On the Observability of Piecewise Linear Systems

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***Abstract*— We study observability in piecewise linear systems. Inspired by recent results on observability analysis in switched linear systems, we give linear-algebraic characterizations of some observability concepts, discuss differences between switched and piecewise linear systems, and conclude with a note on controllability.**

I. INTRODUCTION

We consider autonomous *piecewise linear* (PL) systems in discrete-time, i.e. systems that can be modeled as follows:

$$\begin{aligned} x_{k+1} &= A(\theta_k)x_k \\ y_k &= C(\theta_k)x_k \\ \theta_k &= i,\ x_k \in \chi_i \end{aligned} \tag{1}$$

where x_k (the states) and y_k (the measurements) are in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively, where $A(\cdot)$ and $C(\cdot)$ are real matrices of compatible dimensions, and where $\{\chi(i)\}_{i=1}^s$ is a finite polyhedral partition of the state space $\mathbb{R}^n$. In other words, the $\chi(i)$'s are disjoint convex polyhedra, i.e. subsets of $\mathbb{R}^n$ defined as finite intersections of open and closed half-spaces, whose union covers $\mathbb{R}^n$. Note that since there are only a finite number of polyhedra, they cannot all be polytopes, i.e. bounded polyhedra. We refer to θ_k as the mode in force at time k, thus making every polyhedron induce a different mode. Note the difference with *switched linear* (SL) systems [2], in which the mode θ_k is an exogenous variable that is completely independent of the state history $\{x_j\}_{j=1}^k$.

PL systems were introduced in [16] as approximations of nonlinear systems, and have recently met a surge of interest for their adequacy in modeling a large number of hybrid systems, i.e. systems combining continuous evolution with discrete logic. In this paper, our focus will be set on the following two problems:

1) N-*Observability*: Given an integer N, termed the observation horizon, the PL system (1) is N-observable if the sequence of measurements $(y_1, \ldots, y_N)$ determines the initial state x_1 uniquely for all $x_1 \in \mathbb{R}^n$.
2) *Observability*: The PL system (1) is observable if there exists an observation horizon N such that it is N-observable.

By the statement "$(y_1, \ldots, y_N)$ determines the initial state x_1," we mean that for all $x_1' \in \mathbb{R}^n$, producing the measurements $(y_1', \ldots, y_N')$, if $x_1 \neq x_1'$, then $(y_1, \ldots, y_N) \neq (y_1', \ldots, y_N')$. In other words, the initial state x_1 can be uniquely identified by its first N measurements.

Sontag showed in [16] that observability of general[1] PL systems was undecidable, but that N-observability was decidable. Indeed, it can easily be checked using Linear Programming (LP), although the number of linear programs is exponential in N. He later showed that N-observability was $\mathcal{NP}$-complete in [17], and that observability was undecidable in even the simplest classes of PL systems, such as systems with saturated outputs [18]. As can be imagined, later work has been mainly focused on devising computationally efficient algorithms for checking N-observability. It appears that the main line of work in this regard has emerged from the so-called Mixed Logical Dynamical (MLD) formulation of PL systems, to which any PL system can be translated provided its states and inputs are bounded within a polytope [5]. Since the MLD formulation naturally lends itself to being analyzed through Mixed-Integer Linear Programming (MILP), and since very efficient algorithms have recently been developed for MILP (based, in particular, on branch-and-bound methods), several computational approaches have recently been proposed for analyzing various observability concepts for PL systems. In [4], an algorithm was described for checking "incremental observability," which is a stronger version of observability requiring a minimum amount of distiguishability between states. In [10], an algorithm based on multi-parametric MILP was proposed for computing the "maximal observability region" of a PL system, i.e. the set of initial states that can be uniquely determined by the output. Despite the mild boundedness limitation that the MLD formulation imposes on a PL system, which is actually quite natural since, in practice, states and controls are usually subjected to bounding physical constraints, these methods have become quite popular thanks to their computational efficiency.

It is noteworthy that all of these approaches, whether LP or MILP-based, are purely algorithmic, in that they do not explicitly provide any simple characterization of observability, but rather merely answer the questions in a completely obscure fashion. This is in contrast to linear systems, whose basic properties, such as observability and controllability, have simple, intuitive, linear-algebraic

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[1] That is, PL systems with non-autonomous and affine state update and measurement equations in (1).

characterizations [14]. Unfortunately, it is well known that observability of the linear components of a PL system, i.e. the pairs $(A(i), C(i))$, has absolutely no logical relation to the observability of the PL system [4]. Basically, observability (and controllabililty) can both be gained or lost through the switching. However, it has recently been shown that all observability concepts in SL systems have simple linear-algebraic characterizations, and that, moreover, they are all decidable in the autonomous case [1], [2]. Our objective here is, in light of the latter results, to give similar characterizations for observability of PL systems, and to highlight the main differences along the way. We believe that, short of providing efficient computational tools, the present results give new insight into PL systems, which might have some impact on the study of identification problems, among other things. Moreover, we have chosen to study only the autonomous case for clarity, especially since these results can easily be generalized to non-autonomous PL systems.

Note that the subject of observability in hybrid systems has lately experienced a significant surge of interest resulting in abundant literature, most of which is irrelevant to the problems under consideration here. For instance, while the work in [7], [12], [13], [15], [21] was carried out in a stochastic setting, the work in [3], [6], [8], [9], [20] concerned continuous-time systems, and discrete-time switched linear systems were considered in [1], [2], [11], [19]. All of these problems are different, thereby requiring different analyses.

The outline of this paper is as follows. We start off, in Section II, by establishing some notation, and by giving precise formulations for our problems. We then make preliminary observations in Section III. In Section IV, we study a characterization of N-observability under known modes. In Section V, we examine some sufficient conditions. We then give a note on controllability in Section VI, and end the paper with some concluding remarks.

II. Notation and Problem Formulation

We begin by defining a path θ of length N as a string $\theta = \theta_1\theta_2 \dots \theta_N$ of modes over $\{1, \dots, s\}$, and we denote its length by $|\theta| = N$. We also define Θ_N as the set of all paths of length N. Moreover, we denote by $\theta_{[i,j]}$ the infix of θ between i and j, i.e. $\theta_{[i,j]} = \theta_i\theta_{i+1} \dots \theta_j$.

We now define the following two matrix functions of paths. First, we let

$$\phi(\theta) \triangleq A(\theta_{N-1}) \cdots A(\theta_1) \tag{2}$$

denote the transition matrix of a path θ of length N, and let $\phi(\theta) = I$ when $|\theta| = 1$. Second, we define the observability matrix $\mathcal{O}(\theta)$ of a path θ as

$$\mathcal{O}(\theta) \triangleq \begin{pmatrix} C(\theta_1) \\ C(\theta_2)\phi(\theta_{[1,2]}) \\ \vdots \\ C(\theta_N)\phi(\theta) \end{pmatrix}. \tag{3}$$

Next, we formally define the polyhedra $\{\chi(i)\}_{i=1}^s$ as

$$\chi(i) \triangleq \left\{ x \in \mathbb{R}^n : \begin{array}{l} F_o(i)x < H_o(i) \\ F_c(i)x \leq H_c(i) \end{array} \right\}, \tag{4}$$

where the subscripts o and c refer to open and closed half spaces, respectively, and where the $<$ and $\leq$ signs refer to componentwise strict and weak inequalities, respectively.

Finally, we define the following maps. We let

$$Y_N(x) \triangleq \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \tag{5}$$

$$T_N(x) \triangleq \theta_1 \cdots \theta_N \tag{6}$$

be the stack of measurements and the path of modes visited by system (1) when $x_1 = x$. In other words, $Y_N(x)$, and $T_N(x)$ are the observation vector and mode trajectory of (1) when the initial state is x. Defining the maps

$$\Phi(x, \theta) \triangleq \phi(\theta)x \tag{7}$$

$$\Omega(x, \theta) \triangleq \mathcal{O}(\theta)x, \tag{8}$$

we obtain the following closed form expression for $Y_N(x)$ in terms of $T_N(x)$:

$$Y_N(x) = \Omega(x, T_N(x)). \tag{9}$$

Equation (9) shows that the observation mapping is piecewise linear: it is linear in the state over every region of the state space where $T_N(x)$ is constant, i.e. over

$$\mathcal{X}(\theta) \triangleq \{x \in \mathbb{R}^n : \ T_N(x) = \theta\} \tag{10}$$

for every $\theta \in \Theta_N$. By definition of $\mathcal{X}(\theta)$, the set $\{\mathcal{X}(\theta)\}_{\theta \in \Theta_N}$ clearly forms a partition of $\mathbb{R}^n$. Furthermore, every $\mathcal{X}(\theta)$ is a polyhedron, since it can be expressed as

$$\mathcal{X}(\theta) = \bigcap_{i=1}^{N} \Phi(\cdot, \theta_{[1,i]})^{-1}(\chi(\theta_i)), \tag{11}$$

which is a finite intersection of inverse images of polyhedra by linear mappings, which are polyhedra. For future reference, let the expression of $\mathcal{X}(\theta)$ be as follows:

$$\mathcal{X}(\theta) = \left\{ x \in \mathbb{R}^n : \begin{array}{l} F'_o(\theta)x < H'_o(\theta) \\ F'_c(\theta)x \leq H'_c(\theta) \end{array} \right\}. \tag{12}$$

We can now use the notation defined in the previous subsection to formulate our problems.

Definition 1: The PL system (1) is

1) N-observable if the following map is injective.

$$\begin{array}{ccc} \mathbb{R}^n & \rightarrow & \mathbb{R}^{pN} \\ x & \mapsto & Y_N(x) \end{array} \tag{13}$$

2) observable if there exists an observation horizon N such that it is N-observable. ◊

III. Preliminary Observations

We begin this section by establishing the following observation.

Proposition 1: $Y_N(x)$ determines x if and only if the following hold.

1) $Y_N(x)$ determines $T_N(x)$.
2) $(Y_N(x), T_N(x))$ determines x. $\diamondsuit$

Proof: Assume $Y_N(x)$ determines x. Since x determines $T_N(x)$ by the mapping $T_N(\cdot)$, $Y_N(x)$ thus determines $T_N(x)$. Moreover, $(Y_N(x), T_N(x))$ clearly determines x since $(Y_N(x), T_N(x))$ provides more information than $Y_N(x)$ does.

Assume now that 1) and 2) hold. Then, given $Y_N(x)$, one can determine $T_N(x)$ uniquely by 1), and then determine x from $Y_N(x)$ and $T_N(x))$ by 2). $\square$

What Proposition 1 shows us is that one can break down the determination of x from $Y_N(x)$ in 2 steps: first, the determination of $T_N(x)$, and then the determination of x by using the newly recovered path $T_N(x)$ in combination with $Y_N(x)$, which is the intuitive way to proceed. Note, however, that this is not true in SL systems. In [2, Example 3], a SL system allowing the determination of x, but not that of the mode path $\theta_1 \dots \theta_N$, was analyzed. A way of formally explaining this, in view of Proposition 1, is to note that the map T_N mapping x to $T_N(x) = \theta_1 \dots \theta_N$ is lost in the switched setting.

From now on, we refer to 1) as *Mode Determination*, and to 2) as *N-observability under Known Modes*. It is the latter that admits a linear-algebraic characterization, which we describe in the next section. Meanwhile, we note that:

Theorem 1: $Y_N(x)$ determines $T_N(x)$ if and only if

$$\Omega(\mathcal{X}(\theta), \theta) \cap \Omega(\mathcal{X}(\theta'), \theta') = \emptyset \tag{14}$$

for every pair of paths $\theta \neq \theta' \in \Theta_N$. $\diamondsuit$

Proof: $Y_N(x)$ does not determine $T_N(x)$ if and only if there exist $\theta \neq \theta' \in \Theta_N$, $x \in \mathcal{X}(\theta)$, and $x' \in \mathcal{X}(\theta')$, such that

$$Y_N(x) = Y_N(x'), \tag{15}$$

which, since $T_N(x) = \theta$ and $T_N(x') = \theta'$, and using (9), is equivalent to the existence of $\theta \neq \theta' \in \Theta_N$, $x \in \mathcal{X}(\theta)$, and $x' \in \mathcal{X}(\theta')$, such that

$$\Omega(x, \theta) = \Omega(x', \theta'), \tag{16}$$

hence to

$$\Omega(\mathcal{X}(\theta), \theta) \cap \Omega(\mathcal{X}(\theta'), \theta') \neq \emptyset, \tag{17}$$

which completes the proof. $\square$

Note that given $\theta \neq \theta' \in \Theta_N$, (14) can be checked by Linear Programming. More precisely, since $\Omega(\cdot, \theta)$ is a linear mapping, the image by it of the polyhedron $\mathcal{X}(\theta)$ is a polyhedron. Hence, $\Omega(\mathcal{X}(\theta), \theta) \cap \Omega(\mathcal{X}(\theta'), \theta')$ is the intersection of two polyhedra, hence a polyhedron. Checking (14) therefore boils down to checking the emptiness of a polyhedron, which is decidable. Note, however, that (14) appears naturally when checking N-observability using the direct LP approach, and, therefore, that it does not give any additional insight with respect to previous work.

IV. N-observability under known modes

Let us recall, formally, the definition of N-observability under known modes.

Definition 2: The PL system (1) is N-observable under known modes if the following map is injective.

$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R}^{pN} \times \Theta_N \\ x & \mapsto & (Y_N(x), T_N(x)) \end{array} \tag{18}$$

Now, given a polyhedron $\mathcal{X} \subset \mathbb{R}^n$, we denote by aff$(\mathcal{X})$ its *affine hull*, i.e. the smallest affine subspace containing it, and by lin$(\mathcal{X})$ its *linear hull*, i.e. the linear subspace of $\mathbb{R}^n$ supporting its affine hull (in other words, aff$(\mathcal{X}) = \{v + x \mid x \in \text{lin}(\mathcal{X})\}$ for some $v \in \mathbb{R}^n$). To ease the discussion, we let lin$(\emptyset) \triangleq \emptyset$. We can now prove the following:

Theorem 2: $(Y_N(x), T_N(x))$ determines x if and only if

$$\ker(\mathcal{O}(\theta)) \cap \text{lin}(\mathcal{X}(\theta)) = \{0\} \tag{19}$$

for all $\theta \in \Theta_N$ such that $\mathcal{X}(\theta) \neq \emptyset$. $\diamondsuit$

Proof: $(Y_N(x), T_N(x))$ does not determine x implies that there exist $\theta \in \Theta_N$ and $x \neq x' \in \mathcal{X}(\theta)$ such that

$$Y_N(x) = Y_N(x'). \tag{20}$$

Now, by (9), and since $T_N(x) = \theta$ for all $x \in \mathcal{X}(\theta)$, we get

$$Y_N(x) = \Omega(x, \theta) \tag{21}$$

for all $x \in \mathcal{X}(\theta)$. Therefore, combining (20) and (21), we get

$$O(\theta)(x - x') = 0, \tag{22}$$

hence that $x - x' \in \ker(\mathcal{O}(\theta))$. Since $x - x' \in \text{lin}(\mathcal{X}(\theta))$ by definition of the linear hull, we finally get that $x - x' \in \ker(\mathcal{O}(\theta)) \cap \text{lin}(\mathcal{X}(\theta))$, hence that $\ker(\mathcal{O}(\theta)) \cap \text{lin}(\mathcal{X}(\theta)) \neq \{0\}$ since $x - x' \neq 0$.

Conversely, assume (19) is false, i.e. that there exists $v \in \ker(\mathcal{O}(\theta)) \cap \text{lin}(\mathcal{X}(\theta))$, $v \neq 0$. It follows from the definition of a linear hull that there exists a scalar $\alpha \neq 0$ and $x \in \mathcal{X}(\theta)$ such that

$$x' \triangleq x + \alpha v \in \mathcal{X}(\theta), \tag{23}$$

and $x' \neq x$. But then, by (21) and the expression of Ω (8), $Y_N(x) = Y_N(x')$ because $\mathcal{O}(\theta)x = \mathcal{O}(\theta)x'$, since $O(\theta)\alpha v = 0$. $\square$

The following example illustrates Theorem 2. The partitioning of its state space is depicted in Figure 1.

Example 1: Consider (1), where

$$\begin{array}{l} \chi_1 = \left\{x = (x^1 \ x^2)^T \mid x_2 \geq 0\right\} \\ \chi_2 = \mathbb{R}^n \setminus \chi_1, \end{array}$$

and

$$\begin{array}{ll} C(1) = (1 \ 0) & A(1) = -I \\ C(2) = (1 \ 0) & A(2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{array}$$

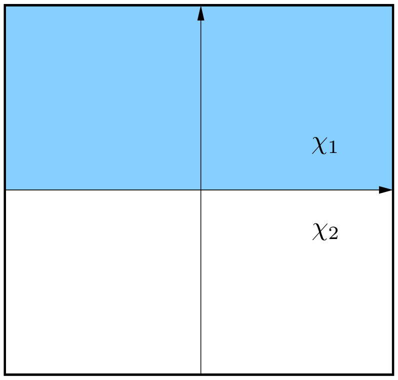

Fig. 1. State space of Example 1.

This system is 2-observable. First,

$$\{\theta \in \Theta_2 \mid \mathcal{X}(\theta) \neq \emptyset\} = \{11, 12, 22\}. \tag{24}$$

Next, noting that both $\mathcal{O}(12)$ and $\mathcal{O}(22)$ are of full rank, it remains to show that $\ker(\mathcal{O}(11)) \cap \text{lin}(\mathcal{X}(11)) = \{0\}$. Since

$$\mathcal{O}(11) = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \tag{25}$$

we have $\ker(\mathcal{O}(11)) = \text{span}(0\ 1)^T$. On the other hand, $\mathcal{X}(11) = \text{span}(1\ 0)^T$, which clearly implies $\ker(\mathcal{O}(11)) \cap \text{lin}(\mathcal{X}(11)) = \{0\}$. Intuitively, if $T_2(x) = 11$, then we know the initial state is on the horizontal axis, and we thus do not need to measure its "height". $\triangle$

Finally, note that there exist various techniques for computing the affine and linear hulls of a polyhedron but, as it turns out, we do not need to compute them. Instead, we can cast (19) as a polyhedron emptiness problem. Even though such an approach may result in much more calculation than the direct approach, which requires only n linear programs per path θ, we have chosen to outline it below for the interested reader.

First, let $k(\theta)$ be the number of weak inequalities in the definition of $\mathcal{X}(\theta)$ in (12), and assume that it is positive. Otherwise, $\mathcal{X}(\theta)$ would be open, and $\text{lin}(\mathcal{X}(\theta))$ would be either empty or equal to $\mathbb{R}^n$, according as $\mathcal{X}(\theta)$ is empty or nonempty, which implies that $\ker(\mathcal{O}(\theta)) \cap \text{lin}(\mathcal{X}(\theta)) = \ker(\mathcal{O}(\theta))$, which reduces (19) to invertibility of $\mathcal{O}(\theta)$.

Now, let $K(\theta) \triangleq \{1, \ldots, k(\theta)\}$. We can then write

$$F'_c(\theta) = \begin{pmatrix} f_1 \\ \vdots \\ f_{k(\theta)} \end{pmatrix}, \quad H'_c(\theta) = \begin{pmatrix} h_1 \\ \vdots \\ h_{k(\theta)} \end{pmatrix}, \tag{26}$$

where f_i and h_i are row vectors for $i \in K(\theta)$. To ease the notation, we omit the dependence of f_i and h_i on θ. Next, for $I \subset K(\theta)$, we define $\mathcal{S}(\theta, I)$ as

$$\mathcal{S}(\theta, I) \triangleq \left\{x \in \mathbb{R}^n : F'_{c,I}(\theta)x = H'_{c,I}(\theta)\right\}, \tag{27}$$

where $F'_{c,I}(\theta)$ (resp. $H'_{c,I}(\theta)$) is the submatrix of $F'_c(\theta)$ (resp. $H'_c(\theta)$) consisting of the rows f_i (resp. h_i), $i \in I$. By convention, we let $\mathcal{S}(\theta, \emptyset) \triangleq \mathbb{R}^n$. We then have

$$\mathcal{X}(\theta) \cap \overline{\mathcal{S}(\theta, I)} = \bigcup_{i \in I} \mathcal{Y}(\theta, i), \tag{28}$$

where $\mathcal{Y}(\theta, i)$ is defined, for every $i \in K(\theta)$, as

$$\mathcal{Y}(\theta, i) \triangleq \left\{x \in \mathbb{R}^n : \begin{array}{rl} F'_o(\theta)x & < H'_o(\theta) \\ F'_{c,K(\theta)\setminus\{i\}}(\theta)x & \leq H'_{c,K(\theta)\setminus\{i\}}(\theta) \\ f_i x & < h_i \end{array}\right\} \tag{29}$$

Again, by convention, we let $\mathcal{X}(\theta) \cap \overline{\mathcal{S}(\theta, \emptyset)} = \emptyset$. Finally, let $\mathcal{H}_i \triangleq \{x \in \mathbb{R}^n : f_i x = 0\}$ denote the hyperplane of $\mathbb{R}^n$ orthogonal to f_i^T, and define, for $I \subset K(\theta)$,

$$L_I \triangleq \bigcap_{i \in I} \mathcal{H}_i, \tag{30}$$

so that we can let

$$\mathcal{I}(\theta) \triangleq \{I \subset K(\theta) : \ \ker(\mathcal{O}(\theta)) \cap L_I = \{0\}\} \tag{31}$$

index a set of subspaces of $\mathbb{R}^n$. Once again, we let $L_\emptyset = \mathbb{R}^n$, by convention. We can now establish the following.

Proposition 2: If $k(\theta) > 0$, then $(Y_N(x), T_N(x))$ determines x if and only if for every $\theta \in \Theta_N$, there exists $I \in \mathcal{I}(\theta)$ such that

$$\mathcal{Y}(\theta, i) = \emptyset \tag{32}$$

for all $i \in I$. $\diamondsuit$

Proof: $\text{lin}(\mathcal{X}(\theta))$ can either be empty or equal to L_I for some $I \subset K(\theta)$. Furthermore, it is easily shown that $\text{lin}(\mathcal{X}(\theta)) \subset L_I$ if and only if

$$\mathcal{X}(\theta) \subset \mathcal{S}(\theta, I') \tag{33}$$

for some I' such that $\mathcal{H}_{I'} \subset \mathcal{H}_I$, and, by virtue of the fact that $\mathcal{S}(\theta, I') \subset \mathcal{X}(\theta)$ by definition, if and only if $\mathcal{X}(\theta) \cap \overline{\mathcal{S}(\theta, I')} = \emptyset$, and by (28), if and only if $\mathcal{Y}(\theta, i) = \emptyset$ for all $i \in I'$. Consequently, by definition of $\mathcal{I}(\theta)$, if $\mathcal{X}(\theta) \neq \emptyset$, then $\ker(\mathcal{O}(\theta)) \cap \text{lin}(\mathcal{X}(\theta)) = \{0\}$ if and only if $\mathcal{Y}(\theta, i) = \emptyset$ for all $i \in I$, for some $I \in \mathcal{I}(\theta)$. Finally, noting that $\mathcal{X}(\theta) = \emptyset$ implies that $\mathcal{Y}(\theta, i) = \emptyset$ for all $i \in K(\theta)$ completes the proof. $\square$

Note that it suffices to restrict the analysis in Proposition 2 to the *minimal* elements of $\mathcal{I}(\theta)$ with respect to the inclusion partial ordering in $2^{K(\theta)}$. Moreover, it is clear that $\emptyset \subset \mathcal{I}(\theta)$ when $\ker(\mathcal{O}(\theta)) = \{0\}$, which, by the previous observation, waves the need to check the emptiness of any $\mathcal{Y}(\theta, i)$, which we already knew: if $\ker(\mathcal{O}(\theta)) = \{0\}$, then the "shape" of $\mathcal{X}(\theta)$ does not matter at all. Sufficiency of the nonsingularity of $\mathcal{O}(\theta)$ is the topic of the next section.

V. Sufficient Conditions

In this section, we study *observability under modes*, defined as follows.

Definition 3: The PL system (1) is observable under known modes if there exists an observation horizon N such that it is N-observable under known modes. $\diamondsuit$

While N-observability under known modes is clearly decidable, by Theorem 2, it is unknown whether observability under known modes is, which is why one may want to consider a sufficient condition such as *trajectory-wise observability*, defined as follows, and wonder whether it is decidable.

Definition 4: The PL system (1) is trajectory-wise observable (TWO) if there exists an observation horizon N such that

$$\text{rank}(\mathcal{O}(\theta)) = n \quad \forall \theta \in \Theta_N : \mathcal{X}(\theta) \neq \emptyset, \tag{34}$$

the smallest one being the index of TWO. $\Diamond$

In fact, it was recently shown that pathwise observability, defined as

Definition 5: The set of pairs $\{(A(1), C(1)), \ldots, (A(s), C(s))\}$ is pathwise observable (PWO) if there exists an observation horizon N such that

$$\text{rank}(\mathcal{O}(\theta)) = n \quad \forall \theta \in \Theta_N, \tag{35}$$

the smallest one being the index of PWO. $\Diamond$

was decidable, the indexes of PWO being bounded by numbers $\mathcal{N}(s, n)$ depending only on s and n [1]. However, it has been known for a while that the indexes of TWO can be arbitrarily large, which makes it impossible, for the time being, to conclude as to its decidability in the same way as for PWO. Even though examples illustrating this fact have appeared in the literature [4], we give the following example for future discussion[2]. Its state space is depicted in Figure 2.

Example 2: Consider (1), where q is a positive integer and λ is a real number satisfying $0 < \lambda < 1$, where

$$\chi_1 = \left\{x = (x^1 \ x^2)^T \mid 0 < \lambda^q x^1 < x^2 \leq x^1\right\}$$
$$\chi_2 = \mathbb{R}^n \setminus \chi_1,$$

and where

$$C(1) = (1\ 0) \quad A(1) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$
$$C(2) = (1\ 0) \quad A(2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This system is TWO with index $q+2$. To see this, first note that $T_2(x) = I$ for any $x \in \chi_2$. On the other hand, every state trajectory originating in χ_1 eventually escapes to χ_2, the escape time k being bounded by $q+1$. While the first rows of $\mathcal{O}(T_{k+1}(x))$ equal $(1\ 0)$, the last one is $(0\ \lambda^{k-1})$, ensuring that $\text{rank}(\mathcal{O}(T_{k+1}(x)) = 2$. $\triangle$

It is easy to explain this disparity between the behaviour of the indexes of PWO and TWO by noting that the index of TWO is nothing other than the smallest integer N such that

$$E_N^u \cap \mathcal{T}_N = \emptyset, \tag{36}$$

[2] Even though χ_2 is not a polyhedron, but a union of polyhedra, we treat it as such for the sake of clarity, and point out that this simplification does not affect the correctness of the analysis.

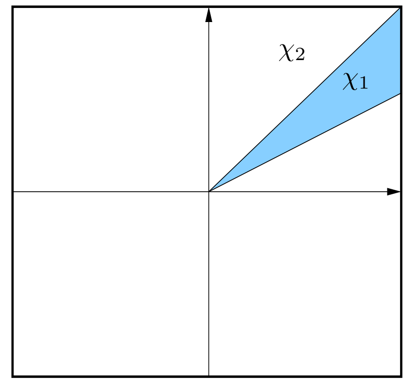

Fig. 2. State space of Example 2.

where $E_N^u \triangleq \{\theta \mid |\theta| \geq N, \ \text{rank}(O(\theta)) < n\}$, i.e., the language containing the "unobservable" paths of length greater than or equal to N, and where $\mathcal{T}_N \triangleq \{\theta \mid |\theta| \geq N, \ \mathcal{X}(\theta) \neq \emptyset\}$, the language containing the "feasible" paths, while the index of PWO is the smallest integer N such that

$$E_N^u = \emptyset. \tag{37}$$

While the boundedness of the indexes of PWO implies that the such an integer is bounded by $\mathcal{N}(s, n)$, E_N^u and $\mathcal{T}_N$ can become disjoint at arbitrary N, which explains the unboundedness of the indexes of TWO. Furthermore, it is clear that the system cannot be PWO when the index of TWO is greater than $\mathcal{N}(s, n)$. Returning to Example 2, it is indeed easy to see that the set of pairs of the system is not PWO: the pair $(A(1), C(1))$ is not even observable.

VI. A NOTE ON CONTROLLABILITY

In this section, we take a quick look at the non-autonomous PL system

$$\begin{aligned} x_{k+1} &= A(\theta_k)x_k + B(\theta_k)u_k \\ \theta_k &= i, \ x_k \in \chi_i, \end{aligned} \tag{38}$$

where the control vectors u_k lie in $\mathbb{R}^m$ and the real matrices $B(\cdot)$ are of compatible dimensions, and we study its controllability (to-the-origin), i.e.

Definition 6: The PL system (38) is controllable if there exists a horizon N such that any state x_1 can be steered to $x_N = 0$ by appropriate choice of $u_1, \ldots, u_{N-1}$. $\Diamond$

In light of the previous study, it would seem natural to believe that *pathwise controllability* (PWC) of the set of pairs $\{(A(1), B(1)), \ldots, (A(s), B(s))\}$, defined as pathwise observability of the set of dual pairs $\{(A(1)^T, B(1)^T), \ldots, (A(s)^T, B(s)^T)\}$, constitutes a sufficient condition for controllability. It turns out that this is not the case, and Example 3 below is a counterexample to the claim.[3] (Its state space is depicted in 3.) Quite surprisingly, one would expect that, as in the observability analysis case, the switching cannot destroy controllability when the

[3] To the best of our knowledge, this is first such counterexample to appear in the literature. For instance, Example V.A.1. in [4] is not PWC.

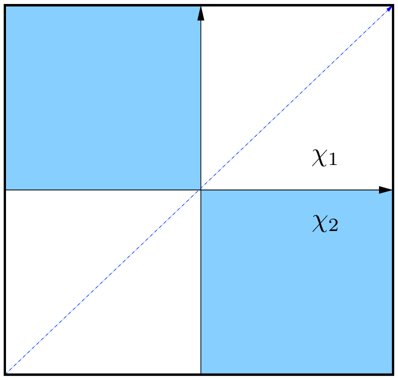

Fig. 3. State space of Example 3. The blue line represents the "direct controllability" subspace of $(A(2), C(2))$, which intersects χ_2 at 0.

underlying set of pairs is PWC, but it can. To explain this, we simply note that controllability has no simple linear-algebraic characterization as N-observability under known modes does.

Example 3: Consider (38), where:

$$\chi_1 = \{x = (x^1\ x^2)^T \mid x_1 > 0,\ x_2 > 0\} \cup \{x = (x^1\ x^2)^T \mid x_1 < 0,\ x_2 < 0\}$$
$$\chi_2 = \mathbb{R}^n \setminus \chi_1,$$

and where

$$B(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad A(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$B(2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad A(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This system is PWC with index 2. However, it is not controllable to the origin. In fact, no non-null state can be driven to 0 in finite time. To see this, we need to consider two cases. First, suppose we want to send $x = (x^1\ \ x^2)^T$ to 0 in one step using $(A(1), C(1))$. We get $0 = (x^2\ \ -x^1)^T + (0\ u)^T$, which implies that $x^2 = 0$, which, in turn, implies that $x \notin \chi_1$. In other words, the "direct controllability" subspace of the pair $(A(1), C(1))$ is disjoint from the subset of the state space where the pair is active. Similarly, steering the state $x = (x^1\ x^2)^T$ to 0 in one step using $(A(2), C(2))$ gives $0 = (-x^1, x^2)^T + (u, -u)^T$, which implies that $x^2 = x^1$, which, in turn, implies that $x \notin \chi_2$, unless $x = 0$. Hence the claim. $\triangle$

VII. Conclusion

We have studied the observability of autonomous piecewise linear systems, and have analyzed its relation to several linear-algebraic criteria. We have, along the way, highlighted some differences between switched linear systems and piecewise linear systems, and posed several open questions that may, in the future, deserve to be answered.

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