Controllability of Prosumer-Based Networks in the Presence of Communication Failures

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Abstract— **Typically, interconnected dynamical systems rely on communication in order to coordinate and compute appropriate control actions. Loss of communication links can exclude key decision makers from providing input and can even alter the system properties. This paper investigates the impact of communication loss on the controllability of a specific networked system, a homogeneous power-grid populated by producer-consumer hybrids. The notion of muteness is introduced in order to characterize the control policy adopted by the nodes which are isolated due to communication loss. We provide results which relates the controllability of such a system with mute nodes to the topology of the underlying electrical network and show that under certain topological conditions, controllability is preserved.**

I. Introduction

Networked control involves the design of distributed control laws for executing given tasks. Some examples of such tasks can be found in multi-robot systems e.g [7], [9], sensor networks e.g [4] and power grids e.g [1], [3]. Typically, the execution of distributed control laws requires the agents to exchange information with each other over a communication network. As such, failure in communication could prevent the agents from completing their task. Isolation occurs when all communication links with an agent is broken and this can be modelled by discarding all the edges inbound to the vertex corresponding to the agent in the communication graph. When faced with such isolation, the isolated agents must respond in an orderly fashion. One course of action is to let the isolated agents not act due to lack of information which results in loss of actuation and consequently, could result in a loss of controllability. This paper explores this connection between controllability and communication failures in a homogenous power grid, modelled as a linear system, and relates it to the underlying network topology. The key results of this paper identifies certain topological conditions on the grid model under which controllability is preserved in presence of isolated agents.

The impact of network topology on networked dynamical systems has been explored before. A commonly explored research theme, in which network topology plays a role, is that of leader selection to ensure controllability of a leader-follower network e.g [8], [10], [12]. Another problem that has been studied in the context of network topology is that of noisy consensus e.g [13]. The focus there is on identifying topologies which allow for robust consensus in the presence of noise. Another key connection between topology and consensus based dynamical models is that of convergence. The rate at which consensus is achieved is tied to the spectral properties of the graph Laplacian which is heavily dependant on the connectivity of the underlying graph.

One of the key notions that allow us to retain a degree of control over a networked dynamical systems even in the presence of communication failures is that of physical coupling. We illustrate the importance of this notion via a counterexample. Consider the problem of coordinating a team of mobile robots. Individual robots in such systems are modelled as autonomous entities capable of making their own decisions. As a result, any robot which is isolated due to communication failure cannot be influenced via any means as they are not *physically coupled* to the rest of the system in any manner. In contrast to this, we investigate controllability of multi-agent systems whose nodes are coupled both physically and via a communication network. In such systems, failure in communication still allows the other nodes in the system to influence the state of the isolated node due to the underlying physical connection between them.

We choose to study the controllability of the power grid as it possesses the cyber-physical structure in which we are interested. The generators and the loads in a power grid are coupled *physically* via tie-lines. They can also exchange information and measurements with each other via a communication network. Typically, power systems are controlled in a centralized manner using AGC (Automatic Generation Control) and SCADA (Supervisory Control And Data Acquisition) systems [6]. But, a drop in the cost of renewable power resources allows for intermittent sources such as wind-mills and solar panels to take a larger part in power generation. This results in distribution of power generation allowing residential customers to take part in power generation blurring the lines between producers and consumers. For a more detailed discussion of this trend, see [5] and [11]. As a result, the grid model we consider is more homogenized and is populated by producer-consumer hybrids known as prosumers. A detailed account of the prosumer based power-grid model can be found in [2].

The paper is structured as follows: section 2 introduces the power grid model and section 3 discusses the notion of muteness in order to characterize the control policy adopted by the nodes which are isolated from the network due to communication failure. section 4 provides a result which characterizes the condition under which the power grid is controllable in the presence of communication failure. section 5 makes a connection between controllability and the electrical topology of the network.

II. Grid model

We briefly describe the power grid model that we shall be using throughout the paper before addressing the issue of communication failure and muteness. A much more detailed discussion of this model can be found in [3].

We consider a set $V = \{1, 2 \dots n\}$ of n power generating agents which are connected to each other electrically via a transimission line. Formally, we can represent the physical layout of this power network by a graph $\mathcal{G}_p = (V, E)$ where $|V| = n$. The presence of an edge (v_i, v_j) indicates that the nodes v_i and v_j are physically coupled. There is a state $x_i \in \mathbb{R}$, the deviation in output power with respect to a scheduled reference, associated with each node v_i in V. We collect the states in a vector associated with each node v_i to obtain the ensemble state given by $x = [x_1, x_2 \dots x_n]^T$. The evolution of x with respect to a discrete time parameter $k \in \mathbb{N}$ is given by the following dynamical model:

$$x(k+1) = A_p x(k) + B_p u(k) \tag{1}$$

where $u(k)$ is a vector of setpoints which controls the power output of each agent at time k. Furthermore,

$$A_p = I - T_s J S \tag{2}$$
$$B_p = T_s J \beta \tag{3}$$

where $J \in \mathbb{R}^{n \times n}$ is called the Jacobian matrix, I is the n-dimensional identity matrix, $T_s \in \mathbb{R}$ is the sampling time, S and β are diagonal matrices of dimension $n \times n$ which encodes certain electrical properties of the each individual agent. Note that A_p and B_p are derived from the Jacobian matrix J. This special structure inherent in the power grid model is what allows us to connect controllability to the grid's electrical layout.

The Jacobian matrix has a sparsity structure which reflects the underlying electrical topology of the network. This is captured by the following relation

$$J_{ij} = 0 \Leftrightarrow (v_i, v_j) \notin E. \tag{4}$$

Note that both A_p and B_p share the same sparsity structure with J as the algebraic operations carried out to obtain them, multiplication by the diagonal matrices, S and β and subtraction with the identity matrix I, do not affect the sparsity structure. Furthermore, the Jacobian, for power systems, is invertible and posseses full rank (For more details, see section 3-D in [3]) . This allows us to establish that the pair (A_p, B_p) is completely controllable.

We also assume that the agents can communicate with each other over a communication network which is represented by a graph $\mathcal{G}_c = (V, E) = \mathcal{G}_p$. The equivalence between the physical and communication graphs imply that the nodes which are connected physically can communicate with each other.

III. Muteness

The work done in [3] provides a distributed method to stabilize the system described by (1) in a distributed manner where the agents iterate over control strategies by exchanging information over the communication network to obtain an optimal stabilizing solution. As a result, any communication failure which isolates an agent from the rest of the system means that the isolated agent cannot participate in the decision making process and is forced to arrive at a control strategy without any information.

Since the isolated agent is still connected to the network physically, and is influenced by the states of the other agents, one approach would be to set its input to zero and relinquish the burden of stablizing the system to the rest of the agents. Consequently, it is important to understand how this loss of actuation affects the controllability of the system. In this section, we introduce the formal notion of muteness in order to discuss the system resulting from the isolated agents not participating in the decision making process.

Let the set $\mathcal{M} \subset V$ denote the set of agents which are isolated due to communication failure. We will refer to the agents belonging to the set $\mathcal{M}$ as **mute**. We assume that each agent in the set $\mathcal{M}$ is unable to communicate with the rest of the network and as a result adopts a "zero-bias" control strategy. The following equation summarizes this control strategy:

$$v_i \in \mathcal{M} \Leftrightarrow u_i(k) = 0 \ \forall k \in \mathbb{N} \tag{5}$$

For convenience, we will assume that $\mathcal{M}$ is given by the last m agents i.e, $\mathcal{M} = \{n-m+1, n-m+2, \dots n\}$ where $m = |\mathcal{M}|$ in order to simplify analysis.After substituting the control strategy adopted by the mute agents given by (5) in the dynamical model (1), we obtain

$$x(k+1) = A_p x(k) + \begin{bmatrix} B_{\mathcal{N}} & B_{\mathcal{M}} \end{bmatrix} \begin{bmatrix} u_{\mathcal{N}}(k) \\ 0_{m \times 1} \end{bmatrix} \tag{6}$$

where $B_{\mathcal{N}}$ is the matrix comprising of the first $n - m$ columns of the B_p, the $B_{\mathcal{M}}$ contains the rest of the columns of B_p which corresponds to the mute agents. Also, $u_{\mathcal{N}} = [u_1, u_2 \dots u_{n-m}]^T$ and $0_{m \times 1}$ is a column vector with m rows containing zeros. Since the inputs corresponding to the columns of $B_{\mathcal{M}}$ is zero, we can simplify the dynamics further and obtain

$$x(k+1) = A_p x(k) + B_{\mathcal{N}} u_{\mathcal{N}}(k). \tag{7}$$

The dynamics given by (7) represents the evolution of the power grid when the mute agents do not participate in the decision making process. It is the controllability of this system that we are interested in.

IV. Controllability result

In this section, we will provide a rank test which allows us to establish the controllability of the system described by (7).

The controllability matrix of the the system defined by (7) is expressed as

$$\Gamma = \begin{bmatrix} B_{\mathcal{N}} & A_p B_{\mathcal{N}} & \dots & A_p^{n-1} B_{\mathcal{N}} \end{bmatrix}. \quad (8)$$

Furthermore, we can define the reduced controllability matrix as follows

$$\hat{\Gamma} = \begin{bmatrix} B_{\mathcal{N}} & A_p B_{\mathcal{N}} & \dots & A_p^{m} B_{\mathcal{N}} \end{bmatrix}, \quad (9)$$

where $m = |\mathcal{M}|$ is the number of mute prosumers.

We present a lemma which shows that it is enough to check the rank of $\hat{\Gamma}$ in order to establish controllability of the system described by (7).

Lemma 1: Let Γ and $\hat{\Gamma}$ be as defined by Equations (8) and (9). Then $\rho(\Gamma) = \rho(\hat{\Gamma})$ where ρ is the rank operator.

Proof: We begin by defining

$$\Gamma_k = \begin{bmatrix} A_p^0 B_{\mathcal{N}} & A_p^1 B_{\mathcal{N}} & \dots & A_p^k B_{\mathcal{N}} \end{bmatrix} \quad (10)$$

It can be easily seen that $\rho(\Gamma_{r+1}) \geq \rho(\Gamma_r)$. This is because Γ_{r+1} is constructed by adding more columns to Γ_r and adding more columns does not decrease the rank. Now, we claim that if $\rho(\Gamma_{r+1}) = \rho(\Gamma_r)$ for some r, then $\rho(\Gamma_{r+n}) = \rho(\Gamma_r)$ for all n. This can be established via induction.

Let $\rho(\Gamma_{r+1}) = \rho(\Gamma_r)$ and let $h = 1$. We will now establish that $\rho(\Gamma_{r+2}) = \rho(\Gamma_r)$. Then,

$$\Gamma_{r+1+h} = \Gamma_{r+2} = \begin{bmatrix} \Gamma_r & A_p^{r+1} B_{\mathcal{N}} & A_p^{r+2} B_{\mathcal{N}} \end{bmatrix}. \quad (11)$$

Now, let v be a column of the matrix $A_p^{r+2} B_{\mathcal{N}}$. Then $v = A_p u$ where u is a column of the matrix $A_p^{r+1} B_{\mathcal{N}}$. Every column of the matrix $A_p^{r+1} B_{\mathcal{N}}$, which represents the last set of columns in the matrix Γ_{r+1}, can be written as a linear combination of the columns of the matrix Γ_r owing to the fact $\rho(\Gamma_{r+1}) = \rho(\Gamma_r)$. This implies that $u = \sum_{i=0}^{r} A_p^i B_{\mathcal{N}} v_i$ for some choice of vectors $v_i \in \mathbb{R}^n$. Then

$$v = A_p \sum_{i=0}^{r} A_p^i B_{\mathcal{N}} v_i = \sum_{i=1}^{r+1} A_p^i B_{\mathcal{N}} v_i \quad (12)$$

$$\implies v = \underbrace{\sum_{i=1}^{r} A_p^i B_{\mathcal{N}} v_i}_{\text{Linear combinations of columns of } \Gamma_r} + A_p^{r+1} B_{\mathcal{N}} v_r \quad (13)$$

Since every column of $A_p^{r+1} B_{\mathcal{N}}$ is a linear combination of the columns of the matrix Γ_r, we can conclude the v, a column of the matrix $A_p^{r+2} B_{\mathcal{N}}$ is a linear combination of columns of the matrix Γ_r. This establishes that every column of the matrix Γ_{r+2} is a linear combination of columns of Γ_r allowing us to conclude that $\rho(\Gamma_{r+2}) = \rho(\Gamma_r)$.

Now, assume $\rho(\Gamma_{r+h}) = \rho(\Gamma_r)$ for some $h > 1$. We will proceed to show that $\rho(\Gamma_{r+h+1}) = \rho(\Gamma_r)$. Note that $\rho(\Gamma_{r+h}) = \rho(\Gamma_r)$ for some $h > 1$ implies that $\rho(\Gamma_{r+h-1}) = \rho(\Gamma_r)$. This can be established by the following argument. We know that $r + h - 1$ is bounded by $r + 1$ as $h > 1$. Then since, the rank operator ρ is monotonic, we have

$$\rho(\Gamma_r) = \rho(\Gamma_{r+1}) \leq \rho(\Gamma_{r+h-1}) \leq \rho(\Gamma_{r+h}) = \rho(\Gamma_r) \quad (14)$$

allowing us to infer that $\rho(\Gamma_{r+h}) = \rho(\Gamma_{r+h-1}) = \rho(\Gamma_r)$. Now, setting $c = r + h - 1$, we have $\rho(\Gamma_c) = \rho(\Gamma_{c+1})$. We can apply the same argument which we used above to establish that $\rho(\Gamma_c) = \rho(\Gamma_{c+2})$ which was to be shown. This combined with base case argument shows that if $\rho(\Gamma_{r+1}) = \rho(\Gamma_r)$ for some r, then $\rho(\Gamma_{r+n}) = \rho(\Gamma_r)$ for all n.

Now, we know that $B_p = T_s J \beta$ is $n \times n$ matrix of rank n as it is a product of two full rank matrices, J and β. This implies all its columns are linearly independent. Then the rank of $B_{\mathcal{N}}$, comprised of the first $n - m$ columns, is $n - m$. Then, $\rho(\Gamma_0) = \rho(B_{\mathcal{N}}) = n - m$. Now, if the $\rho(\Gamma_m) = \rho(\hat{\Gamma}) < n$, then there exists $k < m$ such that $\rho(\Gamma_k) = \rho(\Gamma_{k+1})$. Then, we can establish that $\rho(\Gamma_k) = \rho(\Gamma_m) = \rho(\hat{\Gamma}) = \rho(\Gamma)$ from our previous argument.

If $\rho(\hat{\Gamma}) = n$, then clearly $\rho(\Gamma) = n$, as n is the upper bound for the rank of the controllability matrix. This lets us conclude that $\rho(\hat{\Gamma}) = \rho(\Gamma)$. ■

The above lemma simplifies the problem by allowing us to truncate the controllability matrix and this would allow us to discard a lot of columns if the number of agents n is much greater relative to the number of mute agents m. We can further simplify the problem and draw connections to the underlying physical topology of the system by further exploiting the strucuture of the matrices A_p and B_p.

We recall from Section 2 that $A_p = I - T_p J S$ and $B_p = T_s J \beta$. We can then write $B_{\mathcal{N}} = T_s J \hat{\beta}$ where $\hat{\beta} = [b_1, b_2 \dots b_{n-m}] \in \mathbb{R}^{n \times (n-m)}$ where b_i is the i'th column of the matrix β and $m = |\mathcal{M}|$ is the number of mute agents. In order to proceed, we define a matrix P_m as follows:

$$P_m = \begin{bmatrix} S J \hat{\beta} & \dots & S (JS)^{m-1} J \hat{\beta} \end{bmatrix} \quad (15)$$

The structure of the P_m contains information about the underlying physical graph $\mathcal{G}_p$ and its higher powers. This will later allow us to connect the controllability of the power network to its underlying physical topology. For now, we express P_m as the following block matrix

$$P_m = \begin{bmatrix} G \\ F \end{bmatrix}, \quad (16)$$

where G is a matrix of dimension $(n-m) \times m(n-m)$ and F is a matrix of dimension $m \times m(n-m)$. The matrix F is of quite some importance as it captures the interaction between the mute nodes $\mathcal{M}$ and the non-mute nodes contained in the set $V \setminus \mathcal{M}$.

Theorem 1: The pair $(A_p, B_{\mathcal{N}})$ is completely controllable if the rank of the matrix F is equal to m, where m is the number of mute agents.

Proof: We know that the rank of the controllability matrix Γ is equal to the rank of $\hat{\Gamma}$ from Lemma 1. We also know that $B_p = T_s J \beta$. Then $B_{\mathcal{N}} = T_s J \hat{\beta}$ where $\hat{\beta} = [b_1, b_2 \dots b_{n-m}] \in \mathbb{R}^{n \times (n-m)}$ where b_i is the i'th column of the matrix β.

The sampling time T_s does not affect the rank analysis. So, we set $T_s = 1$ in the following derivation to simplify analysis. Now, we have

$$\rho(\hat{\Gamma}) = \rho(\begin{bmatrix} J\hat{\beta} & (I - JS) J \hat{\beta} & \dots & (I - JS)^m J \hat{\beta} \end{bmatrix})$$

We can expand the terms $(I-JS)^r$ using binomial expansion and obtain

$$\rho(\hat{\Gamma}) = \rho(\begin{bmatrix} J\hat{\beta} & \dots & \sum_{k=0}^{m} c_k^n (-JS)^k J\hat{\beta} \end{bmatrix})$$

where $c_n^k = \frac{n!}{k!(n-k)!}$. Since the coefficients do not contribute the rank analysis, we can drop them and obtain a further simplified expression as follows:

$$\rho(\hat{\Gamma}) = \rho(\begin{bmatrix} J\hat{\beta} & \sum_{k=0}^{1} (-JS)^k J\hat{\beta} & \dots & \sum_{k=0}^{m} (-JS)^k J\hat{\beta} \end{bmatrix}).$$

Note that the first term of the sum $\sum_{k=0}^{r} (-JS)^k J\hat{\beta}$ is always equal to $J\hat{\beta}$. This allows us to drop that term as it is also equal to the first set of columns of the matrix $\hat{\Gamma}$. So, we then obtain

$$\rho(\hat{\Gamma}) = \rho(\begin{bmatrix} J\hat{\beta} & \dots & \sum_{k=1}^{m} (-JS)^k J\hat{\beta} \end{bmatrix}).$$

We can see that the sum $\sum_{k=1}^{r+1} (-JS)^k J\hat{\beta}$ can expressed as the sum of $\sum_{k=1}^{r} (-JS)^k J\hat{\beta}$ and $(JS)^{r+1} J\hat{\beta}$. This allows us to eliminate the summation and only retain the last term further simplifying the expression for the rank as follows:

$$\rho(\hat{\Gamma}) = \rho(\begin{bmatrix} J\hat{\beta} & (-JS)^1 J\hat{\beta} & \dots & (-JS)^m J\hat{\beta} \end{bmatrix}).$$

Finally, we can drop the negative signs (as they do not have an impact on the rank of a matrix) and factor out the Jacobian to obtain

$$\rho(\hat{\Gamma}) = \rho(J \begin{bmatrix} \hat{\beta} & SJ\hat{\beta} & \dots & S(JS)^{m-1} J\hat{\beta} \end{bmatrix}).$$

Since, the Jacobian J is a full rank matrix, it does not reduce the rank of the controllability matrix. So the rank is purely determined by the second term of the product. That is

$$\rho(\hat{\Gamma}) = \rho(\begin{bmatrix} \hat{\beta} & SJ\hat{\beta} & \dots & S(JS)^{m-1} J\hat{\beta} \end{bmatrix}).$$

We set

$$M = \begin{bmatrix} \hat{\beta} & SJ\hat{\beta} & \dots & S(JS)^{m-1} J\hat{\beta} \end{bmatrix}. \tag{17}$$

Note that $\hat{\beta}$ is a truncated diagonal matrix of the form

$$\hat{\beta} = \begin{bmatrix} D \\ 0_{m \times (n-m)} \end{bmatrix}$$

where D is a diagonal matrix of dimension $(n-m) \times (n-m)$. We can use this to express M as a block matrix of the form

$$M = \begin{bmatrix} D & G \\ 0_{m \times (n-m)} & F \end{bmatrix}.$$

where G is a matrix of dimension $(n-m) \times m(n-m)$ and F is a matrix of dimension $m \times m(n-m)$.

Since D is a diagonal matrix, we can take linear combinations of the its columns to eliminate the entries of the matrix G. This allows us to conclude that

$$\rho(M) = \rho(\begin{bmatrix} D & 0_{(n-m) \times m(n-m)} \\ 0_{m \times (n-m)} & F \end{bmatrix}) \tag{18}$$

$$\implies \rho(M) = \rho(D) + \rho(F) = (n-m) + \rho(F). \tag{19}$$

In order for the system to be completely controllable, we require $\rho(F) = m$ which was to be shown. ■

Theorem 1 provides us with a rank test as opposed to topological. One of the primary advantages of a topological characterization as opposed to a rank test is that it aids in the design of the network topology and is therefore of interest. In the next section, we will use the results of Theorem 1 to connect the controllability of (7) to the topology given by $\mathcal{G}_p$.

V. Controllability and Topology

The rank test provided by Theorem 2 involves inspecting the matrix P_m defined by (15). The matrix P_m possess a rich topological structure which encodes information about the physical network G_p and its higher graph powers. In this section, we establish controllability by extracting specific linear submatrices of the matrix P_m and interpret the results from a graph-theoritic viewpoint.

We will seperate our analysis into two cases : $|\mathcal{M}| = 1$ and $|\mathcal{M}| > 1$.

When there is a single mute agent (i.e $|\mathcal{M}| = 1$), we can show that controllablity of (7) can be directly related to the connectivity of the physical network represented by the graph $\mathcal{G}_p$.

***Theorem** 2:* If the graph G_p is strongly connected and $|\mathcal{M}| = 1$, then the pair $(A_p, B_{\mathcal{N}})$ is always completely controllable.

Proof: When the number of mute agents is equal to 1, we can write the matrix P_1 defined by (15) as follows:

$$P_1 = SJ\hat{\beta} = \begin{bmatrix} G \\ F \end{bmatrix} \tag{20}$$

where F is a $1 \times (n-1)$ matrix. Note that F is just the last row of the matrix $SJ\hat{\beta}$. Since both $\hat{\beta}$ and S are diagonal matrices, they do not affect the sparsity structure of the product $P_1 = SJ\hat{\beta}$. So, P_1 inherits its sparsity structure from that of J. Let $u = n$ denote the single element of the set $\mathcal{M}$. Since $\mathcal{G}_p$ is strongly connected, there exists atleast one node v in $V \setminus \mathcal{M}$ such that $(v, u) \in E$. This implies that $\alpha \neq 0$ where α is the element in the v'th position in the vector F. Since, F is a row vector with a non-zero entry α, we can conclude that the rank of F is equal to 1 which is the number of mute agents in the system. This allows us (by Theorem 1) to conclude that the pair $(A_p, B_{\mathcal{N}})$ is completely controllable. ■

So, as long as the isolated node is connected to the system electrically, we can use the other nodes to control the state of the isolated node irrespective of the node's position in the network topology. This shows that the pair $(A_p, B_{\mathcal{N}})$ is always controllable, irrespective of the network topology, when only a single node suffers from communication failure.

For the case $|\mathcal{M}| > 1$, we provide a sufficient condtion under which controllability is preserved. In order to do so, we define the set $\mathcal{N} = V \setminus \mathcal{M}$ where V is set of all agents and $\mathcal{M}$ is the set of mute agents in the network. In the following theorem, we identify topological conditions on the set $\mathcal{M}$ which renders the pair $(A_p, B_{\mathcal{N}})$ controllable.

***Theorem** 3:* Let $G_p = (V, E)$ be the graph representing the electrical network. Let $\mathcal{M} \subset V$ be the set of mute agents.

If there exists an injective map $\phi : \mathcal{M} \to \mathcal{N}$ such that

$$\phi(m) = n \Leftrightarrow (m, n) \in E \wedge (v, n) \notin E \ \forall v \in \mathcal{M} \setminus \{m\}, \tag{21}$$

then the pair $(A_p, B_{\mathcal{N}})$ is completely controllable.

Proof: Let $M \subset V$ be a set of mute agents. Assume that there exists $\phi : \mathcal{M} \to \mathcal{N}$ which satisfies the condition given by (21). Physically speaking, the existence of the map ϕ implies that every mute node $v \in \mathcal{M}$ is electrically connected non-mute node $\phi(v) = m$ which is not connected to any node in the set $\mathcal{M} \setminus \{v\}$.

Once again, we restrict our attention to matrix $\hat{P}_m = SJ\hat{\beta}$ and express it as

$$\hat{P}_m = SJ\hat{\beta} = \begin{bmatrix} \hat{G} \\ \hat{F} \end{bmatrix} \tag{22}$$

where $\hat{F} \in \mathbb{R}^{|\mathcal{M}| \times |\mathcal{N}|}$.

As in the case of the single mute agent case, the matrix $\hat{F}$ encodes the relationship between mute nodes and non-mute nodes. Allowing $\hat{F}_{(i,j)}$ to stand for the element located along the ith row and the jth column of the matrix $\hat{F}$, we can say that

$$\hat{F}_{(i,j)} = 0 \Leftrightarrow (i, j) \notin E \wedge (i \in \mathcal{M}) \wedge (j \in \mathcal{N}). \tag{23}$$

Since ϕ satisfies the condtion (21), for every mute node $m \in \mathcal{M}$, there exists a node $n = \phi(m) \in \mathcal{N}$ such that the column $\hat{F}_{\phi(m)}$ contains zero at all locations except $\hat{F}_{m,\phi(m)}$. Then, the collection of columns $\{F_{\phi(m)} \mid m \in \mathcal{M}\}$ are all mutually orthogonal and therefore linearly independent. Therefore, the matrix $\hat{F}$ contains $|\mathcal{M}|$ linearly independent columns and the rank of $\hat{F}$ is equal to $m = |\mathcal{M}|$. Writing the matrix P_m as defined by (15) as

$$P_m = \begin{bmatrix} SJ\hat{\beta} & \dots & S(JS)^{m-1}J\hat{\beta} \end{bmatrix} = \begin{bmatrix} \hat{G} & \tilde{G} \\ \hat{F} & \tilde{F} \end{bmatrix} \tag{24}$$

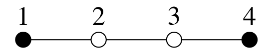
we can see that the rank $\rho(F) = \rho(\begin{bmatrix} \hat{F} & \tilde{F} \end{bmatrix}) = m$. Appealing to Theorem 1, we can conclude that the pair $(A_p, B_{\mathcal{N}})$ is completely controllable. ∎

Theorem 2 and Theorem 3 are results which connect the topology of the physical network to the controllability of the underlying system. While the rank tests establish in Section 4 are more definitive tests for controllability, the topological tests established in this section can be a valuable aid when it comes to designing network topologies as they can be used to identify problematic node configurations and restructure them so that the system is more controllable.

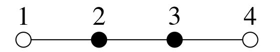
In the next section, we consider different examples of network topologies and apply our results to them in order to determine the controllability of a power grid with that physical topology.

VI. Examples

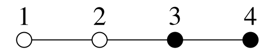
We consider different network topologies and present a brief controllability analysis for each one of them using the results derivied in Sections 5 and 4.



(a) Controllable according to Theorem 3



(b) Controllable according to Theorem 3



(c) Controllability cannot be determined by applying Theorem 3

Fig. 1: Three different configurations of a line graph with different nodes muted (represented by white nodes).

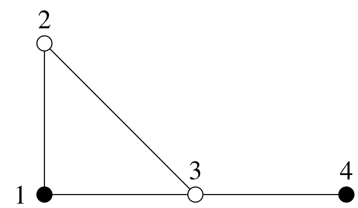


Fig. 2: The controllability of this configuration cannot be detected by applying Theorem 3, but can be inferred from theorem 1

A. A path graph with 4 nodes

The physical topology under consideration is a line graph with four nodes. Figure 1 displays 3 different configurations of line graphs to which we can apply our results. The nodes marked in white in the figures are the muted nodes and the black ones are the unmuted nodes.

In the configuration shown in Figure 1a, we can define a map ϕ as follows:

$$\phi(2) = 1$$
$$\phi(3) = 4$$

The above map satisfies the requirements given by (21) allowing us to apply Theorem 3 and infer that the power grid model with the configuration given 1a is controllable. We can also define a similar ϕ for the configuration given in 1b by mapping the node 1 to 2 and node 4 to 3.

It turns out that there exists no ϕ which satisfies the requirement (21) for configuration 1c. We cannot apply Theorem 3 in this situation. But, it turns out that the configuration given by 1c is actually controllable. This can be seen by computing the higher powers of the matrix P_m given by equation 15 and applying Theorem 1.

B. A 4 node asymmetrical graph

The example shown in the Figure 2 is chosen to illustrate the conservative nature of the result presented in theorem 3. Note that the matrix $\hat{F}$ defined in Theorem 3 would have the following structure:

$$\hat{F} = \begin{bmatrix} f_{21} & 0 \\ f_{31} & f_{34} \end{bmatrix} \tag{25}$$

.

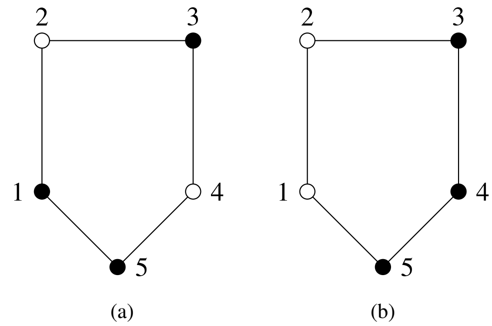


Fig. 3: A cycle graph with 3 non-muted nodes and 2 muted nodes is always controllable

It can be seen that the rank of the matrix $\hat{F}$ is clearly equal to 2 implying that the configuration shown in Figure 2 is controllable. Yet, there exists no map ϕ which will satisfy the requirement given by (21).

C. C_5 : Cycle graph with 5 nodes

Finally, we present an controllablility analysis of a cycle graph with 5 nodes (see fig 3) with any 2 nodes muted as an example of a topology which is resilient to a certain degree of muteness. Note that any 2 nodes in a cycle graph with 5 nodes have a neighboring node which is not connected to the other node. This allows us to construct a ϕ satisfying the requirements given by 21 quite easily and infer that a cycle graph with 5 nodes is always controllable if only 2 nodes are muted.

VII. Simulation

We present a brief controllability analysis of a small power system located in the Flores island using the tools developed in this paper. Flores is an island located in the Azores Archipelago in Portugal with an average electricity demand of the island is about 2 MW (See [3] for more information).

The power grid of the Flores island consists of 3 prosumers connected to each other in a line graph. The system matrices Ap are B_p are given by

$$A_p = \begin{bmatrix} -39.3718 & 20.1086 & 871.2727 \\ 3.9089 & -18.2343 & 0 \\ 36.2953 & 0 & -870.5167 \end{bmatrix} \tag{26}$$

and

$$B_p = \begin{bmatrix} 336.4307 & -40.2171 & 126.9071 \\ -32.5742 & 38.4686 & 0 \\ -302.4605 & 0 & -126.9426 \end{bmatrix} \tag{27}$$

Theorem 2 allows us to infer that this model should be resilient to loss of communication with a single node. It can be verified that this is indeed the case by directly checking the rank of the controllability matrix using the A_p and different truncated versions of the B_p and verify that the rank of the controllability matrix is 3.

VIII. Conclusion

This paper explores the connection between the topology and the controllability of a homogenous power grid in presence of communication failures. A rank test for establishing controllability is presented and certain topological conditions under which the system is controllable are identified.

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