Toward Optimal Control of Switched Linear Systems

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Abstract

We investigate the problem of driving the state of a switched linear control system between boundary states. We propose tight lower bounds for the minimum energy control problem. Furthermore, we show that the change of the system dynamics across the switching surface gives rise to phenomena that can be treated as a decidability problem of hybrid systems. Applying recent results on controller synthesis for hybrid systems with linear continuous dynamics, we provide an algorithm for computing the minimum number of switchings of a trajectory from one state to another, and show that this algorithm is computable for a fairly wide class of linear switched systems.

1 Introduction

The problem investigated in this paper concerns optimal control of hybrid dynamic systems, and it arises in a number of situations where the task is to control a system whose dynamics changes at given parts of the state space. This is the case, for instance, when legged locomotive robotic systems are controlled, where each step-cycle consists of a swing and a stance phase, or when autonomous helicopters make transitions between different flight modes [4].

The reason for introducing an optimization criterion into these types of planning problems is basically twofold. First of all, it is obviously preferable to find as good solutions as possible, with respect to a given performance cost function. The second reason is, however, maybe of more importance, and it is related to the question of feasibility. For these types of switched control systems it is typically not an easy task to even find one set of controls that drives the system between given states. Optimal control techniques thus provide us with means for solving this feasibility problem in a systematic way.

The outline of this paper is as follows: In the next section we define our problem and illustrate some of the features that it exhibits which make it hard to find closed form solutions. We are going to stress its non-convex nature, making global optimization hard. Typically, global non-convex optimization involves having to solve branch and bound problems, where each sub-problem produces a local minimum. In order for such an approach to be fruitful it is vitally important that tight lower bounds can be found in order to be able to terminate the search with certain tolerance margins, which is the topic of Section 3. In Section 4, we try to understand the influence that switching surfaces have on problems that, at first glance, appear to present only minor difficulties. Applying concepts from optimal control [7], mathematical logic [3], and recent results on controller synthesis for classes of hybrid systems with linear continuous dynamics [8, 9] we show that the change of system dynamics across the switching surface gives rise to phenomena that can be treated as a decidability problem of controller synthesis for hybrid systems.

2 Problem Description

The problem that we will examine in this paper can be stated as follows:

Problem 2.1

$$\min_u \int_{t_0}^{t_1} u^T(t)u(t)dt$$

subject to

$$\dot{x} = \begin{cases} A_1x + B_1u & \textit{if } c^Tx > k \\ A_2x + B_2u & \textit{if } c^Tx < k \end{cases}$$
$$x(t_0) = x_0,\ x(t_1) = x_1,$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^p, k \in \mathbb{R}$, *and* (A_1, A_2, B_1, B_2, c) *are constant matrices of compatible dimensions. We furthermore assume, throughout the paper, that* $(A_1, B_1), (A_2, B_2)$ *are controllable pairs, and that* $(c^Tx_0-k)(c^Tx_1-k) < 0$ *as well as* $(c^TAx_0)(c^TAx_1) \neq 0$.

If this problem did not have a discontinuity at $\{x|c^Tx = k\}$ it would be a very standard and well-studied one. However, the switching surface gives rise to a non-convex problem, and we state this fairly obvious fact as a proposition.

Proposition 2.2 *The set of controls in* $PC_p[t_0, t_1]$ *(the space of piece-wise continuous functions :* $[t_0, t_1] \to \mathbb{R}^p$*) that satisfies the boundary conditions, under system dynamics given in Problem 2.1, is in general non-convex.*

The consequences from this fact can be illustrated by the following example.

Example 2.3 (Non-Convexity) *Given*

$$\ddot{x} = \begin{cases} u, & x > 0 \\ -u, & x < 0. \end{cases}$$

The set of controls that drives this system between $(0,1)^T$ and $(0,1)^T$ in time $t_1 = 4$ is non-convex. For instance, $u_1 = -1$ gives a curve that crosses the switching surface once (not counting the boundary points), while $u_2 = -2$ gives a curve that crosses the surface three times while respecting the boundary conditions, as seen in Figure 1. However, any convex combination $u_\lambda = \lambda u_1 + (1-\lambda)u_2$ violates these conditions for every $\lambda \in (0, 1)$. Thus the set of feasible control inputs is non-convex.

As we have pointed out in the introduction already, this makes our general problem very hard to solve. Thus our ambition is to come up with lower bounds that can serve as guides when solving the problem numerically.

However, the switching dynamics introduce yet another phenomena that we have to take into account

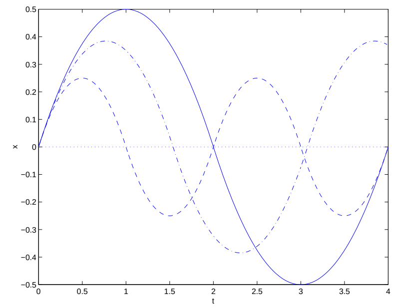

Figure 1: The solid line is generated with $u = -1$ and the dashed line with $u = -2$ in Example 2.3. The dash-dotted line is a non-feasible convex combination of these two and the dotted line shows the switching surface $x = 0$.

when lower bounds are derived, and that is the possibility of sliding solutions. Sliding occurs when the two vector fields on the switching surface both point towards the surface. This gives rise to a an induced flow on the surface in the sense of Filippov [5]. What this corresponds to is an infinite number of switches, which indicates that we in general can not make any assumptions about the curve only crossing the switching surface a finite number of times.

It should be noted that we in Problem 2.1 did not specify the dynamics where $c^Tx = k$. This is due to the fact that we either have an induced flow on this surface, or simply have a scenario where the dynamics on the zero-measure part of the trajectory that lies on the surface does not effect the integral flow [5].

Example 2.4 (Sliding) *Let the system dynamics* $(A_1, B_1), (A_2, B_2)$ *be given by*

$$A_i = \begin{pmatrix} \alpha & \beta_i \\ -\beta_i & \alpha \end{pmatrix},\ i = 1, 2$$
$$\alpha > 0, \beta_1 > 0, \beta_2 = -\beta_1$$
$$c^Tx = x_2,$$

with $B_i, i = 1, 2$ *are arbitrary non-zero vectors in* $\mathbb{R}^n$.

These two systems correspond to clockwise and counter clockwise outgoing spirals that point towards the switching surface. The optimal control that drives this system, with a choice of $\alpha = \beta_1 = 1$, from $(-0.5, -0.5)$ or $(-0.5, 0.5)$ to $(2.5, 0)$ can easily be found to be $\hat{u} = 0$, and the solution is as follows. First, flow along one of the spirals until the switching surface is reached. Then slide along this surface. The sliding solution is given by the unique, convex combination of the two vector fields that remain on the switching surface.

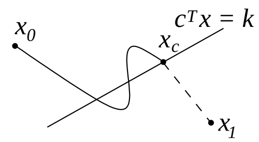

Figure 2: The solid line is the minimum energy trajectory for the first control system between x_0 and the optimal x_c. The dashed line corresponds to the second system. Note that this solution intersects the switching surface without having changed system dynamics, and thus rendering the solution to be infeasible.

A third observation that must be made is that an obvious attempt to solve Problem 2.1 is to find the optimal (t_c, x_c) where $t_c \in [t_0, t_1], x_c = x(t_c)$ under the system dynamics in Problem 2.1, and $c^T x_c = k$. Then the system should simply be driven between x_0 and x_c with a minimum energy controller, and then from x_c to x_1 in a similar way. However, there are no guarantees that this solution is feasible, as seen in Figure 2. Unfortunately we have to choose another route toward solving Problem 2.1.

3 Minimum Energy Control Bounds

The main result that this section presents is that Problem 2.1 has lower bounds that can be calculated by finding the intersection of minimum energy trajectories with appropriate hyperplanes. But, in order to arrive at this result we will need some preliminary results, and we start with the well-known, unconstrained minimum energy controller [1].

Problem 3.1

$$\min_u \int_{t_0}^{t_1} u^T(t)u(t)dt$$

subject to $\dot{x} = Ax + Bu,\ x(t_0) = x_0,\ x(t_1) = x_1$, *where* (A, B) *is a controllable pair.*

Proposition 3.2 *The solution to Problem 3.1 is given by* $\hat{u}(t) = B^T \Phi^T(t_1, t) W^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0]$, *where* $W(t_0, t_1)$ *is the positive definite Reachability Gramian. Furthermore, the optimal value is* $[x_1 - \Phi(t_1, t_0)x_0]^T W^{-1}(t_0, t_1)[x_1 - \Phi(t_1, t_0)x_0]$.

Problem 3.3

$$\min_{x_c, t_c, u} \int_{t_0}^{t_c} u^T(t)u(t)dt$$

subject to

$$\dot{x} = Ax + Bu,\ x(t_0) = x_0,\ x(t_c) = x_c$$
$$c^T x_c = k,\ t_c \in [t_0, t_1]$$

where $c^T x_0 > k$ *and* $c^T A x_0 \neq 0$.

Lemma 3.4 *The optimal value of Problem 3.3 has the lower bound*

$$\min_{t_c \in S} [x(t_c) - \Phi(t_c, t_0)x_0]^T W^{-1}(t_0, t_1)[x(t_c) - \Phi(t_c, t_0)x_0],$$

where

$$x(t_c) = \Phi(t_c, t_0)x_0 + W(t_0, t_1)c \frac{k - c^T \Phi(t_c, t_0)x_0}{c^T W(t_0, t_1)c}$$

and $S = \big(\{t | c^T \Phi(t, t_0)x_0 = k\} \cup \{t | c^T A \Phi(t, t_0)x_0 = 0\} \cup \{t_0, t_1\}\big) \cap [t_0, t_1]$.
It should be noted that unless $c^T x_0 = k$ and $c^T A x_0 = 0$, S is finite which makes the computations numerically tractable. Finiteness of S thus follows directly from our assumptions about x_0 in Problem 3.3 combined with the observation that only a finite number of oscillations can take place around $c^T x = k$ on the finite interval $[t_0, t_1]$.

To prove this lemma we are going to need the following two results atht can be derived directly from the definition of $W(t_0, t)$ and standard quadratic programming respectively.

Lemma 3.5

$$\frac{d}{dt} v^T W^{-1}(t_0, t) v \leq 0 \quad \forall v \in \mathbb{R}^n.$$

Lemma 3.6 *A necessary condition for the optimal solution of the problem*

$$\min_{t,x} \left\{ (x - f(t))^T Q (x - f(t)) \right\}$$

subject to $c^T x = k$, *where* Q *is positive definite is* $c^T f(t) = 0$ *or* $c^T f(t) = k$, *while* $x = f(t) + Q^{-1} c (k - c^T f(t)) / (c^T Q^{-1} c)$.

Let us now return to the proof of Lemma 3.4.
Proof (Lemma 3.4): Since, by Lemma 3.5, we can get a lower bound by simply setting $t = t_1$, which is the upper bound on the interval of consideration.

Now letting $f(t) = \Phi(t, t_0)x_0$ in Lemma 3.6 gives us the necessary conditions $c^T \Phi(t_c, t_0)x_0 = k$ or $c^T A \Phi(t_c, t_0)x_0 = 0$. Including the endpoints in the time interval directly gives us that $t_c \in S$, where S is defined in Lemma 3.4, and thus the lemma holds. ■

We are now ready to state the main theorem that provides us with computationally feasible lower bounds.

Theorem 3.7 *The cost J of Problem 2.1 satisfies the inequality $J \geq v_1^T Q_1 v_1 + v_2^T Q_2 v_2$, where*

$$Q_1^{-1} = W_1(t_0, t_1) = \int_{t_0}^{t_1} \Phi_1(t_1, t) B_1 B_1^T \Phi_1^T(t_1, t) dt$$
$$\Phi_1(t_1, t) = e^{A(t_1 - t)}$$

and Q_2 is chosen similarly with A_1 replaced by $-A_2$, and B_1 by $-B_2$. (For symmetry reasons we view the second system as evolving backwards from the endpoint x_1.) The vectors are $v_1 = [x_{1c} - \Phi_1(t_{1c}, t_0)x_0]$, where t_{1c} and x_{1c} is chosen as in Lemma 3.4. The vector v_2 is chosen in a similar fashion.

Proof: Clearly, every solution (including the optimal) to Problem 2.1 has at least one part of the state space trajectory going from x_0 to $\{x : c^T x = k\}$ and one from $\{x : c^T x = k\}$ to x_1. Denote the two costs associated with these two parts J_1 and J_2. Then $J \geq J_1 + J_2$. But, by applying Lemma 3.4 to these two costs directly gives us lower bounds for J_1 and J_2 respectively. These bounds furthermore coincide with those in Theorem 3.7, which concludes the proof. ∎

If we now return to Example 2.4 we saw that $\hat{u} \equiv 0$, implying that $J = J_1 = J_2 = 0$, and thus the bound is tight in this case.

When it comes to upper bounds the situation becomes slightly different. Obviously any feasible solution to the switched interpolation problem produces an upper bound, but, as pointed out in the introduction, feasibility is not necessarily easy to achieve. In fact, optimization procedures draw some of their importance from the fact that they produce feasible solutions in a systematic way.

An example of finding an upper bound by designing a feasible solution that only crosses the switching surface once can be seen in Example 3.8, as well as in Figure 3.

Example 3.8 (Transversality) *Let*

$$A_1 = A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$B_i = \begin{pmatrix} 0 \\ \beta_i \end{pmatrix}, \; \beta_1 = 3, \; \beta_2 = 1$$
$$c^T = (1, 1), \; k = 2$$
$$x_0^T = (0, 0), \; x_1^T = (2, 2), \; t_0 = 0, \; t_1 = 2.$$

This switching system has a polynomial flow of degree three, when driven by minimum energy controllers, and the different β_i-values can be thought of as gears. If we just pick $x_c = (1, 1)^T, t_c = 1.5$, we get the cost $J = 8.1$ when driving between x_0 and x_c, and x_c and x_1 with minimum energy controllers. This solution is

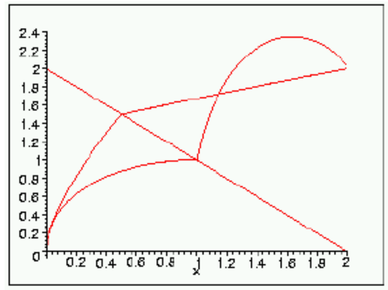

Figure 3: An example with a transversal, piecewise polynomial flow of degree three. Depicted is a feasible solution, $x_c = (1, 1)^T$, and a locally optimal solution, $\hat{x}_c = (0.5, 1.5)^T$.

feasible, as seen in Figure 3, which thus provides us with upper bounds. However, if we pick, numerically, locally optimal $(\hat{x}_c, \hat{t}_c)$ around (x_c, t_c), by for instance checking the Weierstrass-Erdman Corner Condition, we get an even better bound with $\hat{x}_c = (0.5, 1.5)^T, \hat{t}_c = 1.14$, and $\hat{J} = 0.6$.

With this example we conclude the discussion about how to bound the solution to Problem 2.1.

4 Switching Surfaces and Decidability

In this section, we study the influence that switching surfaces have on problems that at first glance appear to present only minor difficulties. Let us take another look at the switched linear system:

$$\dot{x} = \begin{cases} A_1 x + B_1 u & \text{if} \quad x \in M_1 \\ A_2 x + B_2 u & \text{if} \quad x \in M_2 \end{cases}$$
$$M_1 \cap M_2 = \emptyset, \; M_1 \cup M_2 = \mathbb{R}^n,$$
$$x(t_0) = x_0 \in X_0, \; x(t_f) = x_f \in X_f. \quad (1)$$

Given a suitable objective function (minimum energy, time, *etc.*) one would could ask the following list of increasingly difficult questions:

- What is the minimum number of switchings of a trajectory from some $x_0 \in X_0$ to some $x_f \in X_f$?
- Which trajectories have the minimum number of switchings from some x_0 to some x_f?
- How many switchings does the optimal trajectory (w.r.t. some objective function) from x_0 to x_f take?
- Which controllers generate trajectories that minimize the objective?

A natural direction of research is to find classes of switched linear systems for which we can answer one or more of the above questions. In this section, following the work of [6, 8, 9], we use an interesting interplay between classical optimal control [7], and mathematical logic [3, 10] to tackle the first problem and prove the following result:

Theorem 4.1 (Min switching is semi-decidable) *For* normal *switched linear systems the problem of computing the minimum number of switchings of a trajectory from some $x_0 \in X_0$ to some $x_f \in X_f$ is semi-decidable.*

Definition 4.2 (Normal switched linear system) *The switched linear system given in equation (1) is called* normal *if: 1. the systems (A_i, B_i) are* normal. *2. the dynamic matrices A_i are nilpotent or diagonalizable with real rational eigenvalues. 3. X_0, X_f, M_1, M_2 are semi-algebraic sets. 4. the input is constrained to $u \in U$, a compact rectangle with rational vertices.*

The notion of *normality* comes from classical optimal control literature. A linear system (A, B) is normal iff (A, b_i) is completely controllable for each column b_i of B. Semi-algebraic sets are important from both a geometric and mathematical logic perspective. A *semi-algebraic set* is a finite union of *basic* semi-algebraic sets of the form: $\{x \mid f_1(x) < 0, \ldots, f_p(x) < 0, g_1(x) = 0, \ldots, g_q(x) = 0\}$ where $f_i(\cdot)$, $g_i(\cdot)$ are polynomials. Semi-algebraic sets have very nice properties, such as: the closure, interior, convex hull, union, intersection, and the projection of semi-algebraic sets is semi-algebraic. The proper setting for these results is provided by the theory of o-minimal structures [3]. Semi-algebraic sets are exactly those sets definable in the theory reals $(\mathbb{R}, +, -, \cdot, <, 0, 1)$: for each semi-algebraic set M, there exists a *first-order* formula ϕ in the describes the set $M = \{x \in \mathbb{R}^n \mid \phi(x)\}$, where ϕ is a conjunction of disjunctions of (possibly quantified) polynomial inequalities. The theory of reals is known to admit *quantifier elimination*: any first-order formula can be converted to an equivalent quantifier-free formula [10].

Consider the following algorithm for computing the minimum number of switchings of a trajectory from some $x_0 \in X_0$ to some $x_f \in X_f$. Without loss of generality, we take $X_f \subset M_2$.

Algorithm 4.3 (Minimum switchings)

initialize
$S = \overline{M}_1 \cap \overline{M}_2$, $W^0 = \text{Pre}_2(X_f)$, $j = 0$
while $W^j \cap X_0 = \emptyset$
$W^{j+1} = \text{Pre}_1(W^j \cap S) \cup \text{Pre}_2(W^j \cap S)$
$j = j + 1$
end while
set $J = j$

To implement the above algorithm, one needs to encode sets of states, perform intersection, union, test for emptiness, and compute Pre_i of a set. If all these conditions hold for a class of systems, then we say the minimum switchings problem is *semi-decidable* for that class of systems. If in addition, we can guarantee that the algorithm terminates after a finite number of iterations, then we say the problem for that class of systems is *decidable*. If we restrict ourselves to the semi-algebraic sets, then a framework for computing intersection, union, and tests for emptiness of sets is provided by mathematical logic and quantifier elimination. For example, we can test for emptiness of the intersection of two semi-algebraic sets $G = \{x \mid \phi(x)\}$, $H = \{x \mid \psi(x)\}$ by quantifier elimination: $G \cap H \neq \emptyset \quad \Leftrightarrow \quad (\exists x : \phi(x) \wedge \psi(x)) \equiv \mathsf{True}$. The difficulty lies only in the computation of $\text{Pre}_i(\cdot)$ which is now defined:

Definition 4.4 (Constrained predecessor Pre**)** *For a system $\dot{x} = Ax + Bu$ whose state is constrained to $x \in M$ and input constrained to $u \in U$,* Pre(K) *is the set of all initial conditions for which there exists a trajectory which eventually enters K:*

$$\begin{aligned}\text{Pre}(K) \triangleq \{x \in \mathbb{R}^n \mid \exists t > 0,\ \exists [u(\cdot) : [0,t] \to U] :& \\ y(\tau) = e^{A\tau}x + \textstyle\int_0^\tau e^{A(\tau - s)} Bu(s) ds \wedge& \\ y(t) \in K \wedge [\forall \tau : 0 \leq \tau \leq t \Rightarrow y(\tau) \in M]\}.&\end{aligned}$$

In the form described above it is not at all clear if Pre(K) is computable. However, as an immediate corollary to Theorem 4.3 of [9], a very recent result on the computability of the "Reach-Avoid" operator in controller synthesis for normal linear differential games, we have the following:

Theorem 4.5 (Computability of constrained Pre**)** *Given a normal linear system $\dot{x} = Ax + Bu$, where A is nilpotent or diagonalizable with real rational eigenvalues, the input is constrained to $u \in U$ a compact rectangle with rational vertices, and the state is constrained to a semi-algebraic set M, the problem of computing the constrained predecessor* Pre(K) *for a semi-algebraic set K is decidable.*

As an immediate result of Theorem 4.5 and the fact that the theory of reals admits quantifier elimination we have:

Corollary 4.6 (Computability of algorithm) *For the class of normal switched linear systems each iteration of Algorithm 4.3 is computable.*

Lemma 4.7 *If Algorithm 4.3 terminates, then J is the minimum number of switchings of system dynamics that will take us from some $x_0 \in X_0$ to some $x_f \in X_f$.*

Proof: Put $W^0 = \mathrm{Pre}_2(X_f)$. For $j = 0$, if $W^j \cap X_0 \neq \emptyset$, then there exists a trajectory along the dynamics of $\dot{x} = A_2x + B_2u$ which takes x_f to x_0 without switching and we are done. Otherwise, there must be at least $j+1$ switchings.

Compute $W^{j+1} = \mathrm{Pre}_1(W^j \cap S) \cup \mathrm{Pre}_2(W^j \cap S)$, the set of initial conditions for which a trajectory exists along either of the dynamics of the two systems which enters the switching surface at $W_j \cap S$. If $W^{j+1} \cap X_0 \neq \emptyset$, there exists a trajectory from some $x_0 \in X_0$ which passes through some $x \in W^j \cap S$. Since there exist a trajectory from any $x \in W^j \cap S$ to some $x_f \in X_f$ that switches j times, by composition we have a trajectory that switches $j+1$ times which takes x_f to x_0. Otherwise if $W^{j+1} \cap X_0 = \emptyset$, then we set $j = j+1$ and repeat the argument. ■

As an immediate corollary to Corollary 4.6 and Lemma 4.7, we have the main result of this section, Theorem 4.1. Figure 4 shows a conceptual depiction of the computation performed in Algorithm 4.3.

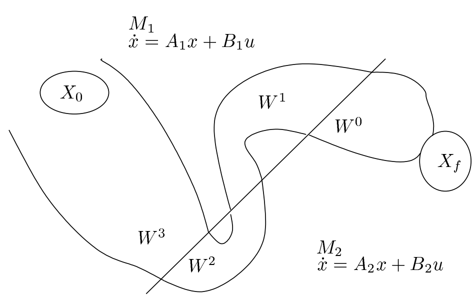

Figure 4: Graphical depiction of reachability algorithm to compute the minimum number of switchings.

From Theorem 4.5, and the proof of Lemma 4.7 it is clear that by a straight-forward modification of Algorithm 4.3 one may obtain the same semi-decidability result on the minimum number of switchings of a trajectory if the switched system were modified to have more than two switching regions: $\dot{x} = A_ix + B_iu,\ \mathit{if}\ x \in M_i$, for $i \in \{1, \ldots, N\}$.

5 Conclusions

In this paper, we studied the problems that arise when trying to drive the states of a switched linear control system between boundary states. The control systems are piecewise linear, with changes occurring on hyperplanes in the state space, and we proposed computationally feasible tight lower bounds for the minimum energy control problem. We also discussed the possibility to find upper bounds, but such bounds were not found for general, switched, linear control systems. As a first step in this direction, we provided an algorithm for computing the minimum number of switchings of a trajectory from one state to another, and showed that this algorithm is computable for a fairly wide class of switched linear systems.

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