Periodic Smoothing Splines

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Abstract

Periodic smoothing splines appear for example as generators of closed, planar curves, and in this paper they are constructed using a controlled two point boundary value problem in order to generate the desired spline function. The procedure is based on minimum norm problems in Hilbert spaces and a suitable Hilbert space is defined together with a corresponding linear affine variety that captures the constraints. The optimization is then reduced to the computationally stable problem of finding the point in the constraint variety closest to the data points.

Key words: spline smoothing; periodic; two-point boundary value problem; constrained optimization

1 Introduction

In this paper we consider the problem of constructing periodic smoothing splines. For interpolating cubic splines there are standard numerical procedures that are quite effective. However the problem of periodic smoothing splines is more general and requires additional machinery. The need for such splines arises whenever there is a need to construct closed curves in the plane [3,9].

We show how this problem can be addressed as an optimal control problem, whose solution is the so-called generalized splines, i.e. belonging to a rich set of splines that include polynomial, trigonometric, and exponential splines [7,11]. It should be noted that the problem of constructing periodic splines has been previously addressed, for example in [1,10,11], and some routines can be found in a spline library [1]. However, apart from the fact that previous results mainly focused on polynomial splines, the solution methods tend to rely heavily on the particular spline-type under investigation in that a unified framework was missing in which a large class of splines could be systematically produced. The view taken in this paper is that the framework of linear systems theory provides the tool needed to address the problem of generalized, smoothing, periodic splines in a unified, systematic, and numerically stable manner.

In this paper we give a very general construction of periodic splines based on Hilbert space methods developed by Martin and collaborators in a series of papers, [6,8,12,14,13]. We use a specific technique developed in [12]. That is, we use the dynamics of a controlled two point boundary value problem to generate the spline curve, given by

$$\dot{x} = Ax + bu, \quad y = cx, \quad x(0) - x(T) = 0$$

with $x \in \mathbb{R}^n$, and $u, y \in \mathbb{R}$, and with the output y defining the spline curve. We assume that the system is controllable and observable. Because we are interested in

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[1] www.physics.lsa.umich.edu/akerlof/Spline/Spline/spline.doc

so-called maximal smoothing (maximal degree of continuity of the derivatives at the interpolation points) we will assume that

$$cb = cAb = \cdots = cA^{n-2}b = 0, \tag{1}$$

or, without loss of generality, we assume that A and b are in control canonical form and $c = (1, 0, \cdots, 0)$.

Moreover, the particular structure assumed on the matrices A, b, c is not necessary, strictly speaking. We assume this structure in order to achieve maximal smoothing, but it should be pointed out already at this point that the framework developed in this paper holds for any minimal (i.e. completely controllable and observable) linear control system with scalar inputs and outputs.

As a final note regarding the system under consideration, it should be stressed that this system is to be thought of as a generator of spline curves. We are not investigating the issue of trajectory planning *per se*. Rather, the tool developed in this paper draws its motivation from the fields of data analysis and statistics, even though the results will provide some new insights into the area of endpoint-constrained, linear optimal control as an added benefit.

2 The Periodic Splines Problems

We are given a controllable linear system $\dot{x} = Ax + bu,\ y = cx$ and a data set $D = \{(t_i, \alpha_i) :\ i = 1, \cdots, N\}$, where we let t_N be equal to the final time T. Our goal is to find a square-integrable control signal $u \in L_2[0, T]$ such that $|y(t_i) - \alpha_i|$ is small for all $i = 1, \ldots, N$, and we require that $y(0) = y(T),\ \dot{y}(0) = \dot{y}(T), \ldots$, up to a suitable order, i.e. we want the spline curve to be smoothly periodic. An easy way to enforce this is to require that $x(0) = x(T)$.

There are essentially two ways in which this problem can be formulated, and in order to see this, we first need to establish some notation. Let

$$f_i(s) = \begin{cases} ce^{A(t_i - s)}b & t_i - s > 0 \\ 0 & t_i - s \leq 0, \end{cases}$$

and let

$$\hat{y} = (y(t_1), \cdots, y(t_N))',\ \hat{\alpha} = (\alpha_1, \cdots, \alpha_N)',$$

$$\beta_i = R^{-1}e^{A' t_i}c',$$

given a positive definite matrix R. In this paper, we use $(\ )'$ to denote transpose, and with this notation, as well as $y_i = y(t_i)$, we have

$$y_i = \langle \beta_i, x_0 \rangle_R + \langle f_i, u \rangle_L,\ i = 1, \ldots, N, \tag{2}$$

where $x(0) = x_0$, and where the inner products are given by $\langle \beta_i, x_0 \rangle_R = \beta_i' R x_0$ and $\langle f_i, u \rangle_L = \int_0^T f_i(t)u(t)dt$.

Problem 1: Let

$$J(u, x_0) = \int_0^T u^2(t)dt + x_0' R x_0 + (\hat{y} - \hat{\alpha})' Q (\hat{y} - \hat{\alpha})$$

where Q and R are positive definite matrices. The problem we are interested in is

$$\min_{u, x_0} J(u, x_0),$$

subject to the constraint $x(0) = x(T)$ as well as the dynamics in Equation (2).

Problem 2: Let

$$J(u, x_0) = \int_0^T u^2(t)dt + (\hat{y} - \hat{\alpha})' Q (\hat{y} - \hat{\alpha}),$$

where Q is a positive definite matrix. The problem is

$$\min_{u, x_0} J(u, x_0),$$

subject to the constraint $x(0) = x(T)$ and the dynamics in Equation (2).

The difference between Problems 1 and 2 is that there is an extra cost associated with the free initial x_0 in Problem 1, through the term $x_0' R x_0$. As we will see, this difference will make Problem 1 easier to solve than Problem 2. The basic idea of the construction of solutions is to define a linear variety $\mathcal{V}$ in a Hilbert space that contains all of the constraints. As we will see in the next section, it is possible to interpret the data as a point (p) in the Hilbert space, which reduces the problem to that of finding the point on the linear variety that is closest (in the sense of the norm in the Hilbert space) to the data point. We know that we can construct this point by finding the orthogonal complement of the linear variety that defines the affine variety and constructing the intersection of the affine variety with the orthogonal complement. In other words, "all" we need to do in order to solve the problem is to compute $\mathcal{V} \cap (\mathcal{V}^\perp + p)$. In this process we follow Luenberger, [4].

Before constructing this intersection two things must be verified. The first is that the linear variety $\mathcal{V}$ is nonempty and the second is that it is closed. Both follow from the fact that $\mathcal{V}$ is the graph of a continuous mapping from an appropriate product space to $\mathbb{R}^N$.

3 Problem 1

We begin by considering the periodic spline as a particular boundary value problem, and for this we will use the methods of [12]. Let the boundary condition be given by

$$x(0) - x(T) = 0. \tag{3}$$

We note that since

$$x(T) = e^{AT}x(0) + \int_0^T e^{A(T-s)}bu(s)ds,$$

the specific dependence on $x(T)$ can be removed and the boundary constraint simply becomes

$$(I - e^{AT})x(0) - \int_0^T e^{A(T-s)}bu(s)ds = 0. \tag{4}$$

We now define the following Hilbert space

$$\mathcal{H} = L_2[0,T] \times \mathbb{R}^n \times \mathbb{R}^N$$

with norm

$$\|(u;x_0;y)\|^2 = \int_0^T u^2(t)dt + x_0'Rx_0 + y'Qy.$$

We define the linear constraint variety, $\mathcal{V} \subset \mathcal{H}$, to be

$$\mathcal{V} = \{(u;x;d) : \ d_i = \langle \beta_i, x\rangle_R + \langle f_i, u\rangle_L, \quad (I - e^{AT})x - \int_0^T e^{A(T-s)}bu(s)ds = 0\}.$$

We first note that $\mathcal{V}$ is a closed subspace of $\mathcal{H}$ since it is the graph of a continuous function restricted to a closed linear variety, and the closed graph theorem applies.

We now construct the orthogonal complement $\mathcal{V}^\perp$. And, from the definition it directly follows that $\mathcal{V}^\perp = \left\{(\tilde{u};\tilde{x};\tilde{d}) : \ \forall (u;x;d) \in \mathcal{V} \ \langle \tilde{u}, u\rangle_L + \langle \tilde{x}, x\rangle_R + \langle \tilde{d}, d\rangle_Q = 0)\right\}$, which allows us to state the following lemma:

Lemma 1

$$\mathcal{V}^\perp = \left\{(\tilde{u};\tilde{x};\tilde{d}) : \tilde{x} = -\sum_{i=1}^N \langle \tilde{d}, e_i\rangle_Q \beta_i + R^{-1}(I - e^{A'T})\lambda, \quad \tilde{u} = -\sum_{i=1}^N \langle \tilde{d}, e_i\rangle_Q f_i - b'e^{A'(T-t)}\lambda\right\}$$

for all $\lambda \in \mathbb{R}^n$, where e_i is the unit vector with zeros everywhere except for the i:th position.

Proof: From the expression of the orthogonal complement above, we directly get that

$$\mathcal{V}^\perp = \{(\tilde{u};\tilde{x};\tilde{d}) : \ \langle \tilde{x} + \sum_{i=1}^N \langle \tilde{d}, e_i\rangle_Q \beta_i, x\rangle + \langle \tilde{u} + \sum_{i=1}^N \langle \tilde{d}, e_i\rangle_Q f_i, u\rangle = 0\},$$

following the construction in [12].

Now, the relationship does not hold for all x and u but only for those x and u for which Equation (4) holds. Multiplying by any $\lambda \in \mathbb{R}^n$ (and its transpose λ'), we can rewrite Equation (4) as

$$\langle R^{-1}(I - e^{AT})'\lambda, x\rangle_R + \langle (-e^{A(T-t)}b)'\lambda, u\rangle_L = 0. \tag{5}$$

From this we conclude that

$$\tilde{x} + \sum_{i=1}^N \langle \tilde{d}, e_i\rangle_Q \beta_i = R^{-1}(I - e^{AT})'\lambda$$

and

$$\tilde{u} + \sum_{i=1}^N \langle \tilde{d}, e_i\rangle_Q f_i = (-b'e^{A'(T-t)})\lambda,$$

and the lemma follows. □

It remains to construct the intersection $\mathcal{V} \cap (\mathcal{V}^\perp + p)$ to find the optimal point, where $p = (0, 0, \hat{\alpha}) \in \mathcal{H}$. This construction is technically more complicated than for the smoothing spline without periodicity constraints, as reported in [12].

The unique point in the intersection is defined as the solution of the following system of four equations in the unknowns u, x_0, y and λ, obtained by identifying x and $\tilde{x}$ with x_0, d with $\hat{y}$, and $\tilde{d}$ with $\hat{y} - \hat{\alpha}$.

$$u = -\sum_{i=1}^N \langle \hat{y} - \hat{\alpha}, e_i\rangle_Q f_i - b'e^{A'(T-s)}\lambda \tag{6}$$

$$x_0 = -\sum_{i=1}^N \langle \hat{y} - \hat{\alpha}, e_i\rangle_Q \beta_i + R^{-1}(I - e^{AT})'\lambda \tag{7}$$

$$0 = (I - e^{AT})x_0 - \int_0^T e^{A(T-s)}bu(s)ds \tag{8}$$

$$y_i = \langle \beta_i, x_0 \rangle_R + \langle f_i, u \rangle_L \tag{9}$$

We begin by eliminating x_0 and u from Equation (9) by substituting Equations (6) and (7). After some manipulation we have

$$y_i = -e_i' GQ(\hat{y} - \hat{\alpha}) - e_i' FQ(\hat{y} - \hat{\alpha}) + \beta_i'(I - e^{AT})'\lambda - \int_0^T f_i(s)b'e^{A'(T-s)}ds\lambda,$$

where $G = \beta' R \beta$ with $\beta = (\beta_1\ \beta_2\ \cdots\ \beta_N)$, and F is the Gramian

$$F = \int_0^T f(s)f'(s)ds \tag{10}$$

with $f(s) = (f_1(s)\ f_2(s)\ \cdots\ f_N(s))'$. Note that since the f_i's are linearly independent, F is invertible. Moreover, since $\beta_i = R^{-1}e^{A't_i}c'$ we can let

$$\beta = R^{-1}(e^{A't_1}c', \cdots, e^{A't_N}c') = R^{-1}E,$$

for the appropriate E, in order to obtain

$$\hat{y} = -GQ(\hat{y} - \hat{\alpha}) - FQ(\hat{y} - \hat{\alpha}) + E'R^{-1}(I - e^{AT})'\lambda + \Lambda\lambda, \tag{11}$$

where

$$\Lambda = -\int_0^T f(s)b'e^{A'(T-s)}ds.$$

We will now use Equation (8) to obtain a second equation in λ and $\hat{y}$. Substituting u in Equation (6) and x_0 in (7) into (8), we have

$$0 = (I - e^{AT})\Big[-\sum_{i=1}^N \langle \hat{y} - \hat{\alpha}, e_i \rangle_Q \beta_i + R^{-1}(I - e^{AT})'\lambda\Big] - \int_0^T e^{A(T-s)}b\Big[-\sum_{i=1}^N \langle \hat{y} - \hat{\alpha}, e_i \rangle_Q f_i - b'e^{A'(T-s)}\lambda\Big]ds.$$

We make the following observation:

$$\sum_{i=1}^N \langle \hat{y} - \hat{\alpha}, e_i \rangle_Q \beta_i = \sum_{i=1}^N \beta_i e_i' Q(\hat{y} - \hat{\alpha}) = R^{-1}EQ(\hat{y} - \hat{\alpha}).$$

Noting $-\sum_{i=1}^N \int_0^T e^{A(T-s)}bf_i(s)e_i'ds = \Lambda'$, it holds that

$$\sum_{i=1}^N \int_0^T -e^{A(T-s)}b\langle \hat{y} - \hat{\alpha}, e_i \rangle_Q f_i(s)ds = \Lambda' Q(\hat{y} - \hat{\alpha}).$$

Using these two constructions we then have

$$0 = (I - e^{AT})(-R^{-1}EQ(\hat{y} - \hat{\alpha})) + (I - e^{AT})R^{-1}(I - e^{AT})'\lambda - \Lambda'Q(\hat{y} - \hat{\alpha}) + \Gamma\lambda \tag{12}$$

where Γ is the controllability Gramian

$$\Gamma = \int_0^T e^{A(T-s)}bb'e^{A'(T-s)}ds. \tag{13}$$

By combining these two expressions (11) and (12) linking $\hat{y}$ and λ, we obtain Equation (14). It can be shown that the coefficient matrix in (14) is invertible under the controllability assumption of the system $\dot{x} = Ax + bu$. Using Equation (14) we can solve for $\hat{y}$ and for λ. These values can be used in Equations (6) and (7) to uniquely determine the optimal control and the optimal initial condition. We see that the optimal estimate of the data is obtained independently of the control.

It is necessary to ask the question "In what sense is the spline periodic?" We state and prove the following theorem in answer to that question.

Theorem 2 *The function $y(t)$ of the above construction can be extended periodically to the entire positive real line and the extension is $2n-1$ times continuously differentiable everywhere with exception of the points $nT : n \in \mathbb{Z}$ where are guaranteed only $n-1$ continuous derivatives.*

We leave the proof to the reader.

An example of this procedure is seen in Figure 1, where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ c = \begin{pmatrix} 1 & 0 \end{pmatrix},\ T = 1$$
$$t_1 = 0.2,\ t_2 = 0.3,\ t_3 = 0.5,\ t_4 = 0.7,\ t_5 = 0.8$$
$$\hat{\alpha} = \begin{pmatrix} 0.8 & 0.2 & 0.5 & 1 & 0.3 \end{pmatrix}'$$
$$Q = 10^4 I_5,\ R = 10^4 I_2,\ (I_p = p \times p \text{ identity matrix}).$$

We assumed in Section 1 that the system $\dot{x} = Ax + bu,\ y = cx$ is controllable and observable. In addition to that it is natural to employ minimal system for generating splines, the controllability guarantees the invertibility of the coefficient matrix in (14) implying the existence and uniqueness of an optimal solution to Problem 1.

$$\begin{pmatrix} I+(G+F)Q & -E'R^{-1}(I-e^{AT})'-\Lambda \\ ((I-e^{AT})R^{-1}E+\Lambda')Q & -(I-e^{AT})R^{-1}(I-e^{AT})'-\Gamma \end{pmatrix}\begin{pmatrix}\hat{y}\\ \lambda\end{pmatrix}=\begin{pmatrix}(G+F)Q\hat{\alpha}\\ ((I-e^{AT})R^{-1}E+\Lambda')Q\hat{\alpha}\end{pmatrix}. \tag{14}$$

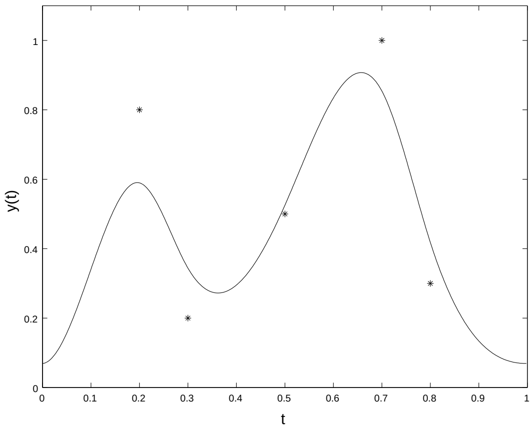

Fig. 1. Periodic spline in Problem 1: Depicted are $y(t)$ (solid line) and $\hat{\alpha}$ (stars).

4 Problem 2

We begin by considering the periodic spline as a particular boundary value problem. We will use the methods related to but not the same as in the above construction. Again let the boundary condition be given by

$$x(0)-x(T)=0, \tag{15}$$

and the dynamics be the same as in Problem 1. Then as before, the boundary constraint is given by (4). Note that the initial data becomes a parameter in the constraint.

We now define a Hilbert space to be

$$\mathcal{H}=L_2[0,T]\times\mathbb{R}^N$$

with norm

$$\|(u;y)\|^2=\int_0^T u^2(t)dt+y'Qy.$$

Since we no longer have a cost associated with the initial condition, x_0 can not be part of the Hilbert space over which the minimum norm problem is solved. In fact, the constraint variety is now affine, parameterized by x_0, and it is given by

$$\mathcal{V}_{x_0}=\{(u;d):\ \langle\beta_i,x_0\rangle_R=d_i-\langle f_i,u\rangle_L,$$
$$(I-e^{AT})x_0-\int_0^T e^{A(T-s)}bu(s)ds=0\}.$$

If we now let $\mathcal{V}$ be the linear constraint variety associated with $x_0=0$ above, we note that, as before, $\mathcal{V}$ is a closed subspace of $\mathcal{H}$. Moreover, from [4], we know that the unique minimizer is given by the intersection $\mathcal{V}_{x_0}\cap(\mathcal{V}^\perp+p)$, where

$$\mathcal{V}^\perp=\{(v;w):\ \forall(u;d)\in\mathcal{V}\ \langle v,u\rangle_L+\langle w,d\rangle_Q=0\}.$$

Using the same technique as for Problem 1 we have

$$\mathcal{V}^\perp=\{(v;w):\ v=-\sum_{i=1}^N\langle w,e_i\rangle_Q f_i-b'e^{A'(T-s)}\lambda\}.$$

We now construct the intersection

$$(\mathcal{V}^\perp+p)\cap\mathcal{V}_{x_0},$$

which is determined by the solution of the following three equations which come from $\mathcal{V}^\perp+p$ and $\mathcal{V}_{x_0}$.

$$u=-\sum_{i=1}^N\langle\hat{y}-\hat{\alpha},e_i\rangle_Q f_i-b'e^{A'(T-s)}\lambda \tag{16}$$
$$\langle\beta_i,x_0\rangle_R=y_i-\langle f_i,u\rangle_L \tag{17}$$
$$(I-e^{AT})x_0=\int_0^T e^{A(T-s)}bu(s)ds \tag{18}$$

We use the first equation to eliminate u from the second and third equation. After some manipulation we have the following system of equations.

$$\begin{pmatrix} I+FQ & -\Lambda\\ \Lambda'Q & -\Gamma\end{pmatrix}\begin{pmatrix}\hat{y}\\ \lambda\end{pmatrix}=\begin{pmatrix}E' & FQ\\ I-e^{AT} & \Lambda'Q\end{pmatrix}\begin{pmatrix}x_0\\ \hat{\alpha}\end{pmatrix} \tag{19}$$

where, as before, F is the Gramian of the f_i's in (10) and Γ is the controllability Gramian in (13). Note that this coefficient matrix is invertible, since

$$\begin{pmatrix} I+FQ & -\Lambda\\ \Lambda'Q & -\Gamma\end{pmatrix}=M_1\begin{pmatrix}Q & 0\\ 0 & -I\end{pmatrix},$$

and the matrix M_1 defined by

$$M_1=\begin{pmatrix}Q^{-1}+F & \Lambda\\ \Lambda' & \Gamma\end{pmatrix}$$

is positive-definite.

We now reconsider u from Equation (16), and obtain

$$u = (-f'(s)Q, \ -b'e^{A'(T-s)}) \begin{pmatrix} \hat{y} - \hat{\alpha} \\ \lambda \end{pmatrix}.$$

Then we can calculate the square integral of u as

$$\int_0^T u^2(s)ds = (\hat{y}' - \hat{\alpha}', \lambda')S \begin{pmatrix} \hat{y} - \hat{\alpha} \\ \lambda \end{pmatrix}, \qquad (20)$$

where

$$S = \begin{pmatrix} QFQ & -Q\Lambda \\ -\Lambda' Q & \Gamma \end{pmatrix}.$$

Now using Equation (19) we have

$$\begin{pmatrix} \hat{y} - \hat{\alpha} \\ \lambda \end{pmatrix} = C \begin{pmatrix} x_0 \\ \hat{\alpha} \end{pmatrix}$$

where

$$C = \begin{pmatrix} I + FQ & -\Lambda \\ \Lambda' Q & -\Gamma \end{pmatrix}^{-1} \begin{pmatrix} E' & -I \\ I - e^{AT} & 0 \end{pmatrix},$$

and thus we have an expression for

$$\int_0^T u^2(s)ds = (x_0', \hat{\alpha}')C'SC \begin{pmatrix} x_0 \\ \hat{\alpha} \end{pmatrix}.$$

Since $\hat{y} - \hat{\alpha}$ is written as

$$\hat{y} - \hat{\alpha} = (I, \ 0)C \begin{pmatrix} x_0 \\ \hat{\alpha} \end{pmatrix},$$

we can now write the total cost $J(u, x_0)$ as a function of the initial position $\hat{J}(x_0)$.

$$\hat{J}(x_0) = (x_0', \hat{\alpha}') \left[C'SC + C' \begin{pmatrix} I \\ 0 \end{pmatrix} Q(I, \ 0)C \right] \begin{pmatrix} x_0 \\ \hat{\alpha} \end{pmatrix}$$

Thus we obtain

$$\hat{J}(x_0) = (x_0', \hat{\alpha}')V \begin{pmatrix} x_0 \\ \hat{\alpha} \end{pmatrix},$$

where

$$V = C' \left[S + \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \right] C,$$

and V can be computed as

$$V = \begin{pmatrix} E' & -I \\ I - e^{AT} & 0 \end{pmatrix}' \begin{pmatrix} Q^{-1} + F & \Lambda \\ \Lambda' & \Gamma \end{pmatrix}^{-1} \begin{pmatrix} E' & -I \\ I - e^{AT} & 0 \end{pmatrix}.$$

The cost function $\hat{J}(x_0)$ is quadratic in x_0, and an optimum x_0 minimizing the cost is obtained as a solution of the following linear equation

$$(I_n, 0)V \begin{pmatrix} x_0 \\ \hat{\alpha} \end{pmatrix} = 0.$$

This equation in x_0 has unique solution if and only if the coefficient matrix of x_0, denoted by V_1,

$$V_1 = (I_n, 0)V \begin{pmatrix} I_n \\ 0 \end{pmatrix}$$

is invertible. Moreover, since V_1 is written as

$$V_1 = \begin{pmatrix} E' \\ I - e^{AT} \end{pmatrix}' \begin{pmatrix} Q^{-1} + F & \Lambda \\ \Lambda' & \Gamma \end{pmatrix}^{-1} \begin{pmatrix} E' \\ I - e^{AT} \end{pmatrix}$$

we see that the solution is unique whenever

$$\text{rank} \left(e^{A't_1}c' \ \cdots \ e^{A't_N}c' \ I - e^{A'T} \right) = n. \qquad (21)$$

The condition for full rank in Equation (21) is interesting. The first N columns arise from an observability problem that was studied in [5]. The problem is to recover the initial data from $\dot{x} = Ax, \ y = cx, \ x(0) = x_0$ when the observations are made at discrete time points, $t_1, \cdots, t_N$. There are no known necessary and sufficient conditions in terms of A, c and $t_1, \cdots, t_N$ for the first N columns to have full rank. However, we will see that the last set of n columns given by $I - e^{A'T}$ simplifies the condition (21).

We introduce the following set of complex numbers

$$\mathcal{I} = \{ j2\pi k/T \mid j = \sqrt{-1}, \ k \in \mathbb{Z} \}.$$

Note that the matrix $I - e^{AT}$ is singular whenever A has an eigenvalue in $\mathcal{I}$. Then we can show that the following lemma holds (the proof omitted).

Lemma 3 *The condition (21) holds if and only if rank* $(c'\ \lambda I - A') = n$ *for all* $\lambda \in \mathcal{I}$.

We have shown that, due to the controllability assumption, (19) has a unique solution in $\hat{y}$ and λ for any initial condition x_0. On the other hand, the assumption of observability, which is equivalent to rank $(c'\ \lambda I - A') = n$ for all $\lambda \in \mathbb{C}$ (e.g.[2]), guarantees the existence of unique optimal x_0 by Lemma 3. Thus, under the assumption of controllability and observability, we obtain the following theorem:

Theorem 4 *There always exists a unique optimal solution to Problem 2.*

The solution to Problem 2 is computed for the same example as in the previous section and the results are shown in Figure 2. Periodic splines can be used to construct closed curves in the plane, and in Figures 3 and 4 we show a result applied to contour modeling of jellyfish for an image frame from a real digital movie file [2]. We used the same second order system (A, b, c) as in the previous examples, and the other parameters are $T = 360$, $N = 36$ and $Q = 10^{-2} I_{36}$. Using standard image processing techniques, the data set $(t_i, \alpha_i),\ i = 1, 2, \cdots, N$ is obtained from the image in Figure 4. Namely, t_i and α_i are obtained as the polar coordinates of sampled boundary pixels with the origin at the centroid of jellyfish, where $t_i = 10 \times (i - 1)$ denotes angles measured every 10 degrees and α_i the radial distance in pixels. The solution to Problem 2 is obtained as in Figure 3, from which the contour is reconstructed as superposed in Figure 4.

Finally, one can note that it may be possible to obtain a solution to this problem as a limiting solution to Problem 1 as $\|R\|$ tends to zero. However, this line of inquiry is not pursued further and we simply state this as a possibility for the future.

5 Conclusions

In this paper we present a method for generating periodic, smoothing splines using linear optimal control. In particular, we show that such curves can be obtained in a quite general fashion by viewing the smoothing cost as an inner product in a suitable space, at the same time as periodicity is enforced by limiting the solutions to lie in a particular linear variety. The resulting methodology is numerically efficient and robust, and it unifies a number of contributions in the general area of smoothing splines through the use of different linear systems as generators of curves of different characteristics, including polynomial, exponential, and trigonometric smoothing splines.

[2] Educational Image Collections, Information-technology Promotion Agency (IPA), Japan. http://www2.edu.ipa.go.jp/gz/

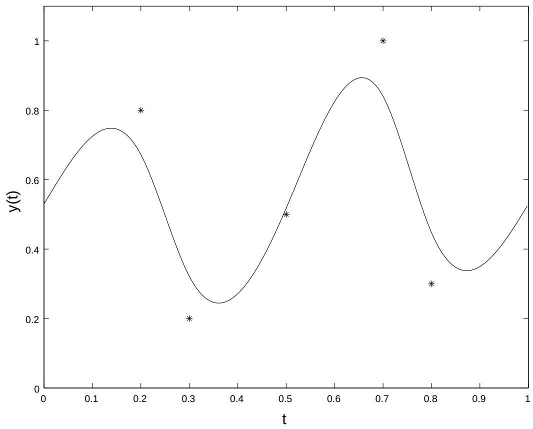

Fig. 2. Periodic spline in Problem 2: Depicted are $y(t)$ (solid line) and $\hat{\alpha}$ (stars).

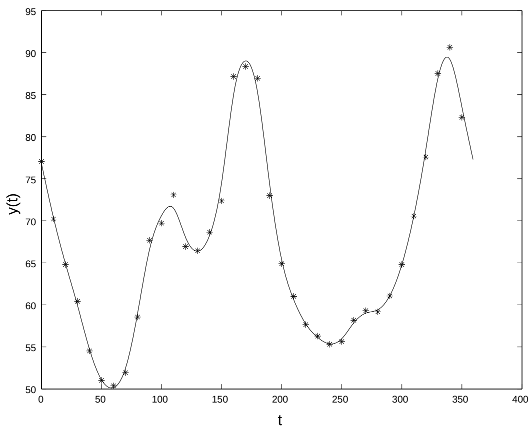

Fig. 3. Periodic spline in Problem 2 for the data from jellyfish image: Depicted are $y(t)$ (solid line) and $\hat{\alpha}$ (stars).

Fig. 4. A jellyfish image frame and the contour reconstructed from the periodic spline $y(t),\ t \in [0, 360]$ in Problem 2 by $(y(t) \cos \frac{2\pi t}{360}, y(t) \sin \frac{2\pi t}{360})$.

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