \end{array}\right],$$

i.e.

$$(\mathcal{T}^\star)^{-1} = \left[\begin{array}{ccc|ccc} \frac{1}{|C_1^M|} & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{|C_s^M|} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & (\mathcal{T}_{uc}^T\ \mathcal{T}_{uc})^{-1} & \end{array}\right].$$

It follows that

$$\mathcal{T}^{-1} = (\mathcal{T}^\star)^{-1} \left[\begin{array}{c} \mathcal{T}_c^T \\ \hline \mathcal{T}_{uc}^T \end{array}\right] = \left[\begin{array}{c} \frac{(\mathcal{T}_c^1)^T}{|C_1^M|} \\ \vdots \\ \frac{(\mathcal{T}_c^s)^T}{|C_s^M|} \\ \hline \mathcal{T}\text{inv}_{uc} \end{array}\right]. \quad (18)$$

Hence

$$\mathcal{T}^i\text{inv}_c = \frac{(\mathcal{T}_c^i)^T}{|C_i^M|}, \quad (19)$$

which proves the lemma. ■

Theorem 1 (Controllable Subspace): Let G be a LS^{2}L network with dynamics (3), and let π_M be its LEP. The controllable subspace of G corresponds to the quotient graph G/π_M.

Proof: It is well known that $\mathcal{L} \in \mathcal{R}^{(n+1)\times(n+1)}$ is such that $\mathcal{L} = \mathcal{D} - \mathcal{H}$, where $\mathcal{D}$ is the diagonal degree matrix and $\mathcal{H}$ is the adjacency matrix. Hence $A = -\mathcal{L}_f = -(\mathcal{D}_f - \mathcal{H}_f)$, where $\mathcal{D}_f$ and $\mathcal{H}_f$ are respectively obtained by taking the first n rows and columns of $\mathcal{D}$ and $\mathcal{H}$. We have

$$\bar{A} = -(\bar{\mathcal{D}}_f - \bar{\mathcal{H}}_f) \quad \text{where} \quad \begin{cases} \bar{\mathcal{D}}_f = \mathcal{T}^{-1}\mathcal{D}_f\ \mathcal{T} \\ \bar{\mathcal{H}}_f = \mathcal{T}^{-1}\mathcal{H}_f\ \mathcal{T} \end{cases}$$

and the matrix $\bar{A}_c$ in (16) can be calculated as

$$\bar{A}_c = -(\bar{\mathcal{D}}_{fc} - \bar{\mathcal{H}}_{fc}), \quad (20)$$

where

$$\bar{\mathcal{D}}_{fc} = \mathcal{T}\text{inv}_c\mathcal{D}_f\ \mathcal{T}_c \quad (21)$$

$$\bar{\mathcal{H}}_{fc} = \mathcal{T}\text{inv}_c\mathcal{H}_f\ \mathcal{T}_c. \quad (22)$$

Since $\mathcal{T}_c$ is equal to the characteristic matrix of π_F, and $\mathcal{T}\text{inv}_c = (\mathcal{T}_c^T\mathcal{T}_c)^{-1}\mathcal{T}_c^T$, (22) corresponds to the adjacency matrix of the quotient graph $\mathcal{H}_f(G/\pi_F)$ (Lemma 9.3.1 in [16]). Moreover, since the degree of nodes belonging to the same set C_i^M is equal, and $\mathcal{T}\text{inv}_c$ satisfies (17), $\bar{\mathcal{D}}_{fc}$ is a diagonal matrix whose diagonal entries represent the degree of nodes belonging to each set C_i^M. It follows that $\bar{A}_c$ in (20) corresponds to $-\mathcal{L}_f(G/\pi_M)$.

Furthermore, with B as in (12), the decomposition (16) is such that each entry $\bar{b}_i$ of the matrix $\bar{B}_c$ satisfies:

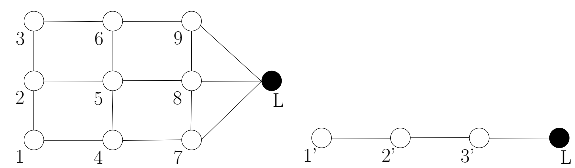
$$\bar{b}_i = \frac{\sum_{k=r_i}^{r_i+r_{(i+1)}} b_i}{|C_i^M|} \quad \text{i.e.} \quad \bar{b}_i = \begin{cases} 1 & \text{if } C_i^M \text{ is connected} \\ & \text{to the leader} \\ 0 & \text{otherwise.} \end{cases}$$

If we define X as the number of sets $|C_i^M|$ connected with the leader, we can conclude that the matrix

$$\left[\begin{array}{c|c} -\bar{A}_c & -\bar{B}_c \\ \hline -\bar{B}_c^T & X \end{array}\right] \quad (23)$$

corresponds exactly to $\mathcal{L}(G/\pi_M)$, which proves the theorem. ■

VII. A Simulation Study



(a) (b)

Fig. 1. The graph of the network and the quotient graph corresponding to controllable part (b).

As an application of the proposed method, consider a network consisting of 9 followers and one leader. As usual, leaders and followers differ in that leaders move autonomously and "herd" the followers, which move using the consensus protocol. Assume moreover that the followers are layed out in a grid, as in Fig. 1(a). Since such structure is a LS^{2}L network, it is not completely controllable, and for this reason we cannot move it from any initial point to any arbitrarily point.

Consider now a translation of the network: due to the fact that the system is not completely controllable, this movement is not feasible. In Fig. 2(a), 2(b), 2(c), we report some steps of a translation process of the entire network, and in Fig. 2(d), 2(e), 2(f), we report the same steps of the same translation, but applied to the quotient graph shown in Fig. 1(b).

We suppose that an external unit tells the leader the trajectory to follow, or that the leader has planning capabilities in order to solve the planning problem. Starting from the initial situation of Fig. 2(a), leader moves along x axis dragging followers, whose disposition (Fig. 2(b), 2(c)) asymptotically converge to the controllable quotient graph (Fig. 2(e), 2(f)).

This result emphasize the importance of a graph theoretic characterization of the controllable part of the network, which enables the designer to fucus directly on the smaller, approximate bisimulation of the original graph, when designing control laws.

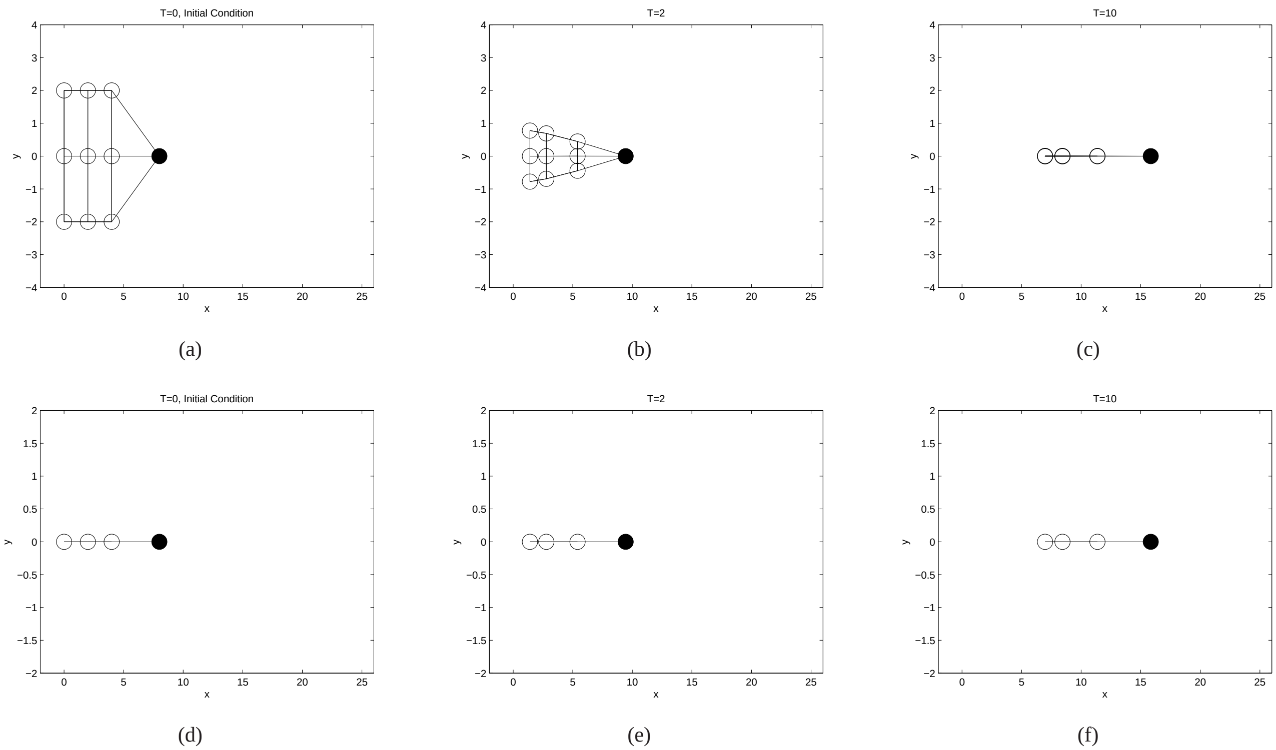


Fig. 2. Translation process of the entire network (a, b, c), and of its quotient graph (d, e, f).

VIII. Conclusions

The problem of controllability of a group of autonomous agents has been considered. A leader-follower linear consensus network has been used to model the interactions among the nodes. It has been shown that when the network is not completely controllable, we can give a graphic theoretic interpretation to the controllability subspace, and that it is possible to construct a smaller completely controllable network that is controllable-equivalent to the original one.

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