NORTH-HOLLAND

A Control Theoretic Model of the Combined Planar Motion of the Human Head and Eye

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ABSTRACT

In this article we investigate the problem of how to model and control the combined motion of the human head and eye. We develop a model of the muscles, based on a simplified physical model and an assumption that the muscles can be modeled as damped springs with a second order linear dynamics. We then find control laws that both make the combined pupil-movement follow a given trajectory, and make the separate head and eye trajectories three times continuously derivable. Our controls also make the energy produced in the movement small, since we believe that to be a reasonable, physical control-criterion. © Elsevier Science Inc., 1998

1. INTRODUCTION

The purpose of this article is two-fold, and the first question, discussed in Section 2, concerns finding a mathematical model of the combined, horizontally rotational movements of the human head and eye. Therefore we first develop simplified physical models of the muscular configurations in the neck and the eye respectively. Then these physical models are translated into systems of differential equations that can be dealt with mathematically.

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The next task, when it comes to finding a mathematical model, is to link the two separate systems, constituted by the head and the eye respectively, together, so that we can move on to the next major problem investigated in this article: How do we combine the movements of the head and the eye in order to follow a moving object with a given, known trajectory, at a constant distance from the head? This question is discussed in Section 3, where control laws are developed for activating the neck and the eye muscles in such a way that the pupil follows the desired trajectory, at the same time as both the head and eye trajectories, viewed separately, are three times continuously derivable.

But any control laws that accomplish this will not do. We try to develop control strategies that minimize the energy produced in the different movements. Therefore we dedicate Section 4 to the optimal control problem, in order to get a feeling for how good the controls from Section 3 are. Two different approaches to the minimization problem are used. The first one concerns trying to optimize the controls from Section 3, while the second one uses Pontryagin's Maximum Principle (PMP) in order to find the best of all possible solutions.

The results are then discussed in Section 5, followed by a presentation of some of the graphs produced in the different simulations that were conducted.

2. HEAD AND EYE ROTATION

2.1. Dynamics of Head Rotation

2.1.1. Muscular anatomy of the neck. Human head movements are controlled by more than 20 pairs of muscles that link the skull, spinal column and shoulder girdle in a complex variety of configurations. Some of the muscles control the facial expressions and other the gross movements of the head. These gross movements are usually categorized as extension (backward head tilt), flexion (forward head tilt), lateral flexion (sideward head tilt), and rotation.

In this paper, concern is only given to the muscles which control the horizontal rotation of the head. This task is mainly handled by five muscles on each side of the body, of which three are so called opposite rotators, meaning that when the left muscle contracts, the neck rotates to the right. Three of these five muscles, the *semispinals*, the *erector spinae* and the *multifidus* are located on the back side of the spine, and they only function as assistant movers in the rotation movement. The other two are the *sternocleidomastoid* and the *splenius.*

The most massive muscles in the neck, such as the sternocleidomastoids and the splenius, have large cross-sectional areas and lever arms, and are thus of greatest importance when it comes to generating torque. The effect of each of these muscles, in terms of head movements, is a function of their generated torque, and this is why the splenius and the sternocleidomastoids are the two most important muscle pairs when studying the head rotation.

The prime movers in the head rotation are the two sternocleidomastoid muscles, which lie on opposite sides of the neck. The actions that these muscles control vary according to whether one or both of the sternocleidomastoid muscles are activated. When only one muscle is activated, the head is tipped toward the shoulder on the same side and is rotated to direct the face toward the opposite side. The sternocleidomastoids are therefore opposite side rotators, so when the sternocleidomastoids on one side and the splenius on the other contracts, the head rotates horizontally toward the contracted splenius. When the muscles act together on the same side, the head is moved forward. The purely horizontal movement is thus produced when only one of the sternocleidomastoid muscles is active, combined with the actions of the other muscles that, among other things, oppose the tipping of the head (see, for example, [1–4]).

What we want to do in this paper is to model the complex behavior of all those muscles, primarily the sternocleidomastoids and the splenius, in such a way that the rotation of the head is given account for in a simple way. Therefore we chose to model the muscles as just one pair of muscles, conducting the same actions as all the five muscles together. These muscles can be thought of as the sternocleidomastoid muscles, even though they really are not these muscles. This is because we are more interested in the principles of the controls behind the muscular contractions, than in finding an exact muscular model at the price of clarity.

2.1.2. Mathematical model of the rotational muscular action. We chose to model these muscles as damped springs with a second order linear dynamics of the form

$$\ddot{x} = -k(x - L) - g\dot{x} + v(t), \tag{1}$$

where L and x are the lengths of the unstretched and the stretched spring respectively, and k and g are frequency and damping parameters of the spring. A controller, $v(t)$, is added to the spring, and the control term is only added to one of the two muscles at a time, since only one muscle is active when the head is rotating. If we let $v_1(t)$ and $v_2(t)$ be the controllers added to the left and the right muscle respectively, this condition can be stated as

$$v_1(t)\, v_2(t) = 0 \quad \forall t. \tag{2}$$

This might not be the best way to model the muscles, since an actual muscle is much more complicated than a spring, but this model suffices to simplify the problem, and works satisfactorily for the purpose of mathematical calculations [5].

If the angle θ is chosen to be the system state variable, as seen in Figure 1, then two different cases can be distinguished:

i) $\theta > 0$

The movement of the head is decided by

$$\ddot{\theta} I = N, \tag{3}$$

where N is the torque, which is given by the difference between the two tangential forces $F_1 \cos \beta$ and F_2. If we now let x_1 and x_2 be the lengths of the left and the right spring respectively, and consider the fact that we now have two springs affecting the lengths simultaneously, we get

$$\begin{aligned}\ddot{x}_2 = {} & k(\cos \beta(x_1 - L) - (x_2 - L)) \\ & + g(\cos \beta \dot{x}_1 - \dot{x}_2) - \cos \beta v_1(t) + v_2(t),\end{aligned} \tag{4}$$

since only the tangential part of the left spring affects the rotation. x_2 is simply given by

$$x_2 = R\theta + L, \tag{5}$$

and therefore

$$\ddot{x}_2 = R\ddot{\theta} \Rightarrow \ddot{\theta} = \frac{\ddot{x}_2}{R}. \tag{6}$$

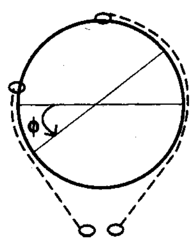


FIG. 1. The two forces producing a rotation of the head.

This gives us

$$\ddot{\theta} = \frac{1}{R}\{(k(x_1 - L) + g\dot{x}_1 - v_1(t))\cos\beta + k(L - x_2) - g\dot{x}_2 + v_2(t)\}. \tag{7}$$

We now want $\ddot{\theta}$ to be a function of θ and $\dot{\theta}$, so x_1, x_2, $\dot{x}_1$, $\dot{x}_2$ and $\cos\beta$ in (7) must be expressed in terms of θ and $\dot{\theta}$. The Law of Cosines directly gives

$$\begin{aligned} x_1^2 &= L^2 + \left(2R\sin\frac{\theta}{2}\right)^2 - 2L\left(2R\sin\frac{\theta}{2}\right)\cos\frac{\theta}{2} \\ &= 4R^2\sin^2\frac{\theta}{2} + L^2 - 2LR\sin\theta, \end{aligned} \tag{8}$$

$$\dot{x}_1 = \frac{R(R\sin\theta - L\cos\theta)\dot{\theta}}{x_1} \tag{9}$$

and $\cos\beta$ is given by

$$\cos\beta = \frac{L\cos\theta - R\sin\theta}{x_1}. \tag{10}$$

Now, let

$$h(\theta) = \frac{x_1\cos\beta}{R} \tag{11}$$

and

$$g(\theta) = -\frac{h(\theta)}{x_1}, \tag{12}$$

which gives us

$$\ddot{\theta} = f(\theta, \dot{\theta}) + g(\theta)v_1(t) + \frac{1}{R}v_2(t), \tag{13}$$

where

$$f(\theta,\dot{\theta}) = -\left(g\dot{\theta} + k\theta\right) + kLg(\theta) + h(\theta)\left[k + \frac{gR(R\sin\theta - L\cos\theta)\dot{\theta}}{L^2 + 4R^2\sin^2\frac{\theta}{2} - 2LR\sin\theta}\right], \tag{14}$$

$$g(\theta) = -h(\theta)\frac{1}{\sqrt{L^2 + 4R^2\sin^2\frac{\theta}{2} - 2LR\sin\theta}} \tag{15}$$

and

$$h(\theta) = \frac{1}{R}(L\cos\theta - R\sin\theta). \tag{16}$$

ii) $\theta < 0$

Symmetry considerations directly gives this case. If we let $\tilde{\theta} = -\theta$ $(\theta < 0)$, we get

$$\ddot{\tilde{\theta}} = f\left(\tilde{\theta},\dot{\tilde{\theta}}\right) + g(\tilde{\theta})v_2(t) + \frac{1}{R}v_1(t). \tag{17}$$

But $\ddot{\tilde{\theta}} = -\ddot{\theta}$, which gives us

$$\ddot{\theta} = -f(-\theta,-\dot{\theta}) - g(-\theta)v_2(t) - \frac{1}{R}v_1(t). \tag{18}$$

If we let $v_2(t) = 0$ when $\dot{\theta} > 0$ and $v_1(t) = 0$ when $\dot{\theta} < 0$, we get the total system to be

$$\ddot{\theta} = \operatorname{sign}(\theta)f\left(|\theta|,\operatorname{sign}(\theta)\dot{\theta}\right) + u(\theta)v(t), \tag{19}$$

where

$$u(\theta) = \begin{cases} \operatorname{sign}(\theta)\, g(|\theta|) & \text{if } \operatorname{sign}(\theta) = \operatorname{sign}(\dot{\theta}) \\ \operatorname{sign}(\theta)\dfrac{1}{R} & \text{if } \operatorname{sign}(\theta) \neq \operatorname{sign}(\dot{\theta}) \end{cases} \tag{20}$$

and

$$v(t) = \begin{cases} v_1(t) & \text{if } \dot{\theta} > 0 \\ v_2(t) & \text{if } \dot{\theta} < 0. \end{cases} \tag{21}$$

This is thus the model that will be used further on for describing the dynamics underlying the horizontal rotation of the head.

2.2. Dynamics of Ocular Motion

2.2.1. Muscular anatomy of the eye. Since the main interest in this paper lies on finding controls that make the eyes and the neck act together in a satisfying way, only monocular vision is being studied. Monocular vision means that we only use one eye, located in the middle of the head, between the actual eyes of a human being, but this is not a serious restriction since the binocular case can be derived in almost the same way as the monocular case [5].

The human eye, which is nearly spherical in shape, is controlled by a set of seven muscles. These are the *evator palpebra*, that raises the upper eyelid, the *superior* and *inferior recti*, which, together with the *superior* and *inferior oblique*, causes an upward or downward rotation of the eye, and finally the *external* and *internal recti*, that causes a horizontal rotation of the eye.

In the eye case, as well as in the neck case, we only focus on the muscle pair, the external and internal recti, that produce the horizontal movements. These two muscles are nicer to work with than the complex muscular configuration in the human neck, since they produce a purely horizontal rotation of the eye, completely without other muscles involved in the action.

2.2.2. Mathematical model of the external and internal recti. The external and internal recti both attach on the so called Annulus of Zinn, behind the eye, and they also attach rather high up on the eye itself, which makes the modeling a bit easier than in the head case, since the geometry is simplified by the fact that the forces, produced by the two muscles, can be assumed to always be tangential to the eye itself (Figure 2).

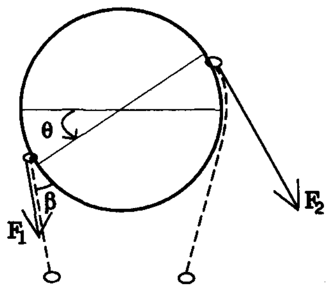


FIG. 2. The geometry of the external and internal recti.

As well as in the neck case, two different cases need to be taken care of, and if ϕ is the system state variable, those cases are:

i) $\phi > 0$

$$\ddot{\phi} I = r(F_1 - F_2), \tag{22}$$

where $I = \frac{2}{5}mr^2$ is the moment of inertia of the disc on which the two forces are acting. These forces are generated by

$$\ddot{x}_2 = k(x_1 - l - (x_2 - l)) + g(\dot{x}_1 - \dot{x}_2) - v_1(t) + v_2(t). \tag{23}$$

We also know that

$$x_1 = l - r\phi \tag{24}$$

and

$$x_2 = l + r\phi. \tag{25}$$

Equations (23–25) give us that

$$\ddot{\phi} = -2(g\dot{\phi} + k\phi) + \frac{1}{r}(v_2(t) - v_1(t)). \tag{26}$$

ii) $\phi < 0$

Symmetry directly gives that

$$-\ddot{\phi} = 2(g\dot{\phi} + k\phi) + \frac{1}{r}(v_1(t) - v_2(t)), \tag{27}$$

so the total system can be expressed as

$$\ddot{\phi} = -2(g\dot{\phi} + k\phi) - \operatorname{sign}(\dot{\phi})\frac{1}{r}v(t), \tag{28}$$

with

$$v(t) = \begin{cases} v_1(t) & \text{if } \dot{\phi} > 0 \\ v_2(t) & \text{if } \dot{\phi} < 0. \end{cases} \tag{29}$$

2.3. The Combined Dynamics

We return to our initial problem: How do we combine the movements of the head and the eye in order to follow an object with a given trajectory at a constant distance from the head?

Figure 3 directly gives the equation

$$\psi(t) = \theta(t) + \gamma(t), \tag{30}$$

if $\psi(t)$ is the tracked object's trajectory.

We had assumed that the object moved at a constant distance, D, from the head, which gives the following:

$$d = D + R. \tag{31}$$

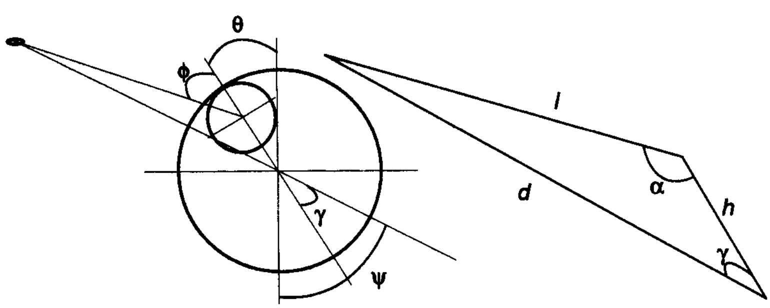


FIG. 3. The geometry behind the combined movement.

We also know that

$$h = R - r \tag{32}$$

and

$$\alpha = \pi - \phi. \tag{33}$$

This, together with the Law of Cosines, gives us

$$d^2 = h^2 + l^2 - 2hl\cos\alpha = h^2 + l^2 + 2hl\cos\phi, \tag{34}$$

$$l^2 = d^2 + h^2 - 2\,dh\cos\gamma \tag{35}$$

and therefore

$$\phi = \arccos\left(\frac{d\cos(\gamma) - h}{\sqrt{d^2 + h^2 - 2\,dh\cos(\gamma)}}\right). \tag{36}$$

We must remember that only the case $\phi > 0$ ($\Rightarrow \gamma > 0$) has been studied. If we allow ϕ to be negative, we get

$$\phi = \mathrm{sign}(\gamma)\arccos\left(\frac{d\cos(\gamma) - h}{\sqrt{d^2 + h^2 - 2\,dh\cos(\gamma)}}\right). \tag{37}$$

Now let

$$\eta(\gamma) = \frac{d\cos\gamma - h}{\sqrt{d^2 + h^2 - 2hd\cos\gamma}}. \tag{38}$$

We then have

$$\phi = \mathrm{sign}(\gamma)\arccos\eta(\gamma), \tag{39}$$

$$\dot{\phi} = -\mathrm{sign}(\gamma)\frac{1}{\sqrt{1 - \eta(\gamma)^2}}\frac{d\eta(\gamma)}{d\gamma}\dot{\gamma} \tag{40}$$

and

$$\ddot{\phi} = -\mathrm{sign}(\gamma)\frac{1}{\sqrt{1 - \eta(\gamma)^2}} \cdot\left(\frac{d^2\eta(\gamma)}{d\gamma^2}\dot{\gamma}^2 + \frac{d\eta(\gamma)}{d\gamma}\ddot{\gamma} + \left(\frac{d\eta(\gamma)}{d\gamma}\gamma\right)^2\frac{\eta(\gamma)}{1 - \eta(\gamma)^2}\right). \tag{41}$$

This can be stated in a more compact form as

$$\ddot{\phi} = F\left(\theta, \psi, \dot{\theta}, \dot{\psi}, \ddot{\theta}, \ddot{\psi}\right), \tag{42}$$

but from (28) we still have the linear equation

$$\ddot{\phi} = -2(g\dot{\phi} + k\phi) - \operatorname{sign}(\dot{\phi})\frac{1}{r}v(t). \tag{43}$$

Combining these two equations makes it possible to impose a control on $\ddot{\theta}$, and then let the control on $\ddot{\phi}$ be given automatically as

$$v_{\text{eye}}(t) = -\operatorname{sign}(\dot{\phi})\, r\left[F\left(\theta, \psi, \dot{\theta}, \dot{\psi}, \ddot{\theta}, \ddot{\psi}\right) + 2(g\dot{\phi} + k\phi)\right]. \tag{44}$$

This way of letting the main tracking be done by the eye is a product of the so called *occulocentric view.* This means that the main tracking is performed by the eye, while the head is just moving in a general way, as seen in the next section. This approach is a rather reasonable one, since the fast *saccadic* movements of the eye make the eye better suited for following fast movements than the head [6].

2.4. Physical Constants

Before we can start doing simulations, the physical constants involved in the modeling need to be determined. The dynamics of ocular motion is discussed in [5], where the following data were used:

$$r \approx 1.27\,\text{cm}, \tag{45}$$

$$m \approx 10.7\,\text{g}, \tag{46}$$

$$l \approx 3\,\text{cm} \tag{47}$$

and

$$\phi_{\max} \approx 1.17\,\text{rad}. \tag{48}$$

Next we need to find the spring parameters k and g. One choice that in [5] proved to work satisfactorily, and that does not make the muscles too stiff is

$$k = 20\,\text{s}^{-2}, \tag{49}$$

$$g = 8\,\text{s}^{-1}. \tag{50}$$

From [7] we get that

$$M \approx 4.5\ \text{kg}, \tag{51}$$

$$R \approx 11\ \text{cm}. \tag{52}$$

Since our neck muscle length, L, is the projection of the muscles that run along the neck onto a two dimensional plane, orthogonal to the neck itself, we get that

$$L \approx 6\ \text{cm}, \tag{53}$$

if we assume that the length is approximately half of the neck's diameter. This finally gives us that

$$\theta_{\max} = \arctan\frac{L}{R} \approx 0.50\ \text{rad}, \tag{54}$$

where $\theta_{\max}$ is the angle of rotation at which the length of the spring has its minimum. This value on $\theta_{\max}$ is somewhat smaller than what you get when you sum up the actual rotations between all the individual cervial joints that are involved in the head rotation [3], but this need not be a serious restriction since we can simply say that we only work with values within $\pm\,\theta_{\max}$.

3. CONTROL LAWS

Now that we have a model for the combined process of activating both the muscles of the neck and of the eye, the next task is to find the control laws. We want the pupil to follow a smooth trajectory, and in order to accomplish this, we need to find controls that make the pupil movement both smooth and completely determined by the tracked object's position. We also want the two separate movements, those of the head and those of the eye, to be continuously derivable at least three times. This smoothness constraint is given by more or less the fact that we want to control models of actual muscles, whose position, velocity, and acceleration appear to be continuously derivable functions of time.

But this constraint is not enough. We do not only want to find any control law that does what we want, we want to find one that does it well. Therefore we need a criterion by which we can determine how good any given solution is. A reasonable approach is to try to minimize the energy produced in the movement, and since the mass of the head, M, is so much larger than the mass of the eye, m, one criterion for finding our control could be that it should make the angular acceleration of the head as small as possible. This would make the energy, given by the torque, small since

$$E = \int_{\theta_0}^{\theta_f} |N\,d\theta| = \int_{\theta_0}^{\theta_f} |\ddot{\theta} I\,d\theta|. \tag{55}$$

3.1. The Linear Trajectory

But we do not want to find the controls by using, for instance, Pontryagins Maximum Principle on

$$\min \int_{\theta_0}^{\theta_f} |\ddot{\theta}\,d\theta| = \min \int_{t_0}^{t_f} |\ddot{\theta}\dot{\theta}|\,dt \qquad (t_0 \leqslant t_f), \tag{56}$$

since such an approach seems rather unphysiological, and we do not want to have to determine the whole movement in advance. Instead we want to divide the trajectory of the head into subparts, where in some parts the head accelerates, and in others the angular acceleration is zero. This is because one obvious control that makes $|\ddot{\theta}|$ small is the one that makes $\ddot{\theta} = 0$. Therefore we want the major part of the trajectory to be of this type.

If we assume that we start at θ_0 and stop at θ_f, what we want is the following scenario:

$$\dot{\theta}(t) = \frac{\theta(t) - \theta_f}{t - t_f} = \text{const} \qquad t \in [t_0, t_f], \tag{57}$$

$$\ddot{\theta}(t) = 0 \qquad t \in [t_0, t_f]. \tag{58}$$

If we recall Equations (19–21), we directly see that

$$\begin{aligned} 0 &= \operatorname{sign}(\theta) f\big(|\theta|, \operatorname{sign}(\theta)\dot{\theta}\big) + u(\theta)\, v_{\text{lin}}(t) \\ v_{\text{lin}}(t) &= -\frac{\operatorname{sign}(\theta) f\big(|\theta|, \operatorname{sign}(\theta)\dot{\theta}\big)}{u(\theta)}. \end{aligned} \tag{59}$$

The general idea can be seen in Figure 4, where the feedback is needed for two reasons. If the model is not perfect, the calculated inverse model would not be the exact inverse of the actual, direct model when implemented in,

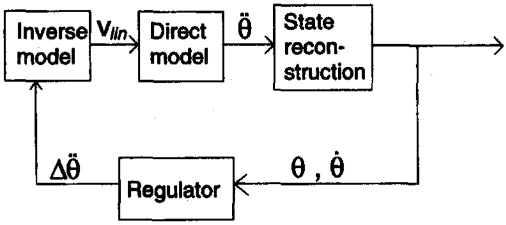


FIG. 4. Block chart for the zero acceleration case.

let's say, a robot, so the feedback would serve as a protection against model errors. We also use the feedback when we are doing simulations as a protection against numerical inaccuracy, but if the inverse model was perfect, there would be no real need for feedback, and the regulator can thus be designed by linear methods, since the model errors are assumably small. We chose to use a feedback on the form

$$\Delta\ddot{\theta} = C\left(\frac{\theta(t) - \theta_f}{t - t_f} - \dot{\theta}(t)\right). \tag{60}$$

3.2. Acceleration Periods

This linear approach is unfortunately not enough. First of all, we assume that we start following the object when the head and the eye both are at rest at some fixed angle, and therefore we need to find controls that can accelerate the systems up to some suitable velocity when the tracking is initiated. Secondly, when the followed trajectories are not well behaved, we have to take into account that that the eye may rotate out of bound if no modification of the head's zero acceleration trajectory is being made. These two cases show that we need to be able to accelerate the head in a controlled way in some situations.

Inspired by the feed forward control system concept in [8], the general idea behind the control laws we chose to use, can be illustrated by the block chart in Figure 5.

Each period of acceleration has a duration of Δt, and it starts at t_a and ends at t_b ($\Delta t = t_b - t_a$), and from the previous period we are given $\theta(t_a)$, $\dot{\theta}(t_a)$ and $\ddot{\theta}(t_a)$. $d^3\theta(t_a)/dt^3$ is given by a simple approximation:

$$\frac{d^3\theta(t_a)}{dt^3} \approx \frac{\ddot{\theta}_{\text{lin}}(t_a) - \ddot{\theta}_{\text{lin}}(t_a - h)}{h}, \tag{61}$$

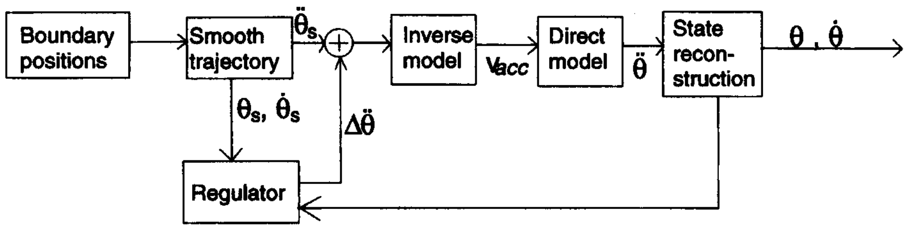


FIG. 5. Block chart for the feed forward acceleration case.

where $\ddot{\theta}_{\text{lin}}$ is the angular acceleration given by the control defined in the previous subsection, producing the constant angular velocity of the head. Therefore $d^3\theta(t_a)/dt^3 \approx 0$, but for the same reasons as for the need of feedback, we use (61) instead of simply letting $d^3\theta(t_a)/dt^3 = 0$. The reason why we need this third derivative in the first place is because we want $\ddot{\theta}$ to be continuously derivable, which makes us need some control over the third derivative.

At the other boundary, we chose a value for $\theta(t_b)$, and, as seen in the following subsections, the value for $\dot{\theta}(t_b)$ will be depending on what acceleration case is being examined. Since the idea is to have $\ddot{\theta} = 0$ most of the time, we let $\ddot{\theta}(t_b) = 0$, and with almost the same approximation as before, we get $d^3\theta(t_b)/dt^3$.

We now move on to deciding what type of function we want to have when we describe the angular acceleration during the acceleration periods. If we try to use a polynomial, which, for calculation reasons, is a good choice, we need a polynomial for describing $\theta(t)$ with a degree of at least seven. This is because we need eight coefficients since we have eight conditions that need to be fulfilled (four conditions at t_a, and four at t_b). If we let

$$\theta(t) = \frac{1}{42}(t - t_a)^7 C_1 + \frac{1}{30}(t - t_a)^6 C_2 + \frac{1}{20}(t - t_a)^5 C_3$$
$$+ \frac{1}{12}(t - t_a)^4 C_4 + \frac{1}{6}(t - t_a)^3 C_5 + \frac{1}{2}(t - t_a)^2 C_6$$
$$+ (t - t_a) C_7 + C_8, \tag{62}$$

we get the following, well-defined equation system:

$$T = \begin{pmatrix} 5\Delta t^4 & 4\Delta t^3 & 3\Delta t^2 & 2\Delta t \\ \Delta t^5 & \Delta t^4 & \Delta t^3 & \Delta t^2 \\ \frac{1}{6}\Delta t^6 & \frac{1}{5}\Delta t^5 & \frac{1}{4}\Delta t^4 & \frac{1}{3}\Delta t^3 \\ \frac{1}{42}\Delta t^7 & \frac{1}{30}\Delta t^6 & \frac{1}{20}\Delta t^5 & \frac{1}{12}\Delta t^4 \end{pmatrix}, \tag{63}$$

$$C = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}, \tag{64}$$

$$D = \begin{pmatrix} \dfrac{d^3\theta(t_b)}{dt^3} - \dfrac{d^3\theta(t_a)}{dt^3} \\ \ddot{\theta}(t_b) - \ddot{\theta}(t_a) - \Delta t \dfrac{d^3\theta(t_a)}{dt^3} \\ \dot{\theta}(t_b) - \dot{\theta}(t_a) - \Delta t \ddot{\theta}(t_a) - \dfrac{1}{2}\Delta t^2 \dfrac{d^3\theta(t_a)}{dt^3} \\ \theta(t_b) - \theta(t_a) - \Delta t \dot{\theta}(t_a) - \dfrac{1}{2}\Delta t^2 \ddot{\theta}(t_a) - \dfrac{1}{6}\Delta t^3 \dfrac{d^3\theta(t_a)}{dt^3} \end{pmatrix} \tag{65}$$

and

$$TC = D. \tag{66}$$

This gives us all we need to know, and if we want to find the control producing this trajectory, we simply use the inverse for $\ddot{\theta}$

$$v_{\text{acc}}(t) = \frac{\ddot{\theta}_{\text{poly}} + \Delta\ddot{\theta} - \text{sign}(\theta) f(|\theta|, \text{sign}(\theta)\dot{\theta})}{u(\theta)}, \tag{67}$$

where $\ddot{\theta}_{\text{poly}}$ is the desired trajectory, and $\Delta\ddot{\theta}$ is the linear feedback. We chose to model the feedback on the form

$$\Delta\ddot{\theta}(t) = C_1\big(\theta_{\text{poly}}(t) - \theta_{\text{actual}}(t)\big) + C_2\big(\dot{\theta}_{\text{poly}}(t) - \dot{\theta}_{\text{actual}}(t)\big). \tag{68}$$

We now have to investigate how the different cases can appear.

3.2.1. Start/ stop. When the pupil starts following the object, it goes from rest at some angle $\psi_0(\theta_0, \phi_0)$, accelerates to the proper velocity and acceleration, and then follows the object's trajectory, with the head having a constant angular acceleration. We know that the activation and deactivation time for the head is approximately 0.05s [9], and therefore we chose

$$\Delta t = 0.1, \tag{69}$$

which makes our model physically possible with large margins.

We also prescribe that

$$\theta(t_b) = \theta_0 - \delta\theta, \tag{70}$$

if the object is moving from the left to the right. The value on $\delta\theta$ depends on, among other things, how big we can allow $|\ddot{\theta}|$ to be. There are different models for finding the acceleration limit for the head, and one is the measure known as the Gadd Sverity Index (GSI). This states that

$$100 > \text{GSI} = \int_0^T a^{2.5}\, dt, \tag{71}$$

where T is the duration of the acceleration pulse. This index gives the following approximate guideline values for safe acceleration.

T	0.1	0.01	0.005
$\ddot{\theta}_{\max}$	40	60	100

This index concerns direct acceleration, G, and not angular acceleration which is considered a much more safe form of acceleration, since the injurious potential to cause cerebral contusion is smaller in a rotational movement, so when it comes to angular acceleration, much higher values are tolerated. Based on a model by Lowenheim, the head can be exposed to as much as 4500 rad/s^2 before being damaged. However, these values are not undisputed, and therefore a safe limit with large margins could be

$$|\ddot{\theta}| < 500\,\text{rad/s}^2. \tag{72}$$

When the head movements are made as quickly as possible, their angular velocities range up to 15 rad/s.

For the eye, the case is somewhat different. The eye is able to accelerate much faster than the head, and in the so called *saccadic* movement, we can have $\dot{\phi}$ as big as 700 rad/s, and $\ddot{\phi}$ up to 10000 rad/s^2.

Given these constraints on $|\ddot{\theta}|$ and $|\ddot{\phi}|$, found in [7, 9–11], we now can move on to actually finding the desired trajectories. In the start case we have

$$\begin{cases} \theta(t_a) = \theta_0 \\ \dot{\theta}(t_a) = 0 \\ \ddot{\theta}(t_a) = 0 \\ \dfrac{d^3\theta(t_a)}{dt^3} = 0 \end{cases} \tag{73}$$

$$\begin{cases} \theta(t_b) = \theta_0 - \delta\theta \\ \dot{\theta}(t_b) = \dfrac{\theta(t_b) - \theta_f}{t_b - t_f} \\ \ddot{\theta}(t_b) = 0 \\ \dfrac{d^3\theta(t_b)}{dt^3} \approx \dfrac{\ddot{\theta}(t_b + h) - \ddot{\theta}(t_b)}{h} \end{cases} \tag{74}$$

and

$$\begin{cases} \phi(t_a) = \phi_0 \\ \dot{\phi}(t_a) = 0 \\ \ddot{\phi}(t_a) = 0 \\ \dfrac{d^3\phi(t_a)}{dt^3} = 0. \end{cases} \tag{75}$$

From Equations (39–41) we also get

$$\begin{cases} \phi(\mathrm{t_b}) = \mathrm{sign}(\gamma)\arccos\gamma \\ \dot{\phi}(\mathrm{t_b}) = -\mathrm{sign}(\gamma)\dfrac{1}{\sqrt{1 - \eta(\gamma)^2}}\dfrac{d\eta(\gamma)}{d\gamma}\dot{\gamma} \\ \ddot{\phi}(\mathrm{t_b}) = \mathrm{F}(\gamma, \dot{\gamma}, \ddot{\gamma}) \\ \dfrac{d^3\gamma(t_b)}{dt^3} \\ \quad \approx \dfrac{F(\gamma(t_b + h), \dot{\gamma}(t_b + h), \ddot{\gamma}(t_b + h)) - F(\gamma(t_b), \dot{\gamma}(t_b), \ddot{\gamma}(t_b))}{h}, \end{cases} \tag{76}$$

with $\gamma(t_b) = \psi(t_b) - \theta_{\text{lin}}(t_b)$. If we use these values, and let $\delta\theta = 0.02$, we get a start trajectory for the head seen in Figure 6.

In the stop case, we have all the information at $t = t_a$ given from the previous period, and at $t = t_b$ we use similar boundary conditions as in the start case.

We now have enough knowledge to cope with a large set of situations (see Figure A.1 in appendix) the trajectory for $\psi(t) = \psi_0 - (t - t_0)$ can be seen with its control signals.

3.2.2. Corrections. So far so good, but if the object's trajectory is on the form $\psi(t) = \psi_0 - (t - t_0)^2$, we find that the purely linear case ($\ddot{\theta} = 0$) makes the eye rotate out of limit if ϕ_0 is close to $\phi_{\max}$. Therefore we need to make corrections in the head movements to prevent this case from happening. This can be done by using the polynomial accelerations from the previous sections. We therefore need to specify $\phi(t_b)$ and $\dot{\phi}(t_b)$ in a smart way. We start by letting

$$\phi(t_b) = \phi(t_a) - \text{sign}(\phi(t_a))\left|\text{sign}(\dot{\psi}(t_a)) - C\,\text{sign}(\phi(t_a))\right|\delta\phi, \quad (77)$$

with $C \in (0.1)$ because when $\text{sign}(\dot{\psi}(t_a)) = \text{sign}(\phi(t_a))$ we need to cancel out the movement induced by the object by making the eye move away from its limit without making the acceleration of the head too big. Now we need the inverses of Equations (39–40) in order to determine the head position

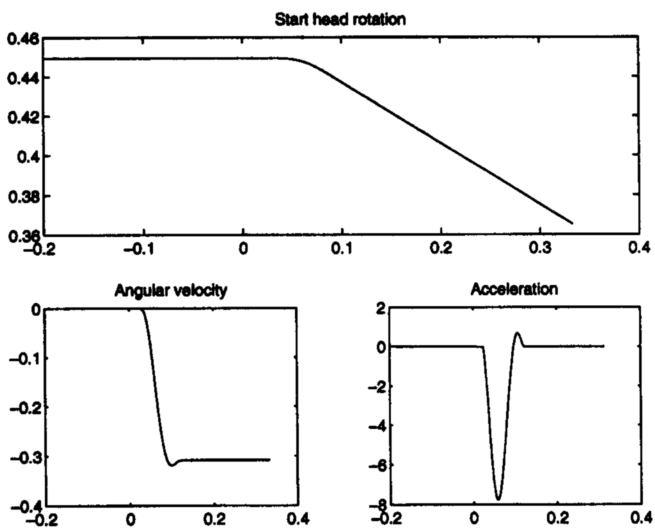


FIG. 6. Head rotation when starting with zero velocity.

and velocity at $t = t_b$. From (34–35) we know that

$$d^2 = l^2 + 2h^2 + 2hl\cos\phi - 2h\sqrt{h^2 + l^2 + 2hl\cos\phi}\,\cos\gamma, \tag{78}$$

with

$$l \approx D + r = \text{const}, \tag{79}$$

which gives us that

$$\gamma = \operatorname{sign}(\phi)\arccos\left(\frac{h + l\cos\phi}{\sqrt{h^2 + l^2 + 2hl\cos\phi}}\right). \tag{80}$$

If we now let

$$\xi(\phi) = \frac{h + l\cos\phi}{\sqrt{h^2 + l^2 + 2hl\cos\phi}} \tag{81}$$

and remember that $\gamma = \psi - \theta$, we get

$$\theta = \psi - \operatorname{sign}(\phi)\arccos\xi(\phi) \tag{82}$$

and

$$\dot{\theta} = \dot{\psi} + \operatorname{sign}(\phi)\frac{1}{\sqrt{1 - \xi(\phi)^2}}\frac{d\xi(\phi)}{d\phi}\dot{\phi}. \tag{83}$$

Equation (82) gives us $\theta(t_b)$, and if we now prescribe for $\dot{\phi}(t_b)$ to be

$$\dot{\phi}(t_b) = -\operatorname{sign}(\phi(t_a))\left|\operatorname{sign}\left(\dot{\psi}(t_a)\right) - \operatorname{sign}(\phi(t_a))\right|, \tag{84}$$

we make the eye rotate away from the unwanted position. This also gives us $\dot{\theta}(t_b)$ according to (83), and if we let $\ddot{\theta}(t_b) = 0$, and calculate $d^3\theta(t_b)/dt^3$ with the same approximation as in the start case, we automatically have both $\ddot{\phi}(t_b)$ and $d^3\phi(t_b)/dt^3$.

But the situation we then end up with when $t > t_b$ can only go on for a limited period of time. If we find that $|\theta|$ is too close to $|\theta_{max}|$, or if we no longer need to actively make the eye move away from its limit, we must force the trajectories back into their normal, zero acceleration states. The case when correction is no longer needed is when the normal head velocity, and the then implied eye velocity, both behave in an acceptable way.

The forcing back to the normal states is done by repeating the polynomial acceleration procedure, with the values at $t = t_a$ given in the same way as in the stop case, and the values at $t = t_b$ as in the start case, with the exception that

$$\theta(t_b) = \theta(t_a) - \operatorname{sign}(\dot{\psi}(t_a))\left|\operatorname{sign}(\dot{\psi}(t_a)) - C\operatorname{sign}(\dot{\theta}(t_a))\right|\delta\theta. \quad (85)$$

This strategy yields results shown in Figure A.2 (in the appendix), for the case $\psi(t) = \psi_0 - (t - t_0)^2$.

4. OPTIMAL CONTROL

Now we have found control laws for controlling the combined rotational movements, and we suspect that these controls yield reasonably good solutions. The method we used for determining how efficient any given solution was, was by calculating the energy produced in the movement. The energy for the trajectories we have studied was calculated by a simple approximation:

$$E = \int_{t_0}^{t_f} |\ddot{\theta}\dot{\theta}|\, I\, dt \approx I \sum_{i=1}^{n} \Delta t_i \left|\ddot{\theta}(t_i)\,\dot{\theta}(t_i)\right|, \quad (86)$$

for a given set of n points on the interval $[t_0, t_f]$, with $t_0 \leqslant t_1 < t_2 < \cdots < t_{n-1} < t_n \leqslant t_f$. This approximative method gave the following results:

$$\begin{aligned} \psi(t) &= \psi_0 - (t - t_0) \Rightarrow E \approx 2.36 \cdot 10^{-3}\ \mathrm{J} \\ \psi(t) &= \psi_0 - (t - t_0)^2 \Rightarrow E \approx 6.42 \cdot 10^{-2}\ \mathrm{J}. \end{aligned} \quad (87)$$

We now need something to compare these values with in order to find out if our solutions can be regarded as satisfactory.

4.1. *The Piecewise Linear Approach*

The search for optimal controls can be conducted in two ways. The first one, investigated in this subsection, is to assume that our previous strategy of letting the head move with piecewise constant angular velocity, is a rather good one. Therefore we must try to find the optimal way of placing the acceleration periods in the time domain, and then compare the found solution with the ones from the previous chapter. If we recall that the produced energy was $6.42 \cdot 10^{-2}$ J when $\psi(t) = \psi_0 - (t - t_0)^2$, we can work with this trajectory and see what we get when we try to minimize the energy produced in the movement. The numerical strategy we are going to use is to divide the time interval into n and m points respectively, and start the correction and the forcing back to normal at these two points. With a discretization choice of

$$\begin{aligned} n = 16, \qquad & i = 1, 2, \ldots, n \\ t_{\text{start}} = & \, i\frac{(t_f - t_0)}{n + 1} \end{aligned} \tag{88}$$

and

$$\begin{aligned} m = n - i, \qquad & j = 1, 2, \ldots, m \\ t_{\text{stop}} = & \, j\frac{(t_f - t_{\text{start}})}{m + 1}, \end{aligned} \tag{89}$$

we find that the best allowed solution gives us

$$E = 6.17 \cdot 10^{-2} \text{ J}. \tag{90}$$

If we compare this with our previous result, $6.42 \cdot 10^{-2}$ J, we can be rather satisfied with our control strategy, since the difference is less than 10%, which must be regarded as a rather good result.

4.2. *The Quadratic Minimization Problem*

The second question we must ask, when it comes to finding optimal controls, is of course whether the piecewise linear approach is in itself a good one. We therefore need to compare our two results,

$$E = 6.42 \cdot 10^{-2} \text{ J} \tag{91}$$

and

$$E = 6.17 \cdot 10^{-2} \text{ J}, \tag{92}$$

with the value we get when we try to find the best of all possible solutions. Since we suspect that $\ddot{\theta}$ will be small at all times, we choose to investigate the minimization problem

$$\min \int_{t_0}^{t_f} \ddot{\theta}^2 \, dt \tag{93}$$

instead of

$$\min \int_{t_0}^{t_f} |\ddot{\theta}\dot{\theta}| \, dt \tag{94}$$

for the sake of simplicity, since the quadratic problem is likely to be less complicated, and hope that we still get a reasonably good solution anyway. The reason for this approach is that we are not really interested in the solution more than as a mere comparison to our previous solutions, in order to get a feeling for how good they are. If we start by letting

$$\begin{cases} x_1 = \theta \\ x_2 = \dot{\theta} \\ x_3 = \phi \\ x_4 = \dot{\phi} \\ x_5 = \psi \\ x_6 = \dot{\psi}, \end{cases} \tag{95}$$

the minimization problem becomes

$$\min \int_{t_0}^{t_f} (\text{sign}(x_1) f(|x_1|, \text{sign}(x_1) x_2) + u(x_1) v)^2 \, dt, \tag{96}$$

under the dynamics

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \operatorname{sign}(x_1) f(|x_1|, \operatorname{sign}(x_1) x_2) + u(x_1) v \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = F(x_1, x_2, \dot{x}_2, x_5, x_6, \dot{x}_6) \\ \dot{x}_5 = x_6 \\ \dot{x}_6 = -2 \\ x_1(t_0) = \theta_0,\ x_2(t_0) = 0,\ x_3(t_0) = \phi_0, \\ x_4(t_0) = 0,\ x_5(t_0) = \psi_0,\ x_6(t_0) = 0, \\ x_1(t_f) = \theta_f,\ x_2(t_f) = 0,\ x_3(t_f) = \phi_f, \\ x_4(t_f) = 0,\ x_5(t_f) = \psi_0 - (t_f - t_0)^2,\ x_6(t_f) = -2(t_f - t_0), \end{cases} \tag{97}$$

if we let

$$\psi(t) = \psi_0 - (t - t_0)^2. \tag{98}$$

This system of differential equations gives the following Hamiltonian function:

$$\begin{aligned} H(x, y, v) = {} & \left(\operatorname{sign}(x_1) f(|x_1|, \operatorname{sign}(x_1) x_2) + u(x_1) v\right)^2 + y_1 x_2 \\ & + y_2\left(\operatorname{sign}(x_1) f(|x_1|, \operatorname{sign}(x_1) x_2) + u(x_1) v\right) + y_3 x_4 \\ & + y_4 F(x_1, x_2, \dot{x}_2, x_5, x_6, \dot{x}_6) + y_5 x_6 - 2 y_6, \end{aligned} \tag{99}$$

where y is the dual state system variable along the optimal trajectory $\hat{x}$. According to Pontryagin's Maximum Principle [12], we have

$$\dot{y}_j = -\frac{\partial H(\hat{x}, y, \hat{v})}{\partial x_j}, \tag{100}$$

which gives the dual system as

$$
\begin{cases}
\dot{y}_1 = \left(\operatorname{sign}(\hat{x}_1)\dfrac{\partial f(|\hat{x}_1|, \operatorname{sign}(\hat{x}_1)\hat{x}_2)}{\partial x_1} + \dfrac{du(\hat{x}_1)}{x_1}\hat{v}\right) \\
\quad \cdot\left(-2(\operatorname{sign}(\hat{x}_1) f(|\hat{x}_1|, \operatorname{sign}(\hat{x}_1)\hat{x}_2) u(\hat{x}_1)\hat{v}) \right. \\
\quad \left. + - y_2 - y_4 \dfrac{\partial F(\hat{x}_1, \hat{x}_2, \hat{\dot{x}}_2, \hat{x}_5, \hat{x}_6, \hat{\dot{x}}_6)}{\partial \dot{x}_2} \right) \\
\quad - y_4 \dfrac{\partial F(\hat{x}_1, \hat{x}_2, \hat{\dot{x}}_2, \hat{x}_5, \hat{x}_6, \hat{\dot{x}}_6)}{\partial x_1} \\
\dot{y}_2 = \operatorname{sign}(\hat{x}_1)\dfrac{\partial f(|\hat{x}_1|, \operatorname{sign}(\hat{x}_1)\hat{x}_2)}{\partial x_2} \\
\quad \cdot\left(-2(\operatorname{sign}(\hat{x}_1) f(|\hat{x}_1|, \operatorname{sign}(\hat{x}_1)\hat{x}_2) + u(\hat{x}_1)\hat{v}) \right. \\
\quad \left. - y_2 - y_4 \dfrac{\partial F(\hat{x}_1, \hat{x}_2, \hat{\dot{x}}_2, \hat{x}_5, \hat{x}_6, \hat{\dot{x}}_6)}{\partial \dot{x}_2} \right) \\
\quad - y_4 \dfrac{\partial F(\hat{x}_1, \hat{x}_2, \hat{\dot{x}}_2, \hat{x}_5, \hat{x}_6, \hat{\dot{x}}_6)}{\partial x_2} - y_1 \\
\dot{y}_3 = 0 \\
\dot{y}_4 = -y_3 \\
\dot{y}_5 = -y_4 \dfrac{\partial F(\hat{x}_1, \hat{x}_2, \hat{\dot{x}}_2, \hat{x}_5, \hat{x}_6, \hat{\dot{x}}_6)}{\partial x_5} \\
\dot{y}_6 = -y_4 \dfrac{\partial F(\hat{x}_1, \hat{x}_2, \hat{\dot{x}}_2, \hat{x}_5, \hat{x}_6, \hat{\dot{x}}_6)}{\partial x_6} - y_5 .
\end{cases}
\tag{101}
$$

According to PMP, the optimal control law is given by

$$
H(\hat{x}, y, \hat{v}) = \min_{v \in \mathscr{U}} H(\hat{x}, y, v), \tag{102}
$$

where $\mathscr{U}$ is the set of allowed control laws. Equation (102) gives us that minimizing $H(\hat{x}, y, v)$ is equivalent to minimizing

$$\tilde{H}(\hat{x}, y, v)$$

$$= u(\hat{x}_1)^2 v^2 + 2\,\mathrm{sign}(\hat{x}_1)\, f(|\hat{x}_1|, \mathrm{sign}(\hat{x}_1)\hat{x}_2) u(\hat{x}_1) v + y_2 u(\hat{x}_1) v$$

$$+ y_4\, \mathrm{sign}(\hat{x}_5 - \hat{x}_1) \frac{1}{\sqrt{1 - \eta(\hat{x}_5 - \hat{x}_1)^2}} \frac{d\eta(\hat{x}_5 - \hat{x}_1)}{d(\hat{x}_5 - \hat{x}_1)} u(\hat{x}_1)\, v. \tag{103}$$

If we derivate this with respect to v, we get a necessary condition for the optimal control, $\hat{v}$, given by

$$0 = \frac{\partial \tilde{H}(\hat{x}, y, \hat{v})}{\partial v}$$

$$= 2\hat{v} u(\hat{x}_1)^2 + 2\,\mathrm{sign}(\hat{x}_1)\, f(|\hat{x}_1|, \mathrm{sign}(\hat{x}_1)\hat{x}_2) u(\hat{x}_1) + y_2 u(\hat{x}_1)$$

$$+ y_4\, \mathrm{sign}(\hat{x}_5 - \hat{x}_1) \frac{1}{\sqrt{1 - \eta(\hat{x}_5 - \hat{x}_1)^2}} \frac{d\eta(\hat{x}_5 - \hat{x}_1)}{d(\hat{x}_5 - \hat{x}_1)} u(\hat{x}_1). \tag{104}$$

If we now let

$$\sigma(\hat{x}_1, \hat{x}_2, \hat{x}_5, y_2, y_4)$$

$$= 2\,\mathrm{sign}(\hat{x}_1)\, f(|\hat{x}_1|, \mathrm{sign}(\hat{x}_1)\hat{x}_2) + y_2$$

$$+ y_4\, \mathrm{sign}(\hat{x}_5 - \hat{x}_1) \frac{1}{\sqrt{1 - \eta(\hat{x}_5 - \hat{x}_1)^2}} \frac{d\eta(\hat{x}_5 - \hat{x}_1)}{d(\hat{x}_5 - \hat{x}_1)}, \tag{105}$$

(104) gives us that

$$\hat{v} = -\frac{\sigma(\hat{x}_1, \hat{x}_2, \hat{x}_5, y_2, y_4)}{2u(\hat{x}_1)}. \tag{106}$$

In order for us to find this $\hat{v}$, we first need to solve a system of first order, coupled, nonlinear differential equations, with $\hat{v}$ given by (106) inserted in the equation system.

This boundary value problem has to be solved numerically, and if we use the shooting method on this problem [13, 14], we end up with a result shown in Figure A.3 (in the appendix), and the quadratic minimization of the produced energy gave us

$$E \approx 1.75 \cdot 10^{-2}\ \mathrm{J}, \tag{107}$$

which is somewhat better than our previous results. However, it should be stressed that this optimal control does not guarantee that the trajectories we end up with, are three times continuously derivable.

5. CONCLUSIONS

In this article, two issues have been discussed. The first one concerns the modeling of the actual system that describes the dynamics behind the neck and the eye muscles. This system was then used in the second part of the article when controls for the muscle movements were developed. Therefore the results from these two parts must be viewed as somewhat separate.

When it comes to the model, the weakest part is probably that of trying to model muscles as second order springs, since an actual muscle has a dynamic that is much more complicated than that. However, this approach has the major advantage that it makes the mathematics reasonably simple. It is also sufficiently complete when it comes to actually start thinking about how to control the head and the eye muscles simultaneously.

The control strategy we chose to use was based on a desire to keep the energy produced in the movement small, since we believed this to be a physically reasonable approach. We therefore let the angular acceleration of the head be zero most of the time, since this would make the energy small. Why the energy, produced by the eye, did not concern us was due to the fact that the mass of the eye is so much smaller than the mass of the head, and therefore the energy produced in the eye movement was assumed to be neglectable. This piecewise linear control strategy proved to be rather successful, when compared to the optimal movement. When minimizing the square of the angular acceleration of the head, the energy in both the piecewise linear case and in the optimal case, were of the same power of ten, which must be regarded as acceptable. However, it must be stressed that our control strategy only works when the tracked object's trajectory is known, since our controls are, among other things, based on a knowledge of the total time that tracking is to be conducted.

When it comes to physical adequacy, it can be worth comparing our results to the trajectories found in Guitton's *Eye–Head Coordination in Gaze Control* [6]. It turns out that our piecewise linear approach is not so bad after all, since actual combined movements seem to have somewhat of the same piecewise linear characteristics as our trajectories, even though they, of course, are more complex. This should not, however, disqualify our model as not being an interesting step toward an understanding of the complex behavior of human head–eye-coordination.

A. SIMULATIONS

In this appendix, the different graphs produced when simulating the movements are presented. The combined movement is calculated with Equation (82) as

$$\psi(t) = \theta + \operatorname{sign}(\phi)\arccos \xi(\phi),$$

while θ and ϕ are given by the dynamics investigated in Section 2.

The controls are presented for each individual muscle, with the notation

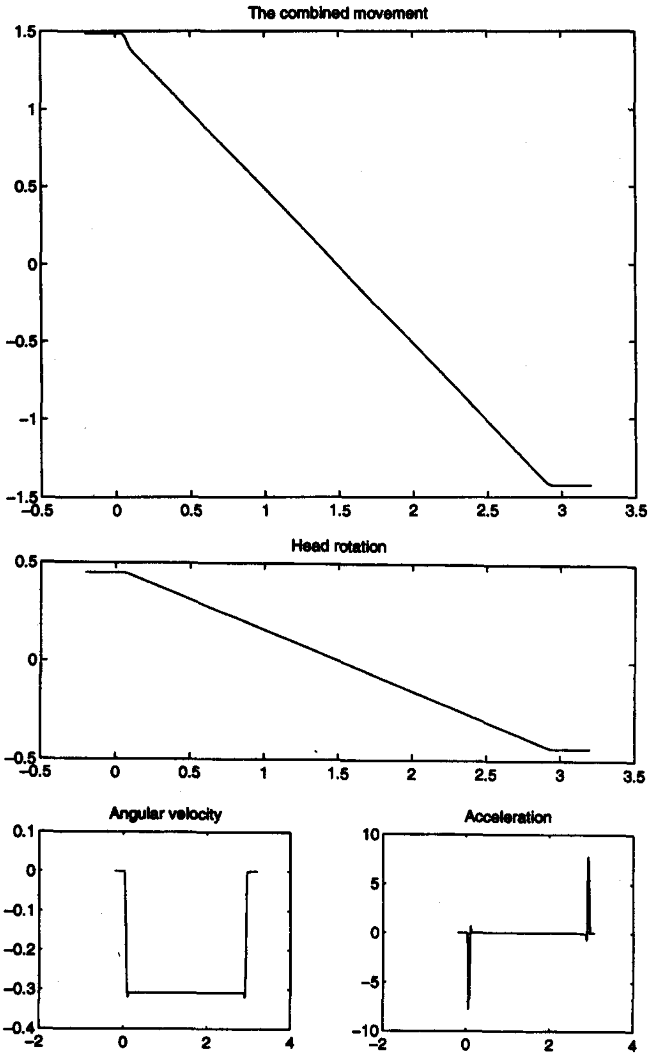
L.H.M-ctrl : Left Head Muscle Control
R.H.M-ctrl : Right Head Muscle Control
L.E.M-ctrl : Left Eye Muscle Control
R.E.M-ctrl : Right Eye Muscle Control.

It can be worth noticing that each pair of muscles have the desired property that

$$v_1(t)\,v_2(t) = 0 \quad \forall t.$$

In each section, the energy produced by the head rotation is also listed, calculated by (86).

A.1. $\psi(t) = \psi_0 - (t - t_0)$



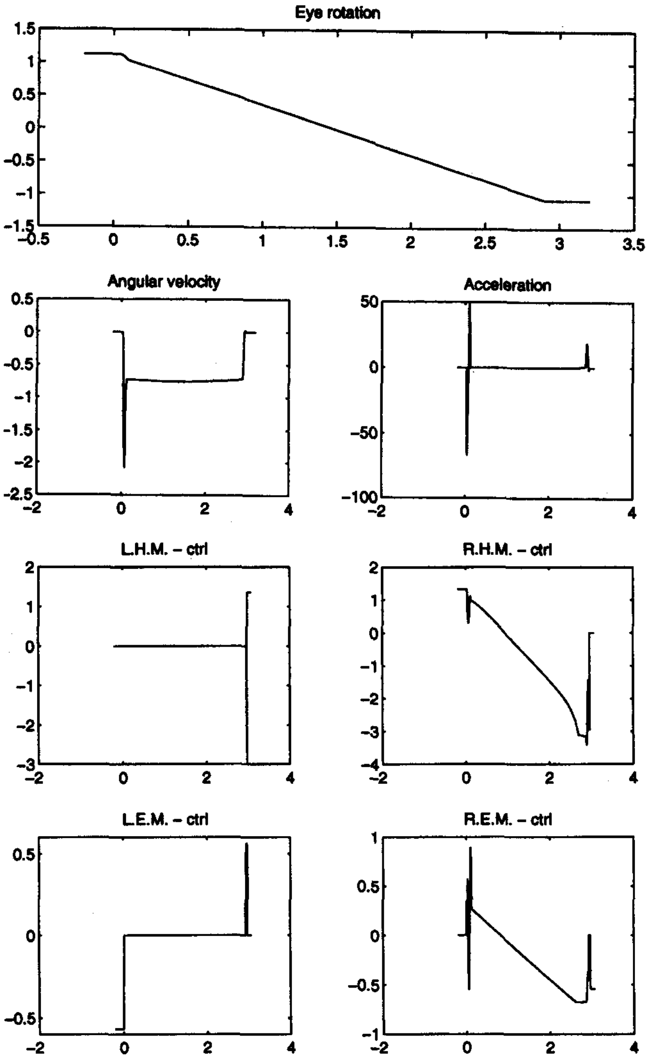
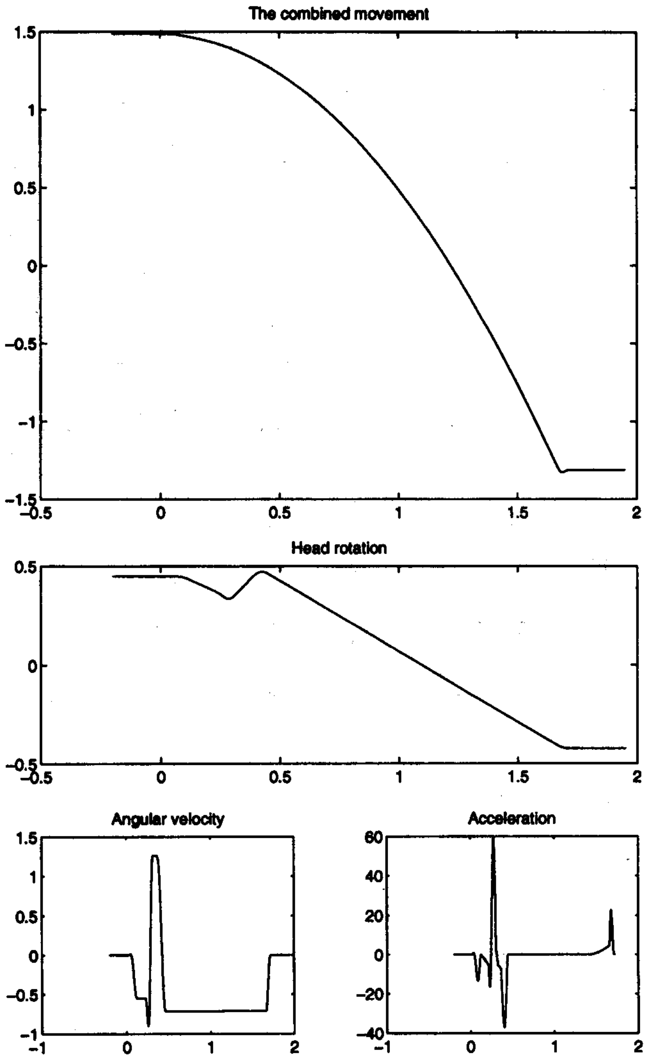


FIG. A.1 (i-iv). Head and eye rotation when $\psi(t) = \psi_0 - (t - t_0)$. No corrections from the zero head acceleration was necessary. The energy produced in the movement was $E \approx 2.36 \cdot 10^{-3}$ J.

A.2. $\psi(t) = \psi_0 - (t - t_0)^2$



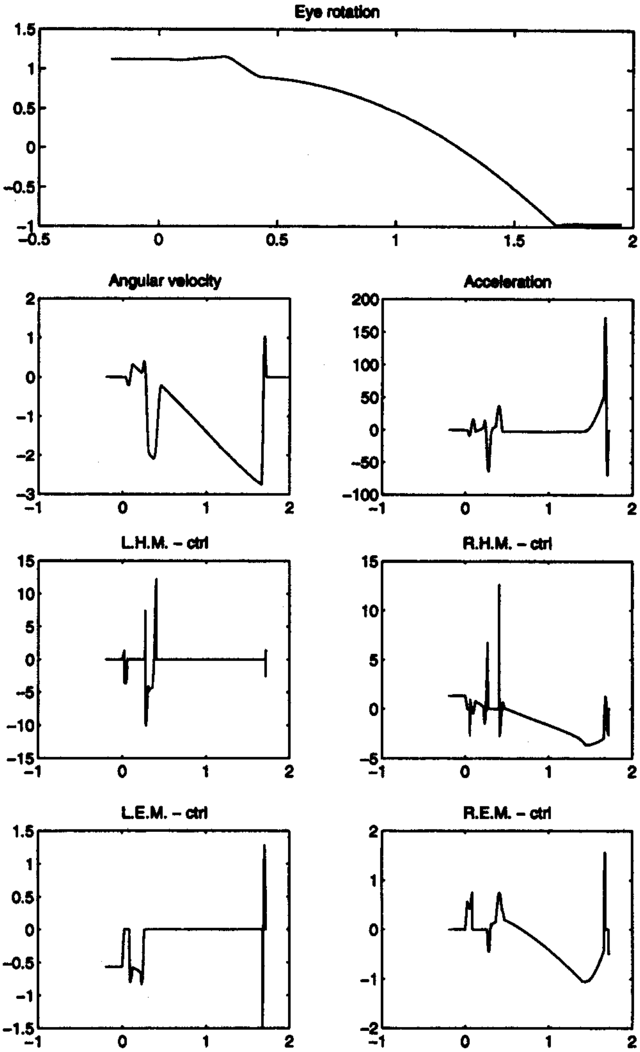
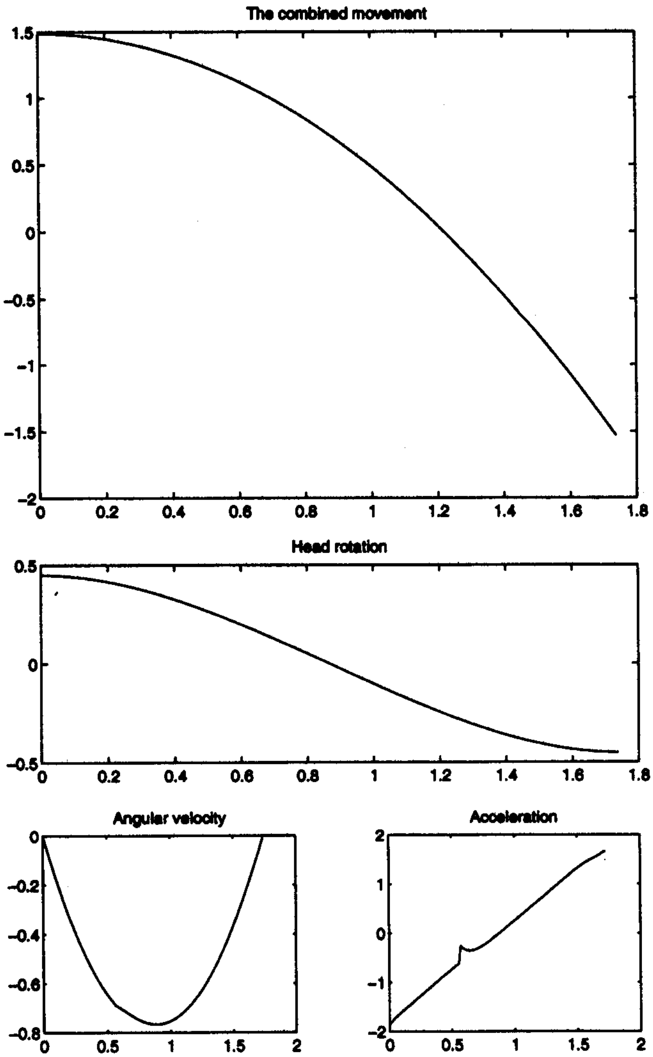


FIG. A.2 (i-iv). Head and eye rotation when $\psi(t) = \psi_0 - (t - t_0)^2$. One correction of the head acceleration was necessary, since the eye was rotating out of bound. The energy produced was $E \approx 6.42 \cdot 10^{-2}$ J.

A.3. $\psi(t) = \psi_0 - (t - t_0)^2$ – PMP



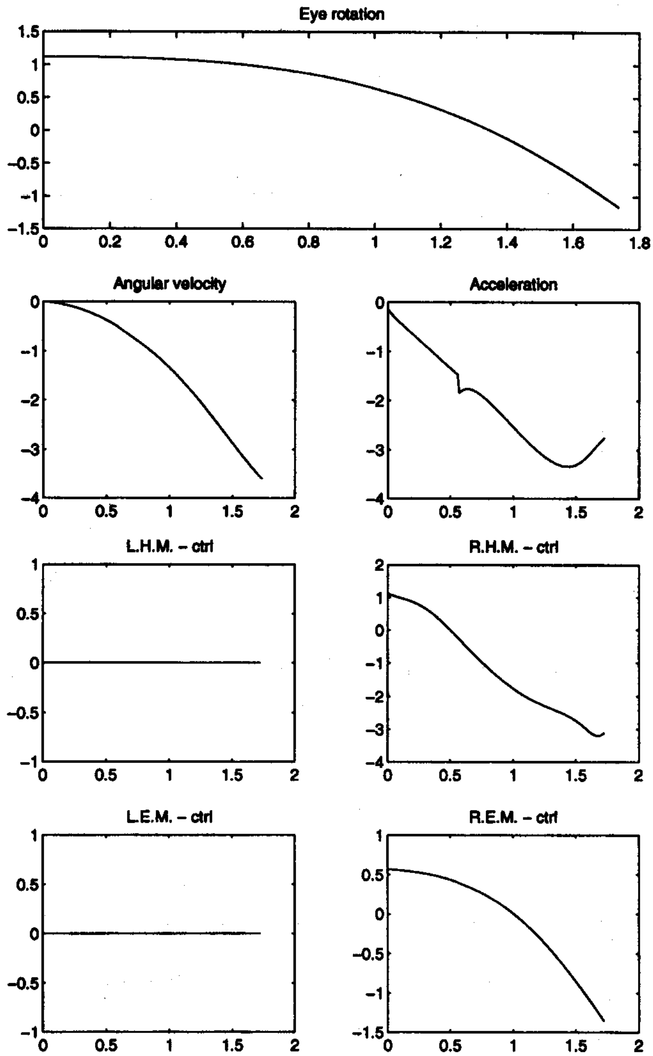


FIG. A.3 (i-iv). Head and eye rotation when $\psi(t) = \psi_0 - (t - t_0)^2$, and when the controls were designed to minimize the square of the angular acceleration of the head, using Pontryagin's Maximum Principle. The energy produced was $E \approx 1.75 \cdot 10^{-2}$ J in this optimal case. This can be compared to our previous $6.17 \cdot 10^{-2}$ J and $6.42 \cdot 10^{-2}$ J, but in this case, we have no guarantee that the trajectories are three times continuously derivable.

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