# ASYMPTOTIC OBSERVERS FOR DISCRETE-TIME SWITCHED LINEAR SYSTEMS [1]

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Abstract: An asymptotic observer design procedure is proposed for discrete-time switched linear systems with exogenous, arbitrary, and unknown mode sequences. The proposed observer consists of two parts: a mode detector, and a continuous observer. It is shown that, under mild conditions, the proposed scheme results in a global asymptotic observer for almost all initial states.

Keywords: Switched Systems, Observers

## 1. INTRODUCTION

The focus of this paper is on the following discrete-time model for *switched linear systems* (SLS):

$$\begin{aligned} x_{k+1} &= A(\theta_k)x_k + B(\theta_k)u_k \\ y_k &= C(\theta_k)x_k \end{aligned} \tag{1}$$

where $x_k$, $y_k$ and $u_k$ are in $\mathbb{R}^n$, $\mathbb{R}^p$ and $\mathbb{R}^m$, respectively, where $A(\cdot)$, $C(\cdot)$ and $B(\cdot)$ are in $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{p\times n}$, respectively, and where the discrete *mode* $\theta_k \in \bar{s} \triangleq \{1,\dots,s\}$, so that $A(\theta_k) \in \{A(1),\dots,A(s)\}$, $C(\theta_k) \in \{C(1),\dots,C(s)\}$ and $B(\theta_k) \in \{B(1),\dots,B(s)\}$, which are known sets of matrices.

The objective is to design a finite-memory, recursive asymptotic state observer for (1), assuming only the measurements $y_k$, $k \geq 1$, are observed, and that the mode sequence $\{\theta_k\}_{k=1}^{\infty}$ is *exogenous*, *arbitrary*, and *unknown*. In other words, it is to design a system producing an estimate $\hat{x}_k$ of $x_k$ based on the knowledge of $y_1,\dots,y_k$ and of $u_1,\dots,u_k$, such that

$$\lim_{k\to\infty} \|x_k - \hat{x}_k\| = 0. \tag{2}$$

State observation in switched linear systems, and in hybrid systems in general, has lately received considerable attention, and many closely related problems to the one posed here have recently been addressed. For *piecewise affine* systems, which are hybrid systems whose modes $\theta_k$ are a picewise constant function of the state, a moving horizon technique was proposed by Ferrari-Trecate et al. (2002), and a piecewise affine observer was studied by Juloski et al. (2003). However, those analyses do not apply here, since the relationship between modes and states makes the output of the system reveal more information about the modes than in the switched case, where the modes are an exogenous, unknown input to the system. Returning to SLS's, observability (Babaali and Egerstedt (2004); Vidal et al. (2002)) means the ability to recover the initial state $x_1$ from a finite number of observations. While it is of course sufficient for the existence of an asymptotic observer, it is not necessary. [2] In Alessandri and Coletta (2001), it was

[1] This work was supported by NSF-CAREER (PECASE) Grant 0132716 and by NSF-CAREER Grant 0237971.

[2] The example at the end of this paper illustrates this fact.

shown how to design Luenberger-like asymptotic observers for the known-modes case, by assigning a common quadratic Lyapunov function to the error dunamics, and in Balluchi et al. (2002), the latter observer was combined with failure detection techniques to produce an asymptotic observer for SLS's with unknown modes. However, because of the delayed detection stemming from the residual-based failure detection techniques used, a minimum sojourn time was required of the mode sequence, and moreover, the observer error could only be eventually-bounded. On the other hand, a discrete output, which is absent in the model considered here, was used in Balluchi et al. (2002) and, later, in Balluchi et al. (2003), in order to recover the modes. However, it turns out that it is possible to estimate the discrete modes instantaneously given only the continuous outputs, as first shown in Ragot et al. (2003), and it is the observer proposed in that paper that we analyze in this paper. Finally, in Babaali et al. (2003), an asymptotic observer was proposed under arbitrary and unknown switching, yet for a subclass of (1), namely systems with constant $A$ matrices, and was based on a direct approach circumventing the need to recover the modes for the purpose of observation.

The outline of this paper is as follows. In section 2, the observer we is described. Its discrete part is then analyzed in Section 3. An illustrative example is then studied in section 4, before some concluding remarks.

## 2. THE OBSERVER

We propose to study the following class of asymptotic observers:

$$\hat{\theta}_k = f(y_{[k-N+1,k]}, u_{[k-N+1,k]}, \hat{\theta}_{[k-N+1,k-1]}) \quad (3)$$
$$\hat{x}_{k+1} = F_{\hat{\theta}_{[1,k]}}(\hat{x}_k, y_k, u_k), \;\; k \geq N, \quad (4)$$

where the positive integer $N$, donoting the *detection horizon*, is a design parameter, where

$$y_{[i,j]} = \begin{pmatrix} y_i \\ y_{i+1} \\ \vdots \\ y_j \end{pmatrix}, \; u_{[i,j]} = \begin{pmatrix} u_i \\ u_{i+1} \\ \vdots \\ u_j \end{pmatrix}, \quad (5)$$

and $\hat{\theta}_{[i,j]} = \hat{\theta}_i \cdots \hat{\theta}_j$. the function $F_{\hat{\theta}_{[1,k]}}$ is linear in its arguments, with gains depending on the mode history $\hat{\theta}_{[1,k]}$. The dependence of $F$ on the mode history is there to enable the consideration of gains that can be computed recursively, as is the case, e.g., in Kalman filtering. The observer (3-4) can be viewed as the interconnection of the following two entities (see Figure 1):

- A *mode detector* (3), which is supposed to return the right mode, i.e., return $\hat{\theta}_k = \theta_k$, given the outputs $y_{[k-N+1,k]}$ and inputs $u_{[k-N+1,k]}$ over a window of length $N$, and the estimates $\hat{\theta}_{[k-N+1,k-1]}$.
- A recursive *continuous observer*

$$\hat{x}_{k+1} = F_{\theta_{[1,k]}}(\hat{x}_k, y_k, u_k), \quad (6)$$

which, under known modes, i.e., when $\theta_{[1,k]}$ is available, should yield estimates $\hat{x}_k$ that converge to $x_k$.

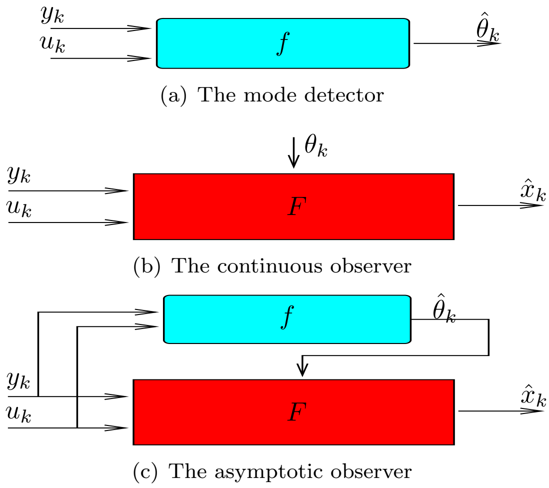


Fig. 1. The asymptotic observer as the interconnection of a mode detector with a continuous observer.

*Definition 1.* The mode detector (3) is *well posed* for $x_1$, $\{u_k\}_{k=N}^{\infty}$, and $\{\theta_k\}_{k=1}^{\infty}$, if $\{\hat{\theta}_k\}_{k=N}^{\infty} = \{\theta_k\}_{k=N}^{\infty}$. $\diamondsuit$

The reason we define well-posedness for specific initial state, input sequence and mode sequence, is that we know, from the results of Babaali and Egerstedt (2004), that it cannot be global. For example, whenever $x_1 = 0$ and $u_k = 0$ for all $k \geq 1$, we get $y_k = 0$ for all $k \geq 1$ no matter what $\{\theta_k\}_{k=1}^{\infty}$ is, which makes it impossible to estimate $\theta_k$. However, we will see that well-posedness is achievable for *almost all* initial states, "almost all" being with respect to Lebesgue measure in $\mathbb{R}^n$. We will concentrate on showing it for such cases, since the initial states are unknown, and thus any condition restricting them to subsets of the state space whose complement is not null would not make sense. Similarly, we define the following desired property of the continuous observer.

*Definition 2.* The continuous observer (6) is convergent if

$$\lim_{k\to\infty} \|x_k - \hat{x}_k\| = 0 \quad (7)$$

for all input sequences $\{u_k\}_{k=1}^{\infty}$, all mode sequences $\{\theta_k\}_{k=1}^{\infty}$, all initial states $x_1 \in \mathbb{R}^n$, and all initial estimates $\hat{x}_1 \in \mathbb{R}^n$. $\diamondsuit$

We clearly have:

*Proposition 1.* If the mode detector (3) is well posed for $x_1$, $\{u_k\}_{k=1}^{\infty}$, and $\{\theta_k\}_{k=1}^{\infty}$, and if the continuous observer (6) is convergent, then (3-4) satisfies

$$\lim_{k\to\infty} \|x_k - \hat{x}_k\| = 0 \tag{8}$$

for all initial estimates $\hat{x}_N$. $\Diamond$

This result allows one to study both the continuous observer and the mode detector separately. In this paper, we are only concerned with the mode detector, which we set up and analyze in the next section. As for the continuous observer, we simply point out that several design procedures exist, among which one can cite the switched Luenberger-like observers of Alessandri and Coletta (2001), and the Kalman filter, for which several convergence results exist and naturally apply here (see, e.g., Kamen (1993); Boutayeb and Darouach (2000); Baras et al. (1988); Deyst and Price (1968)).

## 3. THE MODE DETECTOR

Mode detection has, of course, received some attention in the literature. However, the particular problem considered here, namely recovering the mode $\theta_k$ immediately, given only the measurement and input sequences up to time $k$, has never been studied. The failure detection paradigm, as surveyed, e.g., in Balluchi et al. (2002), is based on the use of residual filters, and is only applicable to systems with slow switching. In Vidal et al. (2002), a switch detection algorithm was proposed for systems with minimum dwell time and known initial state. The mode detector we propose has the following recursive structure:

$$\begin{aligned}\hat{\theta}_k = \Big\{ j \in \bar{s} \;\Big|\; & y_{[k-N+1,k]} \\ & - \mathcal{G}(\hat{\theta}_{[k-N+1,k-1]} j) u_{[k-N+1,k]} \\ & \in \mathcal{R}(\mathcal{O}(\hat{\theta}_{[k-N+1,k-1]} j)) \Big\}.\end{aligned} \tag{9}$$

where

$$\mathcal{G}(\theta) \triangleq \begin{pmatrix} 0 & \cdots & 0 & 0 \\ C(\theta_2)B(\theta_1) & \cdots & 0 & 0 \\ C(\theta_3)A(\theta_2)B(\theta_1) & \cdots & \vdots & 0 \\ \vdots & \cdots & 0 & \vdots \\ C(\theta_N)\Phi(\theta_{[2,N]})B(\theta_1) & \cdots & C(\theta_N)B(\theta_{N-1}) & 0 \end{pmatrix},$$

and

$$\mathcal{O}(\theta) \triangleq \begin{pmatrix} C(\theta_1) \\ \vdots \\ C(\theta_N)A^{N-1} \end{pmatrix} \tag{10}$$

for any path $\theta$ of length $N$ (i.e., a word of length $N$ over $\bar{s}$), and where $\mathcal{R}(M)$ is the column range space of a matrix $M$. First, note the following equivalence

$$\exists x \mid Y = \mathcal{O}x \Leftrightarrow Y \in \mathcal{R}(\mathcal{O}) \Leftrightarrow (\mathcal{O}\mathcal{O}^{\{1\}} - I)Y = 0,$$

where the vector $Y$ and the matrix $M$ are given, and where $\mathcal{O}^{\{1\}}$ is a $\{1\}$-inverse of $\mathcal{O}$ as defined, e.g., in Rao and Mitra (1971). This gives both an interpretation and a way to compute (9). Indeed, it follows from (1) and from our notation that

$$\begin{aligned} y_{[k-N+1,k]} = {} & \mathcal{O}(\theta_{[k-N+1,k]})x_k \\ & + \mathcal{G}(\theta_{[k-N+1,k]})u_{[k-N+1,k]}, \end{aligned}$$

which, whenever $\hat{\theta}_{[k-N+1,k-1]} = \theta_{[k-N+1,k-1]}$, implies by (9) that

$$\theta_k \in \hat{\theta}_k. \tag{11}$$

Therefore, when the mode estimates history is correct up to time $k-1$, if the mode detector returns a *singleton* at time $k$, i.e., if $\mathrm{card}(\hat{\theta}_k) = 1$, then it must be the right mode. In the sequel, we study the well posedness of our mode detector, i.e., whether or not, and when, (9) returns a singleton for all $k \geq 1$. Note that the existence of a detection horizon $N$ making the mode detector well posed is of critical concern. Finally, it is noteworthy that the online computational complexity of the mode detector is linear in $Ns$, making it an efficient detector.

In Section 3.1, we review some preliminary results. In Sections 3.2 and 3.3, we study the autonomous and non-autonomous cases, respectively. Finally, in Section 3.4, we establish the decidability of the criterion established for the well-posedness of the mode detector.

### *3.1 Preliminaries*

In this section, we shall recall several results on *discernibility* from Babaali and Egerstedt (2004). We first define the function $Y$ as

$$Y(\theta, x) \triangleq \mathcal{O}(\theta)x, \tag{12}$$

and recall the definition of discernibility:

*Definition 3.* (Discernibility). A path $\theta$ is discernible from another path $\theta'$ of the same length if

$$\rho([\mathcal{O}(\theta)\mathcal{O}(\theta')]) > \rho(\mathcal{O}(\theta')), \tag{13}$$

where $[\mathcal{O}(\theta)\mathcal{O}(\theta')]$ denotes the horizontal concatenation of $\mathcal{O}(\theta)$ and $\mathcal{O}(\theta')$, and where the degree $d$ of discernibility is defined as

$$d = \rho([\mathcal{O}(\theta)\mathcal{O}(\theta')]) - \rho(\mathcal{O}(\theta')). \tag{14}$$

We then say that $\theta$ is $d$-discernible from $\theta'$. $\Diamond$

The following proposition is now in order:

*Proposition 2.* $Y(\theta, x) \notin \mathcal{R}(\mathcal{O}(\theta'))$ for generic $x \in \mathbb{R}^n$ iff $\theta$ is discernible from $\theta'$. $\Diamond$

The proof of Proposition 2 can be found in Babaali and Egerstedt (2004), and is based on showing that

$$\dim(c(\theta, \theta')) = n - d, \tag{15}$$

where

$c(\theta, \theta') \triangleq \{x \in \mathbb{R}^n \mid \exists x' \in \mathbb{R}^n : \ \mathcal{O}(\theta)x = \mathcal{O}(\theta')x'\}$

is the *state subspace of conflict of $\theta$ with $\theta'$*, which can be furthermore expressed as $c(\theta, \theta') = \mathcal{O}(\theta)^{-1}(C(\theta, \theta'))$, where $C(\theta, \theta') \triangleq \mathcal{R}(\mathcal{O}(\theta)) \cap \mathcal{R}(\mathcal{O}(\theta'))$ is the *output subspace of conflict of $\theta$ from $\theta'$*.

### *3.2 The Autonomous Case*

In this section, we assume that $u_k = 0$ for all $k \geq 1$. We define *backward discernibility* as follows:

*Definition 4.* (Backward Discernibility (BD)). A mode $j$ is backward discernible from another mode $j'$ if there exists an integer $N$ such that for any path $\lambda$ of length $N$, $\lambda j$ is discernible from $\lambda j'$. The smallest such integer $N$ is the index of BD of $j$ from $j'$. $\Diamond$

We can finally establish the main result of this section:

*Theorem 1.* Assume $u_k = 0$ for all $k \geq 1$. If the matrices $A(j)$ are all invertible, then the following are equivalent.

(1) Every mode is BD from any other mode.
(2) There exists a decision horizon $N$ such that, for all $\{\theta_k\}_{k=1}^{\infty}$, the mode detector is well posed for almost all $x_1$. $\Diamond$

*Proof:* First, note that we have

$$y_{[k-N+1,k]} = Y(\theta_{[k-N+1,k]}, x_{k-N+1}), \tag{16}$$

and therefore that, assuming $\hat{\theta}_l = \theta_l$ for all $l < k$, we get

$$j \in \hat{\theta}_k \Leftrightarrow \\ Y(\theta_{[k-N+1,k]}, x_{k-N+1}) \in \mathcal{R}(\mathcal{O}(\theta_{[k-N+1,k-1]}j)),$$

by definition of $\hat{\theta}_k$, and, therefore, by our established notation, that

$$\left\{x_{k-N+1} \in \mathbb{R}^n \;\middle|\; j \in \hat{\theta}_k;\ \hat{\theta}_l = \theta_l,\ l < k\right\} = \\ c(\theta_{[k-N+1,k-1]}\theta_k, \theta_{[k-N+1,k-1]}j). \tag{17}$$

Now, fix $\{\theta_k\}_{k=1}^{\infty}$, and define

$$\phi(\theta) \triangleq A(\theta_{N-1})A(\theta_{[N-2]}) \cdots A(\theta_1), \tag{18}$$

for any path $\theta$ of length $N$. We then have that the set of initial states destroying the well-posedness of the mode detector can be expressed as:

$$\begin{aligned}
&\chi(\{\theta_k\}_{k=1}^{\infty}) \\
&\triangleq \left\{x_1 \in \mathbb{R}^n \;\middle|\; \exists k \geq N :\ \mathrm{card}(\hat{\theta}_k) > 1\right\} \\
&= \cup_{k=1}^{\infty} \left\{x_1 \in \mathbb{R}^n \;\middle|\; \mathrm{card}(\hat{\theta}_k) > 1;\ \hat{\theta}_l = \theta_l,\ l < k\right\} \\
&= \cup_{k=1}^{\infty} \cup_{j \neq \theta_k} \left\{x_1 \in \mathbb{R}^n \;\middle|\; j \in \hat{\theta}_k;\ \hat{\theta}_l = \theta_l,\ l < k\right\} \\
&= \cup_{k=1}^{\infty} \cup_{j \neq \theta_k} \phi(\theta_{[1,k-N+1]})^{-1} \\
&\qquad \left\{x_{k-N+1} \in \mathbb{R}^n \;\middle|\; j \in \hat{\theta}_k;\ \hat{\theta}_l = \theta_l,\ l < k\right\} \\
&= \cup_{k=1}^{\infty} \cup_{j \neq \theta_k} \phi(\theta_{[1,k-N+1]})^{-1} \\
&\qquad c(\theta_{[k-N+1,k-1]}\theta_k, \theta_{[k-N+1,k-1]}j).
\end{aligned}$$

Since $A(j)$ is invertible for all $j \in \bar{s}$, $\phi(\theta)$ is invertible for all $\theta$, and we get that $\chi(\{\theta_k\}_{k=1}^{\infty})$ has null Lebesgue measure if and only if

$$\dim(c(\theta_{[k-N+1,k-1]}\theta_k, \theta_{[k-N+1,k-1]}j)) < n \tag{19}$$

for all $k \geq N$ and all $j \neq \theta_k$.

Now, since $\{\theta_k\}_{k=1}^{\infty}$ is arbitrary, we thus get that a necessary and sufficient condition for the mode detector to be well posed for almost all $x_1$ is

$$\dim(c(\lambda i, \lambda j)) < n \tag{20}$$

for all $\lambda$ of length $N$ and $i \neq j$, which is equivalent to backward discernibility with an index smaller than or equal to $N$. Furthermore, it is readily seen that the smallest detection horizon guaranteeing such well-posedness is the largest index of BD over all pairs of modes. $\Box$

### *3.3 The Non-Autonomous Case*

Here, we waive the assumption that $\{u_k\}_{k=1} = 0$, we define

$$Y(\theta, x, U) \triangleq \mathcal{O}(\theta)x + \mathcal{G}(\theta)U,$$

and we note that

$$y_{[k-N+1,k]} = Y(\theta_{[k-N+1,k]}, x_{k-N+1}, u_{[k-N+1,k]}).$$

Recalling the following classic result from linear algebra,

*Theorem 2.* The intersection of $V + v$ and $V' + v'$ is either empty or equal to $V \cap V' + w$ for some $w$, in which case it has the dimension of $V \cap V'$. $\Diamond$

we realize that, while the $\mathcal{G}(\theta)U$ terms cannot increase the degree of discernibility, they can achieve something impossible in the non-autonomous case: they can render the affine output subspaces of $\theta$ and $\theta'$, i.e., $\mathcal{R}(\mathcal{O}(\theta)) + \mathcal{G}(\theta)U$

and $\mathcal{R}(\mathcal{O}(\theta'))+\mathcal{G}(\theta')U$, totally disjoint. Therefore, the inputs cannot increase the "size" of the undesired initial states, and we have:

*Theorem 3.* Assume that the matrices $A(j)$ are all invertible. If every mode is BD from any other mode, then there exists a decision horizon $N$ such that the mode detector is well posed for almost all $x_1$ and all input sequences. $\Diamond$

### 3.4 Decidability

In this section, we establish the decidability of backward discernibility. The proof is based on the following result establishing the decidability of pathwise observability, which was proven, independently, in Gurvits (2002) and Babaali and Egerstedt (2003):

*Theorem 4.* There exist naturals numbers $N(s,n)$ such that if $\theta$ is a path of length $N(s,n)$, then there exists a prefix $\theta^0$ of $\theta$ (i.e., $\theta = \theta^0\theta^1$ for some $\theta^1$) and a path $\theta'$ of arbitrary length such that [3]

$$\mathcal{R}(\mathcal{O}(\theta^0\theta')) \subset \mathcal{R}(\mathcal{O}(\theta^0)), \tag{21}$$

and thus $\rho(\mathcal{O}(\theta^0\theta')) = \rho(\mathcal{O}(\theta^0)) \leq \rho(\mathcal{O}(\theta))$. $\Diamond$

What this result shows is that the index of pathwise observability of any SLS is less than or equal to $N(s,n)$. We will establish that the index of BD of any two modes is less than or equal to $N(s,2n)$, in the reversible case. In other words,

*Theorem 5.* (Decidability of Backward Discernibility). If all matrices $A(j)$ are invertible, then BD is decidable, as the index of BD is smaller than or equal to $N(s,2n)$ given in theorem 4. $\Diamond$

The proof is similar to that of the decidability of *forward discernibility* given in Babaali and Egerstedt (2004), but is sufficiently different to be presented (note that forward discernibility was shown to be decidable for arbitrary $A$ matrices). We first recall the following technical lemma from Babaali and Egerstedt (2004):

*Lemma 3.* Let $\theta$ and $\theta'$ be two different paths of the same length, and $\lambda$ be any path of length $N$. The degree of discernibility of $\theta\lambda$ from $\theta'\lambda$ is greater than or equal to the degree of discernibility of $\theta$ from $\theta'$. $\Diamond$

*Proof:* It is easily shown, by elementary linear algebra, that

$$\rho([\mathcal{O}(\theta\lambda)\mathcal{O}(\theta'\lambda)]) - \rho([\mathcal{O}(\theta)\mathcal{O}(\theta')]) \geq \rho(\mathcal{O}(\theta\lambda)) - \rho(\mathcal{O}(\theta)).$$

In other words, the rank of the concatenation must increase by at least the increase in rank of each path. $\square$

*Proof of Theorem 5:* First, since every matrix $A(j)$ is invertible, we can write

$$[O(\lambda j)\mathcal{O}(\lambda j')]^r = [O'(j\lambda)\mathcal{O}'(j'\lambda)] \times \begin{pmatrix} \Phi(\lambda)^{-1} & 0 \\ 0 & \Phi(\lambda)^{-1} \end{pmatrix},$$

where $O'(\theta)$ is the observability matrix of $\theta$ computed by replacing $A(j)$ with $A(j)^{-1}$, and where $M^r$ denotes the $pN \times 2n$ matrix $M$ written with the size $p \times n$ blocks in reverse order. Therefore, we can refocus our attention on proving that if there exists $N$ such that

$$\rho([O(j\lambda)\mathcal{O}(j'\lambda)]) > \rho(\mathcal{O}(j\lambda)) \tag{22}$$

for all $j$, $j'$, and all $\lambda$ of length $N$ (where we have replaced $\mathcal{O}'$ with $\mathcal{O}$ for ease of exposition) then it must be true for $N(s,2n)$.

Now, fix two modes $j$ and $j'$, and suppose there exists a path $\lambda$ of length $N(s,2n)$ such that $\lambda j$ is not discernible from $\lambda j'$. In what follows, we abuse language and say that the degree of discernibility of $\lambda j$ from $\lambda j'$ is zero.

Note that the matrices $[\mathcal{O}(\lambda)\mathcal{O}(\lambda)]$ are produced by the following set of $s$ pairs:

$$\begin{pmatrix} A(i) & 0 \\ 0 & A(i) \end{pmatrix}, \ \begin{pmatrix} C(i) & C(i) \end{pmatrix}, \ i \in \{1,\ldots,s\}.$$

Therefore, by Theorem 4, there exists $\lambda^0$, a prefix of $\lambda$, and a path $\mu$ of arbitrary length, such that $\mathcal{R}([\mathcal{O}(\lambda^0\mu)\mathcal{O}(\lambda^0\mu)]) \subset \mathcal{R}([\mathcal{O}(\lambda^0)\mathcal{O}(\lambda^0)])$, which, by (Babaali and Egerstedt, 2003, Lemma 4) and upon some manipulation, implies that

$$\mathcal{R}([\mathcal{O}(j\lambda^0\mu)\mathcal{O}(j'\lambda^0\mu)]) \subset \mathcal{R}([\mathcal{O}(j\lambda^0)\mathcal{O}(j'\lambda^0)]). \tag{23}$$

By Lemma 3, Equation (23) implies that the degree of discernibility of $j\lambda^0\mu$ from $j'\lambda^0\mu$ is equal to that of $j\lambda^0$ from $j'\lambda^0$, which, again by Lemma 3, is smaller than or equal to that of $j\lambda$ from $j'\lambda$, proving that $j\lambda^0\mu$ is not discernible from $j'\lambda^0\mu$, which completes the proof since $\mu$ is of arbitrary length. $\square$

## 4. AN ILLUSTRATIVE EXAMPLE

Consider the following example, which is not initial state observable (in the sense of Babaali and Egerstedt (2004)), and for which, to the best of the author's knowledge, no asymptotic observer design procedure is applicable. Even though it is quite trivial, it serves the purpose of explaining

[3] $\mathcal{R}(M)$ denotes the row range space of a matrix $M$.

the previous analysis. The system is autonomous, has two modes, and the parameters are:

$$C(1) = (1 \quad 0) \qquad C(2) = (2 \quad 0)$$
$$A(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \; A(2) = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

This system admits the Luenberger-like gains (see Alessandri and Coletta (2001)) $L(1) = (0.5\ 0)^T$ and $L(2) = (1.5\ 0)^T$ resulting in the same error dynamics $E(1) = E(2) = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$, and therefore admits a convergent continuous observer. This system is furthermore backward discernible with index 1. To see this, it suffices to compare the rank of $[\mathcal{O}(\lambda i)\ \mathcal{O}(\lambda j)]$ to that of $\mathcal{O}(\lambda i)$ for any pair of modes $i$ and $j$, and any path $\lambda$ of length 1. Therefore, the mode detector will be well posed for almost any $x_1$. Specifically, whenever $x_1 \notin \chi(\{\theta_k\}_{k=1}^{\infty}) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \;\middle|\; b = 0 \right\}$, which has Lebesgue measure 0. Intuitively, if the initial state is not on the horizontal axis, then the modes can successfully be inferred from time $k = 2$ upward.

## 5. CONCLUSION

We have described an asymptotic observer design approach for switched linear systems with unknown and arbitrary modes, and have shown that it results in an asymptotically decaying observer error for almost all initial states. The subsets of Lebesgue measure zero in question are countable unions of proper subspaces of the finite-dimensional state space, and may actually be dense, making it likely for the mode detector to fail. A solution to remedy this fact, which is currently under investigation, is to select the inputs $u_k$ in order to guarantee immediate detection of the current mode for all initial states and mode sequences, which is possible according to the results of Babaali and Egerstedt (2004).

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