POSITIVSTELLENSATZ CERTIFICATES FOR NON-FEASIBILITY OF CONNECTIVITY GRAPHS IN MULTI-AGENT COORDINATION [1]

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Abstract: In this paper, we discuss how to obtain certificates for the non-feasibility of connectivity graphs arising from multi-agent formations. We summarize some previous work on connectivity graphs and their feasibility. Next, we introduce the Positivstellensatz to show how it can be used to better understand the space of all connectivity graphs for a fixed number of vertices, which had previously been understood as only a subspace of the space of all graphs. We study one particular class of graphs and use our methodology to prove some more results in the feasibility of connectivity graphs.

Keywords: Connectivity graphs, Positivstellensatz, multi-agent coordination.

1. INTRODUCTION

The problem of coordinating multiple mobile robots is one in which a finite representation of the configuration space appears naturally, namely by using graph-theoretic models for describing the local interactions in the formation. In other words, graph-based models can serve as a bridge between the continuous and the discrete when trying to manage the design-complexity associated with formation control problems. Notable results along these lines have been presented in (Saber *et al.*, 2003; Jadbabaie *et al.*, 2003; Mesbahi, 2002; Muhammad *et al.* ACC, 2004). The conclusion to be drawn from these research efforts is that a number of questions can be answered in a natural way by abstracting away the continuous dynamics of the individual agents. In (Muhammad *et al.* CDC, 2004; Muhammad *et al.* AMC, 2004), the authors have presented a detailed study of graphs that arise due to the limited sensory perception or communication of individual agents in a formation. We showed that the graphs that can represent formations do in fact correspond to a proper subset of all graphs, denoted by the set of connectivity graphs. We presented several examples of graphs that fail to exist as connectivity graphs. An important observation was that these invalid graphs begin to appear only when the number of agents are greater than 4. The examples of invalid connectivity graphs in (Muhammad *et al.* CDC, 2004) were studied using simple geometrical arguments and the method was found to be suitable for studying only certain types of graphs. What has been missing is a computational method that gives a feasibility or non-feasibility certificate for any arbitrary graph, thereby making the characterization of the set of connectivity graphs complete. In this paper, we take the first steps towards such a compu-

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tational method. We find that this new method has strong connections with emerging methods in semi-definite programming and the Positivstellensatz for semi-algebraic sets. We also note, that this new formulation of the problem leads to ideas that are helpful for studying formation planning under sensory and communication constraints.

This paper is organized as follows. We first summarize our previous work on connectivity graphs (Section 2). Then we formulate the feasibility problem of connectivity graphs as that of a semi-algebraic set (Section 3) and explain how to use the Positivstellensatz for obtaining certificates of non-feasibility.

2. FORMATIONS AND CONNECTIVITY GRAPHS

Graphs can model local interactions between agents, when individual agents are constrained by limited knowledge of other agents. In this section we summarize some previous results (Muhammad *et al.* CDC, 2004) of a graph theoretic formalism for describing formations in which the primary limitation of perception for each agent is the limited range of its sensor. Suppose we have N such agents with identical dynamics evolving on $\mathbb{R}^2$. Each agent is equipped with a range limited sensor by which it can sense the position of other agents. All agents have identical sensor ranges δ. Let the position of each agent be $\mathbf{x}_n \in \mathbb{R}^2$, and its dynamics be given by

$$\dot{\mathbf{x}}_n = f(\mathbf{x}_n, u_n),$$

where $u_n \in \mathbb{R}^m$ is the control for agent n and $f : \mathbb{R}^2 \times \mathbb{R}^m \to \mathbb{R}^2$ is a smooth vector field. The configuration space $\mathcal{C}^N(\mathbb{R}^2)$ of the agent formation is made up of all ordered N-tuples in $\mathbb{R}^2$, with the property that no two points coincide, i.e. $\mathcal{C}^N(\mathbb{R}^2) = (\mathbb{R}^2 \times \mathbb{R}^2 \times \ldots \mathbb{R}^2) - \Delta$, where $\Delta = \{(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) : \mathbf{x}_i = \mathbf{x}_j \text{ for some } i \neq j\}$. The evolution of the formation can be represented as a trajectory $\mathcal{F} : \mathbb{R}_+ \to \mathcal{C}^N(\mathbb{R}^2)$, usually written as $\mathcal{F}(t) = (\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots \mathbf{x}_N(t))$ to signify time evolution. The spatial relationship between agents can be represented as a graph in which the vertices of the graph represent the agents, and the pair of vertices on each edge tells us that the corresponding agents are within sensor range δ of each other.

Let $\mathcal{G}_N$ denote the space of all possible graphs that can be formed on N vertices $V = \{v_1, v_2, \ldots, v_N\}$. Then we can define a function $\Phi_N : C^N(\mathbb{R}^2) \to \mathcal{G}_N$, with $\Phi_N(\mathcal{F}(t)) = \mathcal{G}(t)$, where $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t)) \in \mathcal{G}_N$ is the *connectivity graph* of the formation $\mathcal{F}(t)$. $v_i \in \mathcal{V}$ represents agent i at position x_i, and $\mathcal{E}(t)$ denotes the edges of the graph. $e_{ij}(t) = e_{ji}(t) \in \mathcal{E}(t)$ if and only if $\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq \delta \; i \neq j$. In other words,

$$\Phi_N(\mathcal{F}(t)) = (\{v_1, \ldots v_N\}, \{(v_i, v_j) \mid i \neq j \text{ and } \|\mathbf{x}_i(t) - \mathbf{x}_j(t)\| \leq \delta\})$$

These graphs are *simple* by construction i.e. there are no loops or parallel edges. The graphs are always undirected because the sensor ranges are identical. The motion of agents in a formation may result in the removal or addition of edges in the graph. Therefore $\mathcal{G}(t)$ is a dynamic structure. Lastly and most importantly, every graph in $\mathcal{G}_N$ is not a connectivity graph. The last observation is not as obvious as the others, and it has been analyzed in detail in (Muhammad *et al.* CDC, 2004). We summarize this work below.

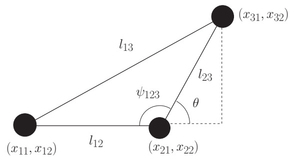

Fig. 1. Depicted are three robots and their inter-robot distances.

A *realization* of a graph $\mathcal{G} \in \mathcal{G}_N$ is a formation $\mathcal{F} \in \mathcal{C}^N(\mathbb{R}^2)$, such that $\Phi_N(\mathcal{F}) = \mathcal{G}$. An arbitrary graph $\mathcal{G} \in \mathcal{G}_N$ can therefore be *realized* as a connectivity graph in $\mathcal{C}^N(\mathbb{R}^2)$ if $\Phi_N^{-1}(\mathcal{G})$ is nonempty. We denote by $\mathcal{G}_{N,\delta} \subseteq \mathcal{G}_N$, the space of all possible graphs on N agents with sensor range δ, that can be realized in $\mathcal{C}^N(\mathbb{R}^2)$. Let us start by analyzing this space for small values of N. Consider the situation in Figure 1, where the 3 agents are positioned at the points marked by circles. Let each position $\mathbf{x}_i$ be given by its Cartesian coordinate pair $(x_{i1}, x_{i2})^T$. For notational convenience let $\|\mathbf{x}_1 - \mathbf{x}_2\| = l_{12}$, $\|\mathbf{x}_2 - \mathbf{x}_3\| = l_{23}$ and $|\mathbf{x}_1 - \mathbf{x}_3\| = l_{13}$. Also let θ and ψ_{123} be the angles shown in the figure. In general, any connectivity graph on N vertices imposes various constraints on the relative positions of individual agents in the configuration space $\mathcal{C}^N(\mathbb{R}^2)$. In the case of a connected graph on 3 vertices, the constraints on positions x_1, x_2 and x_3 correspond to a single constraint on the angle ψ_{123}, when the the agents are positioned as shown in Figure 1. This simple observation will subsequently lead to some interesting properties of the connectivity graphs and their realizations. Suppose we are considering the line graph on 3 vertices in Figure 3, then the given geometrical configuration corresponds to this graph if $l_{12} \leq \delta, l_{23} \leq \delta$, and $l_{13} > \delta$. Moreover we can write

$$\begin{aligned} l_{13}^2 &= (l_{12} + l_{23}\cos\theta)^2 + (l_{23}\sin\theta)^2, \\ &= l_{12}^2 + l_{23}^2 + 2 l_{23} l_{12} \cos\theta. \end{aligned}$$

If $l_{13} > \delta$ then

$$\cos\theta > \frac{\delta^2 - l_{12}^2 - l_{23}^2}{2 l_{23} l_{12}}.$$

It is easy to see that the term on the right has a minimum corresponding to the maximum values of $l_{12} = l_{23} = \delta$. Therefore $\cos\theta > -\frac{1}{2}$ which means that $\theta \in [-\frac{2\pi}{3}, \frac{2\pi}{3}]$. Therefore the smaller angle between l_{12} and l_{23} satisfies $\psi_{123} = \pi - \theta > \frac{\pi}{3}$, for all $0 < l_{12}, l_{23} \leq \delta$ and $l_{13} > \delta$. Hence, whenever we have two edges e_{ij} and e_{ik} in a connectivity graph that share a vertex v_i in such a way that there is no edge between vertices v_j and v_k, then

$$\psi_{j,i,k} \triangleq \cos^{-1}\left(\frac{< \mathbf{x}_j - \mathbf{x}_i, \mathbf{x}_i - \mathbf{x}_k >}{\|\mathbf{x}_j - \mathbf{x}_i\|\|\mathbf{x}_i - \mathbf{x}_k\|}\right) > \frac{\pi}{3} \quad (1)$$

Now, denote by S_N the "star graph" in $\mathcal{G}_N$ i.e.

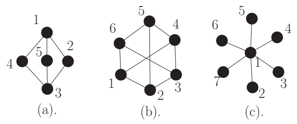

Fig. 2. Graphs $\diamondsuit_5$, $\diamondsuit_6$ and S_7, that are not connectivity graphs

the graph which has $N-1$ vertices $v_2, v_3 \ldots v_N$ of degree 1 and one vertex v_1 with degree $N-1$. An example of such a graph is shown in Figure 2.a. Also denote by $\diamondsuit_5$ and $\diamondsuit_6$, the graphs in $\mathcal{G}_5$ and $\mathcal{G}_6$ respectively, as drawn in Figures 2.a and 2.b. It is easy to see that the graph $\diamondsuit_5 \in \mathcal{G}_5$ does not belong to $\mathcal{G}_{5,\delta}$. This is because, if it is realizable, then the angles ψ_{415}, ψ_{512}, ψ_{123}, ψ_{235}, ψ_{534} and ψ_{341} are all greater than $\frac{\pi}{3}$. Therefore, $\psi_{415} + \psi_{512} + \psi_{123} + \psi_{235} + \psi_{534} + \psi_{341} > 6\left(\frac{\pi}{3}\right) = 2\pi$. But since $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \mathbb{R}^2$ are vertices of a polygon of 4 sides, $\psi_{415} + \psi_{512} + \psi_{123} + \psi_{235} + \psi_{534} + \psi_{341} = 2\pi$, which is a contradiction. By similar geometrical arguments, we can see that $\diamondsuit_6 \notin \mathcal{G}_{N,6}$. Similarly $\mathcal{S}_N \in \mathcal{G}_N$ does not belong to $\mathcal{G}_{N,\delta}$ for $N > 6$ (See (Muhammad *et al.* CDC, 2004) for details). There are of course other examples of realizable and non-realizable connectivity graphs. If a graph is completely disconnected, it means that the distance between any two agents in the formation is separated by more than δ. This can easily be achieved by placing the vertices one by one in such a way that $\mathbf{x}_i$ does not belong to $\bigcup_{j=1}^{i-1} \mathcal{B}_\delta(\mathbf{x}_j)$, where $\mathcal{B}_\delta(\mathbf{x})$ is the closed ball of radius δ centered at $\mathbf{x}$. This observation can be further generalized as follows. A graph $\mathcal{G} \in \mathcal{G}_{N,\delta}$ if and only if each of its connected component $\mathcal{G}_i \in \mathcal{G}_{M_i}$ is realizable in some $\mathcal{G}_{M_i,\delta}$, $M_i < N$.

Theorem: $\mathcal{G}_{N,\delta}$ is a proper subspace of $\mathcal{G}_N$ if and only if $N \geq 5$.

Proof: In order to prove that $\mathcal{G}_{N,\delta}$ is a proper subspace of $\mathcal{G}_N$ for some N, it is enough to show that $\Phi : C^N(\mathbb{R}^2) \to \mathcal{G}_N$ is not onto. Therefore we need to provide a graph $\mathcal{G} \in \mathcal{G}_N$ such that $\Phi^{-1}(\mathcal{G}) = \emptyset$. From the discussion above, we have examples of graphs that are not realizable in $\mathcal{G}_{5,\delta}$ and $\mathcal{G}_{6,\delta}$. For $N \geq 7$ the star graphs $\mathcal{S}_N$ provide

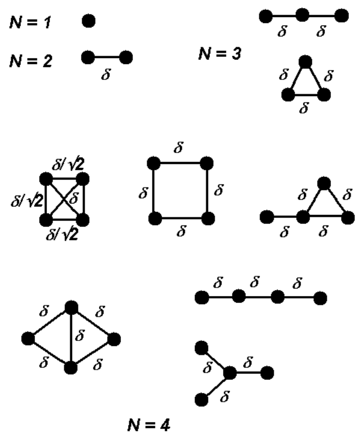

Fig. 3. Possible realizations for all $G \in \mathcal{G}_{N,\delta}$ for $N \leq 4$

the examples of graphs that cannot be realized as connectivity graphs in $\mathcal{G}_{N,\delta}$. This proves that $\mathcal{G}_{N,\delta}$ is a proper subspace of $\mathcal{G}_N$ if $N \geq 5$. To prove that every graph in $\mathcal{G}_N$, for $N < 5$, is realizable in $\mathcal{G}_{N,\delta}$, we have to enumerate all possible graphs for $N < 5$ and give realizations for each graph. Since we are dealing with a small number $N(< 5)$, the enumeration strategy works well. The number of possible graphs to check can be further reduced by noting that we need to consider only connected graphs. These graphs together with their realizations are given in Figure 3, which completes the proof. ∎

It follows from this theorem that if each connected component $\mathcal{G}_i$ of a graph $\mathcal{G} \in \mathcal{G}_N$ belongs to $\mathcal{G}_{M_i}$, $M_i < 5$ then the graph has a realization in $\mathcal{G}_{N,\delta}$. Formations can produce a wide variety of graphs for N vertices. This includes graphs that have disconnected subgraphs or totally disconnected graphs with no edges. However the problem of switching between different formations or of finding interesting structures within a formation of sensor range limited agents can only be tackled if no sub-formation of agents is totally isolated from the rest of the formation. This means that the connectivity graph $\mathcal{G}(t)$ of the formation $\mathcal{F}(t)$ should always remain *connected* (in the sense of connected graphs) for all time t.

3. FEASIBILITY OF FORMATIONS AND THE POSITIVSTELLENSATZ

Some previous results on the set of connectivity graphs $\mathcal{G}_{N,\delta} \subseteq \mathcal{G}_N$ have been summarized above. In the arguments for proving the above theorem, some simple geometrical arguments have been used to show whether a given connectivity graph has any feasible realization. Although, these arguments give sufficient conditions for certain graphs, they cannot be easily applied to all graphs. In fact, it will be interesting to know the answer to the following question: Given *any* arbitrary graph $\mathcal{G} \in \mathcal{G}_N$, can it be realized as a connectivity graph in $\mathcal{C}^N(\mathbb{R}^2)$? Recall that each connectivity graph $(\mathcal{V}, \mathcal{E})$ for the formation $(\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_N) \in \mathcal{C}^N(\mathbb{R}^2)$ can be described by $N(N-1)/2$ relations of the following form:

(1) $\|\mathbf{x}_i - \mathbf{x}_j\| \leq \delta$, if $e_{ij} \in \mathcal{E}$,
(2) $\|\mathbf{x}_i - \mathbf{x}_j\| > \delta$, if $e_{ij} \notin \mathcal{E}$.

Let $\mathbf{x}_i = (x_i, y_i)$ for all $1 \leq i \leq N$, then each of these relations can be written as inequality constraints, $\{f_k \geq 0\}$, where each $f_k \in \mathbb{R}[x_1, y_1, \ldots, x_N, y_N]$, a polynomial in $2N$ variables over the real numbers. Therefore the realization problem is equivalent to asking if there exist $x_1, y_1, \ldots, x_N, y_N$ such that the following inequality constraints are satisfied.

$$\begin{aligned} \delta^2 - (x_i - x_j)^2 - (y_i - y_j)^2 \geq 0, &\text{ if } e_{ij} \in \mathcal{E}, \\ (x_i - x_j)^2 + (y_i - y_j)^2 - \delta^2 > 0, &\text{ if } e_{ij} \notin \mathcal{E}, \end{aligned}$$

where $1 \leq i < j \leq N$. The following is an important result in studying problems of feasibility for semi-algebraic sets (Stengle, 1974).

Positivstellensatz: Given polynomials $\{f_1, \ldots f_s\}$, $\{g_1, \ldots, g_t\}$ and $\{h_1, \ldots, h_u\}$, all in $\mathbb{R}[x_1, \ldots x_n]$, the following are equivalent:

(1) The set $\{x \in \mathbb{R}^n : f_i(x) \geq 0, h_i(x) = 0, g_i(x) \neq 0, i = 1 \ldots s, j = 1 \ldots t, k = 1 \ldots u\}$ is empty.
(2) There exist polynomials $f \in \mathcal{P}(f_1, \ldots f_s), g \in \mathcal{M}(g_1, \ldots g_t), h \in \mathcal{I}(h_1, \ldots h_u)$ such that $f + g^2 + h = 0$,

where $\mathcal{P}(f_1, \ldots f_s)$ is the *Cone* generated by the polynomials $\{f_i\}$, $\mathcal{M}(g_1, \ldots g_t)$ is the *Monoid* over $\{g_i\}$ and $\mathcal{I}(h_1, \ldots h_u)$ is the *Ideal* generated by $\{h_i\}$. See (Stengle, 1974; Hungerford, 2004) for further details on these algebraic objects. Before using the Positivstellensatz, let us first perform the following simplifications. Note that if a formation $(\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_N) \in \mathcal{C}^N(\mathbb{R}^2)$ is feasible, and $A : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry, then $(A\mathbf{x}_1, A\mathbf{x}_2, \cdots A\mathbf{x}_N)$ is also feasible. In fact, this induces an isometry of $C^N(\mathbb{R}^2)$ as well. Let us define an isometry A by

$$A(\mathbf{x}) = R(-\psi)(\mathbf{x} - \mathbf{x}_1),$$

where

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and ψ is the angle of the vector $\mathbf{x}_2 - \mathbf{x}_1$ w.r.t the positive x-axis in $\mathbb{R}^2$. With this choice, $A\mathbf{x}_1 = 0$ and $A\mathbf{x}_2 = [\|\mathbf{x}_2 - \mathbf{x}_1\|, 0]^T$. The existence of such an isometry implies that we can study the non-feasibility problem by ignoring the 3 variables x_1, y_1, and y_2. We therefore setup the quadratic forms in the following way. Let $M = 2N - 2$ and $\mathbf{x} = [x_2, x_3, y_3, x_4, y_4 \ldots x_N, y_N, 1]^T \in \mathbb{R}^M$. If $e_{ij} \in \mathcal{E}$, then the inequality

$$\delta^2 - (x_i - x_j)^2 - (y_i - y_j)^2 \geq 0,$$

can be written as $\mathbf{x}^T A_{ij} \mathbf{x} \geq 0$, where $A_{ij} \in \mathbb{R}^{M \times M}$. If $j > i \geq 3$, then A_{ij} has the following form,

$$\left(\begin{array}{|cc|c|cc|c} & & & & & \\ \hline -1 & 0 & & 1 & 0 & \\ 0 & -1 & & 0 & 1 & \\ \hline & & & & & \\ \hline 1 & 0 & & -1 & 0 & \\ 0 & 1 & & 0 & -1 & \\ \hline & & & & & \delta^2 \end{array}\right) \begin{array}{c} \text{Index} \\ 2i-4 \\ 2i-3 \\ \\ 2j-4 \\ 2j-3 \\ 2N-2 \end{array}$$

and all entries not explicitly written are zeros. It is easy to see that for terms involving $\mathbf{x}_1$ and $\mathbf{x}_2$, we have similar matrices with more zeros at the appropriate slots. In particular, if $e_{12} \in \mathcal{E}$ then the relation between $\mathbf{x}_1$ $\mathbf{x}_2$ simplifies to $x_2^2 \leq \delta^2$, with $A_{12} = \mathrm{diag}(-1, 0, \ldots, 0, \delta^2)$. Similarly, if $e_{lm} \notin \mathcal{E}$, then the inequality

$$(x_l - x_m)^2 + (y_l - y_m)^2 - \delta^2 > 0,$$

can be written as $\mathbf{x}^T B_{lm} \mathbf{x} > 0$. For $m > l \geq 3$ we have

$$B_{lm} = \left(\begin{array}{|cc|c|cc|c} & & & & & \\ \hline 1 & 0 & & -1 & 0 & \\ 0 & 1 & & 0 & -1 & \\ \hline & & & & & \\ \hline -1 & 0 & & 1 & 0 & \\ 0 & -1 & & 0 & 1 & \\ \hline & & & & & -\delta^2 \end{array}\right) \begin{array}{c} \\ 2l-4 \\ 2l-3 \\ \\ 2m-4 \\ 2m-3 \\ 2N-2 \end{array}$$

We will have a total of $N(N-1)/2$ such quadratic forms. The non-feasibility problem is therefore equivalent to asking if the set $X = \{\mathbf{x} \in \mathbb{R}^M \mid \mathbf{x}^T A_{ij} \mathbf{x} \geq 0,\ 1 \leq i < j \leq N,\ e_{ij} \in \mathcal{E}, \mathbf{x}^T B_{lm} \mathbf{x} > 0,\ 1 \leq l < m \leq N,\ e_{lm} \notin \mathcal{E}\}$ is empty. It is worthwhile to note that if the formation is feasible, the above simplification based on the isometry A, bounds the set X in the following way. We give the following standard definition.

The *diameter* D of a graph is defined as the longest graph geodesic between any two graph vertices of a graph.

$$D(G) = \max_{v_i, v_j \in \mathcal{V}} d(v_i, v_j),$$

where $d(v_i, v_j)$ is the graph geodesic between vertices v_i and v_j, given by the minimum length of the paths connecting them. In some sense, it is longest shortest path in a graph. Similar to this quantity, we can define the *graph radius* by,

$$\rho(G) = \max_{v_i \in \mathcal{V}} d(v_i, v_1),$$

We note that $\rho(G) \leq D(G)$. By placing $\mathbf{x}_1$ at $(0, 0)$, we therefore have $\|\mathbf{x}_i\| \leq \rho(G)\delta$ for $2 \leq i \leq N$. If $\mathbf{x} \in X$ then,

$$\mathbf{x}^T\mathbf{x} \leq (N-1)^2\delta^2\rho(G)^2 + 1. \tag{2}$$

Let $\mathcal{D} = \{\mathbf{x} \in \mathbb{R}^M \mid \mathbf{x}^T\mathbf{x} \leq (N-1)^2\delta^2\rho(G)^2 + 1\}$, then $X \subseteq \mathcal{D}$. Therefore for checking non-feasibility, we restrict our search to $\mathcal{D}$. Eq. (2) gives a useful bound on $\mathbf{x}$ and is particularly helpful for numerical computations. Moreover, it tells us that the search for a non-feasibility certificate is restricted to a bounded set. Unfortunately despite being bounded, X is not compact. We will see later that this complicates the search for the certificates.

It should be noted that all semi-algebraic constraints on the set X are quadratic. Moreover $A_{ij} = A_{ij}^T$ and $B_{lm} = B_{lm}^T$. For quadratic constraints, the Positivstellensatz has been shown to be equivalent to the celebrated $\mathcal{S}$-procedure (Parrilo *et al.*, 2003). The $\mathcal{S}$-procedure transforms the Positivstellensatz into a linear matrix inequality (LMI) problem.

Theorem: Given symmetric $n \times n$ matrices $\{A_k\}_{k=0}^m$, the following are equivalent:

(1) The set $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^T A_1\mathbf{x} \geq 0, \mathbf{x}^T A_2\mathbf{x} \geq 0, \cdots, \mathbf{x}^T A_m\mathbf{x} \geq 0, \mathbf{x}^T A_0\mathbf{x} \geq 0, \mathbf{x}^T A_0\mathbf{x} \neq 0\}$ is empty.

(2) There exist non-negative scalars $\{\lambda_k\}_{k=1}^m$ such that $-A_0 - \sum_{k=1}^m \lambda_k A_k \geq 0$.

If $\{\lambda_k\}_{k=1}^m$ exist then let $\mathcal{Q} = -A_0 - \sum_{k=1}^m \lambda_k A_k \geq 0$, $g = \mathbf{x}^T A_0\mathbf{x}$ and

$$f = (\mathbf{x}^T\mathcal{Q}\mathbf{x})(\mathbf{x}^T A_0\mathbf{x}) + \sum_{k=1}^m \lambda_k(\mathbf{x}^T A_0\mathbf{x})(\mathbf{x}^T A_k\mathbf{x}).$$

Since $g \in \mathcal{M}(\mathbf{x}^T A_0\mathbf{x})$ and $f \in \mathcal{P}(\mathbf{x}^T A_0\mathbf{x}, \mathbf{x}^T A_1\mathbf{x}, \cdots \mathbf{x}^T A_m\mathbf{x})$, $f + g^2 = 0$ as desired.

Note that any strict constraint $\mathbf{x}^T C\mathbf{x} > 0$ can be formulated as $\mathbf{x}^T C\mathbf{x} \geq 0, \mathbf{x}^T C\mathbf{x} \neq 0$. Therefore, if we ignore the strictness of all but one of the constraints given by $\mathbf{x}^T B_{lm}\mathbf{x} > 0$, the $\mathcal{S}$-procedure gives the certificates of non-feasibility for a given graph. Unfortunately, the strictness of more than inequality makes the $\mathcal{S}$-procedure *lossy* (Boyd *et al.*, 1994) i.e. it only becomes a sufficient condition. Therefore, this procedure can only work for the limited case described below.

Consider the complete graph $\mathcal{K}_N$. As discussed in (Muhammad *et al.* AMC, 2004), $\mathcal{K}_N$ is always feasible. It will be interesting to find if there is an example of an infeasible graph obtained by removing exactly one edge e, from K_N. We will denote such graph by $K_N - e$. Without loss of generality let $e = e_{12}$. By the procedure described above, the non-feasibility question can be answered by setting up $N(N-1)/2 - 1$ quadratic forms of the type $\mathbf{x}^T A_{ij}\mathbf{x} \geq 0$, $3 \leq i < j \leq N$ and one quadratic form of the type $\mathbf{x}^T B_{12}\mathbf{x} \geq 0$, where $B_{12} = \text{diag}(1, \cdots, -\delta^2)$. Clearly this can be a candidate for testing the $\mathcal{S}$-procedure described above. In fact, it can be seen that the corresponding LMI

$$\mathcal{Q} = -B_{12} - \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \lambda_{ij}A_{ij} \tag{3}$$

can never be non-negative definite for any combination of $\lambda_{ij} > 0$. To prove this, note that $\mathcal{Q}$ can be written as

	x_2	x_3	y_3	x_4	y_4	.	x_i	y_i	.	x_N	y_N	1
x_2	a_{22}	a_{23}	0	a_{24}	.	.	a_{2i}	0	.	a_{2N}	0	0
x_3	a_{23}	a_{33}	0	a_{34}	0	.	a_{3i}	0	.	a_{3N}	0	0
y_3	0	0	a_{33}	0	a_{34}	.	.	a_{3i}	.	0	a_{3N}	0
x_4	a_{24}	a_{34}	0	a_{44}	0	.	a_{4i}	0	.	a_{4N}	0	0
y_4	0	0	a_{34}	0	a_{44}	.	0	a_{4i}	.	0	a_{4N}	0
$\vdots$												$\vdots$
x_i	a_{2i}	a_{3i}	0	a_{4i}	0	.	a_{ii}	0	.	a_{iN}	0	0
y_i	0	0	a_{3i}	0	a_{4i}	.	0	a_{ii}	.	.	a_{iN}	0
$\vdots$	.	.	.	.	.	.	.	.	.	.	.	$\vdots$
x_N	a_{2N}	a_{3N}	0	a_{4N}	0	.	a_{4i}	0	.	a_{NN}	0	0
y_N	0	0	a_{3N}	0	a_{4N}	.	0	a_{4i}	.	0	a_{NN}	0
1	0	0	0	0	0	.	0	0	.	0	0	a_{MM}

where the non-zero off-diagonal entries are

$$a_{mn} = -\lambda_{mn}, \; m \neq n,$$

and the diagonal entries are given by

$$\begin{aligned} a_{ii} &= \quad -1 + \sum_{j=3}^{N} \lambda_{2j}, \qquad i = 2. \\ a_{ii} &= \lambda_{1i} + \sum_{k=2}^{i-1} \lambda_{ki} + \sum_{k=i+1}^{N} \lambda_{ik}, 2 < i \leq N. \\ a_{ii} &= \quad \delta^2(1 - \sum_{k=1}^{N-1}\sum_{j=k+1}^{N} \lambda_{kj}) \qquad i = M. \end{aligned}$$

Notice that the feasibility question is independent of δ. If G is feasible for $\delta_1 > 0$ then it is feasible for all $\delta > 0$. For simplicity let $\delta = 1$. Since $\lambda_{ij} > 0$, $\mathcal{Q} \neq 0$. So we need to show that $\mathcal{Q} \not> 0$. Note that this is *not* equivalent to showing that $\mathcal{Q} < 0$. A necessary condition for $\mathcal{Q} = [q_{ij}] > 0$ is that

$$\begin{aligned} q_{ii} &> 0 \\ q_{ii} + q_{jj} &> 2q_{ij} \end{aligned}$$

Now consider,

$$\begin{aligned} a_{22} + a_{MM} &= -1 + \sum_{j=3}^{N} \lambda_{2j} + 1 - \sum_{i=1}^{N-1}\sum_{k=i+1}^{N} \lambda_{ik} \\ &= -\sum_{j=1}^{N} \lambda_{2j} - \sum_{i=3}^{N-1}\sum_{k=i+1}^{N} \lambda_{ik} \end{aligned}$$

If $\mathcal{Q} > 0$ then $a_{22} + a_{MM} > 0$, but $\lambda_{ij} > 0$ for all $1 \leq i < j \leq N$, so that $a_{22} + a_{MM} < 0$, which is a contradiction. So $\mathcal{Q}$ is never non-negative definite for any choice of λ_{ij}. By the $\mathcal{S}$-procedure, the non-negativity condition of $\mathcal{Q}$ is both necessary and

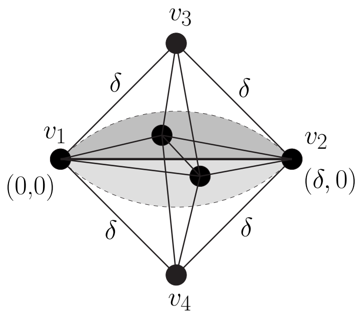

Fig. 4. Geometric construction of a feasible formation for $K_N - e$.

sufficient. Therefore, the set is always non-empty and we have proved the following result.

Proposition: $\mathcal{K}_N - e$ is feasible for all N.

Motivated by this result, one can also give a geometric construction to provide feasible formations for $\mathcal{K}_N - e$ for all N. Consider the realization of the graph depicted in the lower left corner of Fig. 3. This is a realization $\mathcal{K}_4 - e$. Now consider Fig.4. The shaded region is the intersection of discs of radius δ, centered at x_3 and x_4. Any node placed in this shaded region will be connected to all other nodes of the graph. Therefore, we can pack any number of nodes in this shaded region, getting a realization for $\mathcal{K}_N - e$ for arbitrary value of N.

For the general case, where we have multiple strict inequalities, we present the following result for odd number of strict constraints.

Proposition: Let $Q \in \mathbb{N}$ be odd. Given symmetric $n \times n$ matrices $\{A_k\}_{k=0}^{P}$, the set $Y = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^T A_0 \mathbf{x} \geq 0, \cdots, \mathbf{x}^T A_{Q-1}\mathbf{x} \geq 0,$
$\mathbf{x}^T A_Q \mathbf{x} \geq 0, \cdots, \mathbf{x}^T A_P \mathbf{x} \geq 0,$
$\mathbf{x}^T A_0 \mathbf{x} \neq 0, \cdots, \mathbf{x}^T A_{Q-1}\mathbf{x} \neq 0, \}$ is empty if for all $0 \leq q \leq Q-1$, there exist non-negative scalars $\{\lambda_k^{(q)}\}$, $1 \leq k \leq P, k \neq q$ such that

$$-A_q - \sum_{k=1,k\neq q}^{P} \lambda_k^{(q)} A_k \geq 0.$$

Proof: Let $\mathcal{Q}_q = -A_q - \sum_{k=1,k\neq q}^{P} \lambda_k^{(q)} A_k$. If $\mathcal{Q}_q \geq 0$ then let

$$g(\mathbf{x}) \triangleq \prod_{q=0}^{Q-1} (\mathbf{x}^T A_q \mathbf{x}) \in \mathcal{M}(\mathbf{x}^T A_0, \cdots, \mathbf{x}^T A_{Q-1}\mathbf{x}),$$

$$f(\mathbf{x}) \triangleq \prod_{q=0}^{Q-1} (\mathbf{x}^T A_q \mathbf{x})(\mathbf{x}^T \mathcal{Q}_q \mathbf{x}) + \cdots$$
$$\cdots + \sum_{i_1,\cdots i_{Q-1}} \lambda_{j_1}^{i_1} \cdots \lambda_{j_{Q-1}}^{i_{Q-1}} (x^T A_{i_1} x) \dots (x^T A_{i_{Q-1}} x)$$

Here the details on the indexes $i_1, \dots, i_{Q-1}$ and $j_1, \dots, j_{Q-1}$ have been omitted for brevity. Clearly, $f \in \mathcal{P}(\mathbf{x}^T A_0, \cdots, \mathbf{x}^T A_P \mathbf{x})$ so that $f + g^2 = 0$ as desired. ∎

Note that this result only provides a sufficient condition for the non-feasibility of Y for odd number of strict constraints only. Finding a necessary criterion, and finding a criterion for even number of strict constraints is still a topic of research. Once we have such a method, it will be possible to give a complete characterization of the entire class of connectivity graphs, significantly improving the characterization given in the Theorem given in Section 2.

4. CONCLUSIONS

The feasibility of connectivity graphs can be studied using tools from semi-definite programming and algebraic geometry. The positivstellensatz can be used to give explicit certificates of non-feasibility. These computational methods give a way to study the feasibility questions for arbitrary graphs in a more systematic way.

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