PERGAMON

Automatica 37 (2001) 1057–1064

automatica

www.elsevier.com/locate/automatica

Brief Paper

Optimal trajectory planning and smoothing splines☆

Magnus Egerstedt[a,*,1], Clyde F. Martin[b,2]

[a]Division of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138, USA

[b]Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409, USA

Received 30 August 1999; revised 24 February 2000; received in final form 12 December 2000

Abstract

In this paper, some of the relationships between optimal control and trajectory planning are examined. When planning trajectories for linear control systems, a demand that arises naturally in air traffic control or noise contaminated data interpolation is that the curve passes close to given points, or through intervals, at given times. In this paper, we produce these curves by solving an optimal control problem for linear control systems, while driving the output of the system close to the waypoints. We furthermore show how this optimal control problem reduces to a finite, quadratic programming problem, and we thus provide a constructive, yet theoretically sound framework for producing a rich set of curves called smoothing splines.

Keywords: Trajectory planning; Linear systems; Optimal control; Splines

1. Introduction

In this paper, we consider the problem of finding the control that drives the output of a given linear control system close to predefined points, or through intervals, at given times. The input applied to the system will be chosen in such a way that it minimizes a quadratic cost functional, and the resulting output curve will be a so-called smoothing spline.

Classical polynomial splines and the splines developed in Martin, Tomlinson, and Zhang (1997) and Wahba (1990) are interpolating splines, i.e. they are required to pass through specific points at specific times. In most applications, including trajectory planning, this is overly restrictive. We are usually content if the trajectory passes close to an assigned point at an assigned time. We thus dedicate this article to the development of such smoothing splines which are reflective of the dynamics of the underlying system.

These types of relaxed interpolation problems exhibit properties that are desirable for two different reasons. First of all, small deviations from the waypoints can result in a significant decrease in the cost, and secondly, when the data that we work with is noise contaminated, it is not even desirable to interpolate through these points exactly. Inspired by Luo and Wahba (1997), we can incorporate these aspects into our proposed method, and by not demanding exact interpolation, we end up with smoothing splines instead of the standard splines (Egerstedt & Martin, 1998).

Even though there already exist good numerical tools for solving more general optimal control problems (Rehbock, Teo, Jennings, & Lee, 1999), the main contribution in this paper is that we provide closed form solutions, or solutions that only require a minimum of numerical computations, which presents an advantage over, for example, iterative dynamic programming techniques (Bertsekas, 1995, and Nash & Sofer, 1996), or standard linear-quadratic optimal control methods (Chen, 1989). Our proposed methods can thus be used in time critical applications, as well as offer some new insight into the interpolation problems addressed in this paper, since we show how they can be raised and solved in a natural way as control theoretic problems.

☆This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor K.L. Teo under the direction of Editor Tamer Basar.

* Corresponding author. Tel.: + 1-617-496-2942; fax: + 1-617-496-6404.

E-mail address: magnuse@hrl.harvard.edu (M. Egerstedt).

[1] Sponsored in part by the Swedish Foundation for Strategic Research through its Centre for Autonomous Systems at KTH.

[2] Supported in part by NASA Grants NAG 2-902 and NAG 2-899 and by NSF Grant DMS 9628558.

0005-1098/01/$ - see front matter
PII: S0005-1098(01)00055-3

The outline of this paper is as follows: In Section 2 we briefly discuss some facts about linear control systems, and show how they can be used to arrive at an optimal control problem, suitable for producing the output curves discussed above. We then proceed, in Sections 3 and 4, to actually solving the problem of producing the smoothing splines, using standard optimal control techniques together with mathematical programming, reducing the problem into a convex, easy-to-solve, quadratic programming problem. In Section 3 we focus on driving the curves close to given points in the state space, while in Section 4 we add hard interval interpolation constraints to our problem formulation. We then, in Section 5, discuss some differentiability properties of the generated curves. We conclude with an example that shows how our approach can be used to produce paths under velocity, acceleration and jerk limitations.

2. Problem formulation

In this paper, we study linear, single input, multiple output control systems of the form

$$\dot{x} = Ax + bu,$$
$$y = C^T x, \tag{1}$$

where $x \in \Re^n$, $y \in \Re^p$, $u \in \Re$, and (A, b, C) are constant matrices of compatible dimensions. We furthermore, without loss of generality, let $x(0) = 0$.

Assumption 1. *The system (A, b, C) is both controllable and observable.*

The problem that we want to solve is the following: Given intervals and interpolation times $([\alpha_i, \beta_i], t_i)$, $i = 1, \dots, m$, how do we drive the outputs of the system through these intervals

$$y(t_i) \in [\alpha_i, \beta_i], \quad i = 1, \dots, m, \tag{2}$$

where $\alpha_i, \beta_i \in \Re^p$, while interpolating close to $\xi_i \in \Re^p$ at the interpolation times, t_i, $i = 1, \dots, m$? In (2) the inclusions should be taken componentwise, and one natural choice of interpolation points could for instance be $\xi_{ij} = (\beta_{ij} + \alpha_{ij})/2$, $i = 1, \dots, m$, $j = 1, \dots, p$, even though they can be any points in $\Re^p$ since exact interpolation is not required. We furthermore want to accomplish this while minimizing the following cost functional

$$\frac{1}{2}\int_0^T \rho u^2(t)\,\mathrm{d}t, \tag{3}$$

over $u \in L^2[0, T]$.

It is a well known fact (see for example Brockett (1970)) that the problem of interpolating the outputs of the system through given, distinct points, while minimizing (3), has a solution if the system (1) is minimal, which was the case here due to Assumption 1. However, since we do not demand exact interpolation, existence and uniqueness results must still be established in this case before we can proceed to actually trying to find the solution to this optimal control problem.

3. Smoothing splines

Initially, we neglect the interval interpolation constraints and focus on penalizing deviations from the desired points quadratically. What we want to do is to produce what in the statistics literature is referred to as smoothing splines (Luo & Wahba, 1997, and Wegman & Wright, 1983), by minimizing

$$\min_{u \in L^2[0,T]} \frac{1}{2}\int_0^T \rho u^2(t)\,\mathrm{d}t + \frac{1}{2}\sum_{i=1}^m (y(t_i) - \xi_i)^T \tau_i (y(t_i) - \xi_i)$$
$$= \min_{u \in L^2[0,T]} J(u), \tag{4}$$

where $\tau_i = \mathrm{diag}(\tau_{i1}, \dots, \tau_{ip})$ is a diagonal, positive semidefinite $p \times p$ weight matrix, where the jth diagonal element describes how important it is that the jth output interpolates close to ξ_i's jth component at time t_i. The parameter $\rho > 0$ furthermore controls the amount of smoothing used for producing the output curve.

We now define the following set of basis functions

$$g_i(t) = \begin{cases} C^T e^{A(t_i - t)} b & t \leqslant t_i \\ 0 & t > t_i \end{cases} \quad i = 1, \dots, m. \tag{5}$$

Lemma 2. *The set of functions $\{g_i(t)\}_{i=1}^m$ are linearly independent.*

This is an obvious fact since the different g_i's vanish at different times.

Based on this new set of basis functions, it is possible to rewrite $J(u)$ as

$$J(u) = \frac{1}{2}\int_0^T \rho u^2(t)\,\mathrm{d}t + \frac{1}{2}\sum_{i=1}^m \left(\int_0^T g_i(s)u(s)\,\mathrm{d}s - \xi_i\right)^T \tau_i \left(\int_0^T g_i(s)u(s)\,\mathrm{d}s - \xi_i\right)$$
$$= \frac{1}{2}\int_0^T \rho u^2(t)\,\mathrm{d}t + \frac{1}{2}\left(\int_0^T g(t)u(t)\,\mathrm{d}t - \xi\right)^T \mathscr{T}\left(\int_0^T g(t)u(t)\,\mathrm{d}t - \xi\right),$$

where $g(t) = (g_1(t)^T, \dots, g_m(t)^T)^T$, $\xi = (\xi_1^T, \dots, \xi_m^T)^T$, and $\mathscr{T} = \mathrm{diag}(\tau_{11}, \dots, \tau_{1p}, \dots, \tau_{m1}, \dots, \tau_{mp})$ is a positive semidefinite weight matrix. We use $\succcurlyeq$ and $\succ$ to denote positive semidefiniteness and positive definiteness, respectively.

Lemma 3. *$J(u)$ is convex in u.*

Since J is a closed, quadratic function in u, convexity follows immediately since $\rho > 0$, $\mathscr{T} \succcurlyeq 0$. From convexity, results on existence and uniqueness now follow from standard infinite dimensional optimization theory. (See for example Luenberger (1969).)

Theorem 4. *There exists a unique, minimizing $u \in L^2[0, T]$, and it is given by the u_0 that makes the Fréchet differential of J vanish for all increments $h \in L^2[0, T]$.*

The Fréchet differential, with increment $h \in L^2[0, T]$, is given by

$$\delta J(u;h) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (J(u + \varepsilon h) - J(u))$$
$$= \int_0^T \left(\rho u(t) + g(t)^T \mathscr{T} \left(\int_0^T g(s) u(s) \, ds - \xi \right) \right) h(t) \, dt. \tag{6}$$

For this differential to vanish for all $h \in L^2[0, T]$, it has to hold that

$$\rho u(t) + g(t)^T \mathscr{T} \left(\int_0^T g(s) u(s) \, ds - \xi \right) = 0.$$

Setting $u(t) = \eta^T g(t)$, which is a finite dimensional parameterization of the problem, gives the optimal η as

$$\eta = (\rho I + \mathscr{T} G)^{-1} \mathscr{T} \xi, \tag{7}$$

where $G \in \Re^{mp \times mp}$ is the Gramian

$$G = \int_0^T g(s) g(s)^T \, ds$$

and thus

$$u_0(t) = g(t)^T (\rho I + \mathscr{T} G)^{-1} \mathscr{T} \xi \tag{8}$$

is the unique minimizer in $L^2[0, T]$.

It is worth stressing that the optimal control is a linear combination of the g_i's. We thus reduce a non-parametric problem to the problem of calculating parameters in a finite dimensional space. Some results from applying this method can be seen in Figs. 1 and 2.

4. Interval interpolation

We now add the interval interpolation constraints to the optimal control problem, demanding that

$$y(t_i) \in [\alpha_i, \beta_i], \quad i = 1, \ldots, m, \tag{9}$$

and formulate the following key lemma:

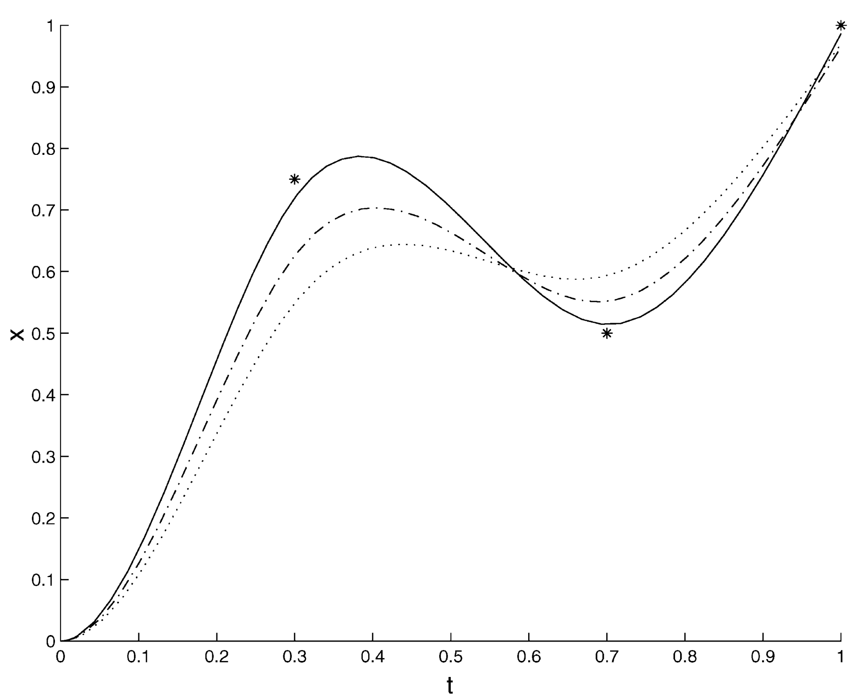


Fig. 1. A smoothing spline with different smoothing parameters, ρ, is shown. The parameters are 1/10000 (solid), 1/2000 (dash–dotted), and 1/1000 (dotted). The τ_i's are all equal to one, and the stars correspond to the different waypoints. The underlying second order system was given by $\ddot{x} = u$ and $y = x$.

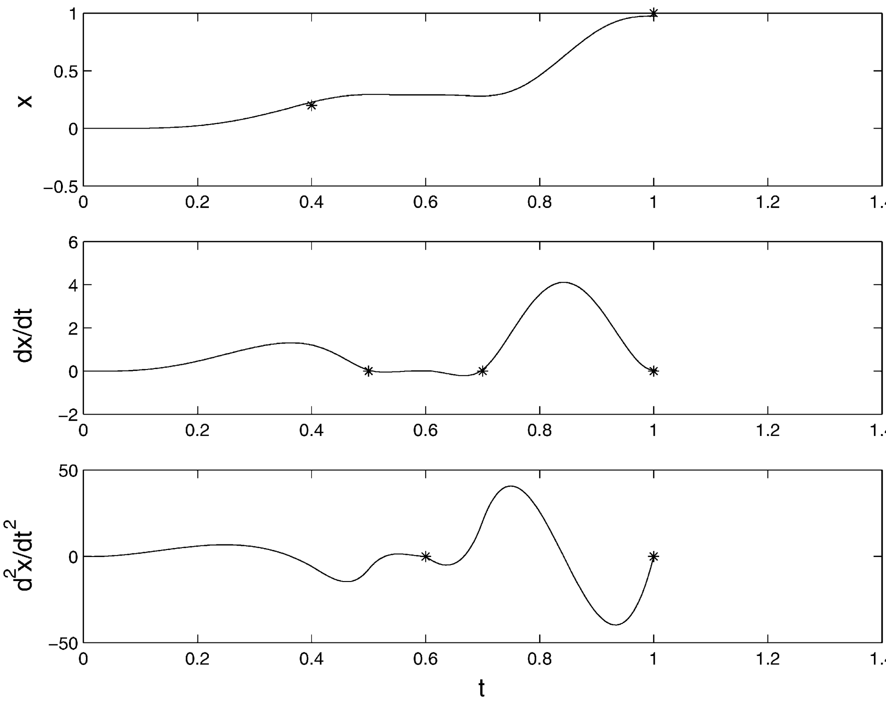


Fig. 2. Interpolation through points specified simultaneously for the position (upper graph), velocity (middle graph) and acceleration (lower graph), while controlling the jerk directly. The trajectory passes closely to the waypoints since we chose to let the τ_i's be large, relative to ρ, giving a high priority to interpolation rather than smoothing.

Lemma 5. *The set of controls that make y satisfy the interval interpolation constraints is a non-empty, closed and convex subset of* $L^2[0,T]$.

Proof. Let $\{\sigma_k\}_{k=1}^N$ be such that

$$\sum_{k=1}^{N} \sigma_k = 1, \quad \sigma_k \geqslant 0, \quad k = 1,\dots,N.$$

Given a set of feasible outputs, $y_k(t_i) \in [\alpha_i, \beta_i]$, $i = 1,\dots,m$, $k = 1,\dots,N$, then obviously

$$\sum_{k=1}^{N} \sigma_k y_k(t_i) \in [\alpha_i, \beta_i], \quad i = 1,\dots,m.$$

But if y_k is generated by the underlying control u_k, such that the interpolation constraints are satisfied, then

$$\sum_{k=1}^{N} \sigma_k y_k(t_i) = \sum_{k=1}^{N} \sigma_k \int_0^{t_i} C^T e^{A(t_i - s)} b u_k(s)\,\mathrm{d}s = \int_0^{t_i} C^T e^{A(t_i - s)} b \sum_{k=1}^{N} \sigma_k u_k(s)\,\mathrm{d}s.$$

Thus any convex combination of controls that individually make the outputs satisfy the constraints is feasible with respect to the interval interpolation constraints, implying that the set of such feasible u_k's is convex. That it is non-empty follows from the minimality assumption on (A,b,C). (See for example Brockett (1970).)

Now, assume that $\{u_k\}$ is a sequence of controls that each satisfies the constraints. Due to the compactness of $[0,t_i]$, the limit still satisfies the constraints since, for every control in the sequence, it holds that

$$\int_0^{t_i} C^T e^{A(t_i - s)} b u_k(s)\,\mathrm{d}s \in [\alpha_i, \beta_i], \quad i = 1,\dots,m,$$

and hence

$$\lim_{k\to\infty} \int_0^{t_i} C^T e^{A(t_i - s)} b u_k(s)\,\mathrm{d}s = \int_0^{t_i} C^T e^{A(t_i - s)} b u^\star(s)\,\mathrm{d}s \in [\alpha_i, \beta_i], \quad i = 1,\dots,m.$$

That the limit, $u^\star$, stays in $L^2[0,T]$ follows from the fact that $L^2[0,T]$ is a Banach space. We thus have closedness and the lemma follows. □

Once again, minimizing a convex functional over a convex set gives the following standard results, that for instance can be found in Luenberger (1969):

Theorem 6. *There exists a unique* $u_0 \in L^2[0,T]$ *such that* $J(u)$ *is minimized while* $y(t_i) \in [\alpha_i, \beta_i]$, $i = 1,\dots,m$.

Theorem 7 (Lagrange Duality Theorem). *Minimizing* $J(u)$, *subject to the constraints* $y(t_i) \in [\alpha_i, \beta_i]$, $i = 1,\dots,m$, *is* *equivalent to finding*

$$\max_{\lambda,\gamma\geqslant 0}\ \min_{u\in L^2[0,T]}\ L(u,\lambda,\gamma), \tag{10}$$

where L is the Lagrangian

$$\begin{aligned}L(u,\lambda,\gamma) &= \frac{1}{2}\int_0^T \rho u^2(t)\,\mathrm{d}s + \frac{1}{2}\sum_{i=1}^m \left(\int_0^T g_i(s)u(s)\,\mathrm{d}s - \xi_i\right)^T \\ &\quad \times \tau_i\left(\int_0^T g_i(s)u(s)\,\mathrm{d}s - \xi_i\right) \\ &\quad + \sum_{i=1}^m \lambda_i^T\left(\alpha_i - \int_0^T g_i(s)u(s)\,\mathrm{d}s\right) \\ &\quad + \sum_{i=1}^m \gamma_i^T\left(\int_0^T g_i(s)u(s)\,\mathrm{d}s - \beta_i\right) \\ &= \frac{1}{2}\int_0^T \rho u^2(t)\,\mathrm{d}t + \frac{1}{2}\left(\int_0^T g(s)u(s)\,\mathrm{d}s - \xi\right)^T \\ &\quad \times \mathcal{T}\left(\int_0^T g(s)u(s)\,\mathrm{d}s - \xi\right) \\ &\quad + \lambda^T\alpha - \lambda^T\int_0^T g(s)u(s)\,\mathrm{d}s - \gamma^T\beta \\ &\quad + \gamma^T\int_0^T g(s)u(s)\,\mathrm{d}s,\end{aligned}$$

and $\lambda,\gamma\in\Re^{pm}$ are the Lagrange multipliers.

Setting $L_{\lambda,\gamma}(u)=L(u,\lambda,\gamma)$, and calculating $\delta L_{\lambda,\ \gamma}(u;h)=0$, $\forall h\in L^2[0,T]$, gives that

$$\begin{aligned}&\rho u(t) + g(t)^T\mathcal{T}\left(\int_0^T g(s)u(s)\,\mathrm{d}s - \xi\right) \\ &\quad + g(t)^T(-\lambda+\gamma) = 0.\end{aligned} \tag{11}$$

If we, as before, set $u(t)=\eta^T g(t)$, we in this case get the optimal coefficients

$$\eta = (\rho I + \mathcal{T}G)^{-1}(\mathcal{T}\xi + \lambda - \gamma). \tag{12}$$

Now, inserting the optimal control, $u_0(t)=\eta^T g(t)$, into the Lagrangian gives that, after some calculations, it reduces to

$$\begin{aligned}&-\tfrac{1}{2}(\lambda^T-\gamma^T)G(\rho I+\mathcal{T}G)^{-1}(\lambda-\gamma) \\ &\quad + \xi^T\mathcal{T}G(\rho I+\mathcal{T}G)^{-1}(\lambda-\gamma) + \alpha^T\lambda - \beta^T\gamma + \mathcal{C},\end{aligned}$$

where $\mathcal{C}$ is a constant term, not depending on λ or γ. Thus maximizing $L(u_0,\lambda,\gamma)$ over λ and γ is a quadratic programming problem. This has a unique, optimal solution if $G(\rho I+\mathcal{T}G)^{-1}$ is positive definite, i.e. if the programming problem is convex.

Assumption 8. $ImC^T = \Re^p$.

Lemma 9. *Given Assumption* 8,

$$G(\rho I + \mathcal{T}G)^{-1} \succ 0.$$

Proof. It is a well-known fact that, given Assumption 8, the Gramian, G, is positive definite. We now rewrite $G(\rho I+\mathcal{T}G)^{-1}=(\rho G^{-1}+\mathcal{T})^{-1}$, and since $\mathcal{T}$ is diagonal with non-negative diagonal elements, we have $\mathcal{T}\succcurlyeq 0$, and thus $\rho G^{-1}+\mathcal{T}\succ 0$ since $G\succ 0\Rightarrow G^{-1}\succ 0$ and $\rho>0$. □

We have thus proved the following theorem:

Theorem 10. *The optimal Lagrange multipliers are uniquely determined by*

$$\min_{\zeta\geqslant 0}\frac{1}{2}\zeta^T H\zeta + F^T\zeta, \tag{13}$$

where

$$\zeta = \begin{pmatrix}\lambda\\ \gamma\end{pmatrix}\in\Re^{2pm}, \quad H = \begin{pmatrix} M & -M \\ -M & M\end{pmatrix},$$

$$M = G(\rho I + \mathcal{T}G)^{-1},$$

$$F^T = (-\xi^T\mathcal{T}M - \alpha^T, \xi^T\mathcal{T}M + \beta^T).$$

Thus, in order to find the optimal u_0, we just need to solve this finite, convex, quadratic programming problem, for which there already exist efficient algorithms (Nash & Sofer, 1996). An example of applying this interval interpolation method can be seen in Fig. 3.

5. Properties of the solution

In this section, we will discuss some of the properties that the optimal curves exhibit, and we will begin by discussing the optimal control function, u_0, before we consider the actual output spline function.

Assumption 11. $C^TA^ib = 0$ *for* $i=0,\dots,q$, *for some integer* $q\geqslant 1$.

First we note that since, on the open intervals (t_i,t_{i+1}), the optimal control is of the form

$$u_0(t) = \eta_i^T C^T e^{A(t_i-t)}b + \cdots + \eta_m^T C^T e^{A(t_m-t)}b,$$

there can be discontinuities in the control or its derivatives only at the points t_i. The degree of differentiability at these points is given by the degree of differentiability of the g_i's, which is the degree up to which the derivatives of the g_i's are equal to zero at the points t_i, $i=1,\dots,m$, since the basis functions are piecewise entire.

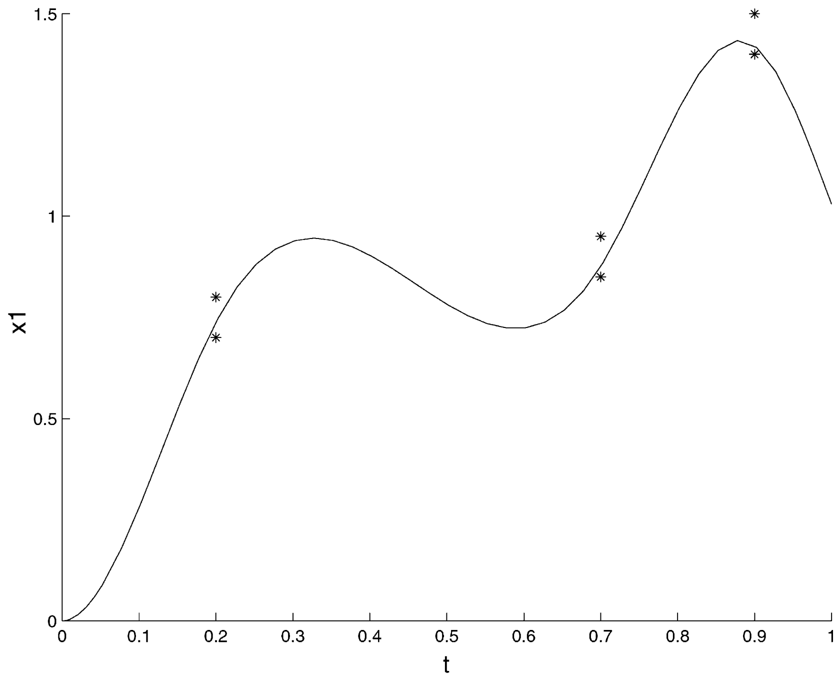


Fig. 3. Interpolating through intervals while penalizing deviations from the mid points of the intervals. Here, a second order system with both eigenvalues equal to -1 was used to produce the scalar output $y = x_1$, with $\dot{x}_1 = -x_1 + x_2$, $\dot{x}_2 = -x_2 + u$.

Recalling the definition of the g_i's gives that the kth derivative of $g_i(t)$ is

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\, g_i(t) = \begin{cases} (-1)^k C^T A^k e^{A(t_i - t)} b & t \leqslant t_i \\ 0 & t > t_i. \end{cases} \tag{14}$$

Thus the kth derivative is zero at t_i if and only if $C^T A^k b = 0$, and we state this as a lemma.

Lemma 12. *The optimal control is differentiable of degree* q, *given Assumption* 11.

Remark 13. The optimal control may have a higher degree of differentiability at a given node, t_i, if the coefficients in front of the g_j's in the optimal control are equal to zero for $j = i, \ldots, m$.

Theorem 14. *$y(t)$ is differentiable of degree $2q + 2$.*

Proof.

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t C^T e^{A(t-s)} b u_0(s)\,\mathrm{d}s = \int_0^t C^T A e^{A(t-s)} b u_0(s)\,\mathrm{d}s.$$

Similarly, it holds that

$$\frac{\mathrm{d}^{q+1}}{\mathrm{d}t^{q+1}}\int_0^t C^T e^{A(t-s)} b u_0(s)\,\mathrm{d}s = \int_0^t C^T A^{q+1} e^{A(t-s)} b u_0(s)\,\mathrm{d}s,$$

since $C^T A^i b = 0$ for $i \leqslant q$. Now it follows that

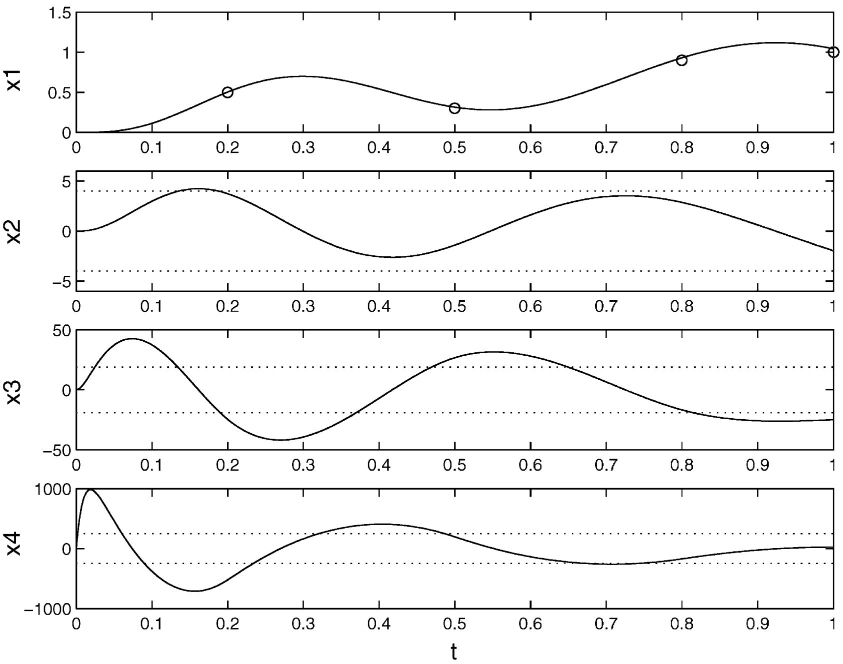
$$\frac{\mathrm{d}^{q+1+l}}{\mathrm{d}t^{q+1+l}}\int_0^t C^T e^{A(t-s)} b u_0(s)\,\mathrm{d}s = \int_0^t C^T A^{q+i+l} e^{A(t-s)} b u_0(s)\,\mathrm{d}s + \sum_{i=0}^{l-1} C^T A^{q+1+i} b \,\frac{\mathrm{d}^{l-1-i}}{\mathrm{d}t^{l-1-i}} u_0(t).$$

Thus we see that in the expression above, $l_{\max} = q + 1$, which implies that $y(t)$ has a total of $2q + 2$ continuous derivatives. The theorem then follows. □

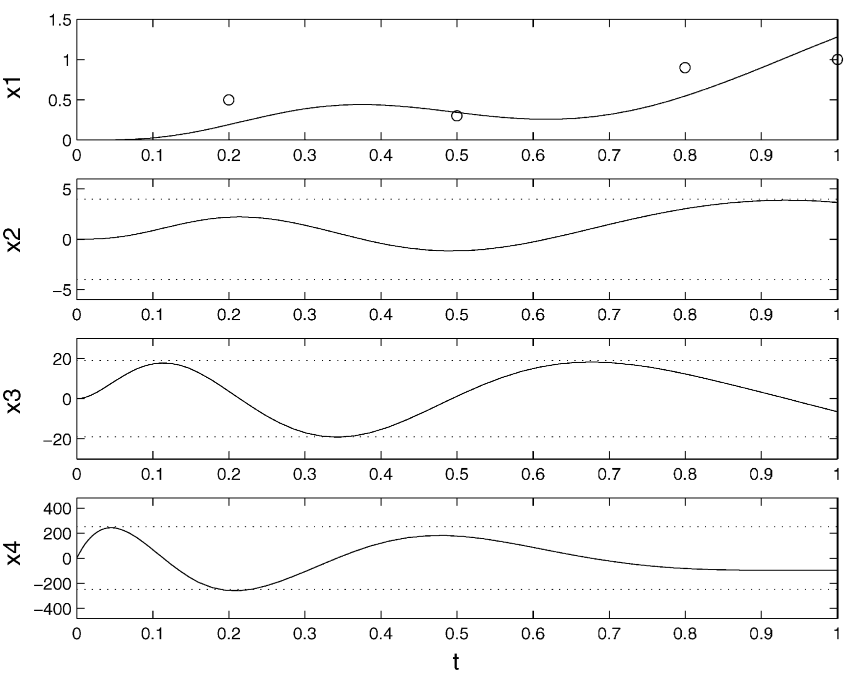
6. Example—velocity, acceleration and jerk saturation

We will now see how our approach can be applied to the problem of planning trajectories under velocity, acceleration, and jerk saturations. If we assume that our control system is given on the control canonic form

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u,$$
$$y = x, \tag{15}$$



(a) No saturation



(b) Saturation

Fig. 4. The position is driven close to the waypoints, corresponding to the circles in the graphs. In the upper graph, an example without any saturation on the velocity, acceleration and jerk is depicted. In the lower graph interval interpolation constraints on the velocity, acceleration and jerk are added at the distinct interpolation times, corresponding to the dotted lines in the graphs.

where $s^4 + a_1 s^3 + \cdots + a_4$ is the characteristic polynomial of the system matrix. It should be noted that even though this system violates Assumption 11, our method still works. The violation of the assumption just means that our resulting output curve will not have the nice differentiability properties discussed in the previous section.

What we want to do is to drive the position, the first state variable, close to given, predefined points. We do not care what $x_j(t_i)$ are for $j = 2, 3, 4$, and we simply set

$\tau_i = \mathrm{diag}(\tau_{i1}, 0, 0, 0)$. If we do not impose any further constraints on the system, such a trajectory can be seen in Fig. 4(a), where the numerical data is given by

$$x_1(0.2) = 0.5, \quad x_1(0.5) = 0.3, \quad x_1(0.8) = 0.9, \quad x_1(1) = 1,$$

and the figure displays the result where $a_i = 0$, $i = 1, \ldots, 4$ in (15).

We now add interval interpolation constraints on the velocity, acceleration and jerk, corresponding to physical saturations:

$$x_1(t_i) \in [-\mathscr{M}_\infty, \mathscr{M}_\infty], \quad x_2(t_i) \in [-4, 4],$$

$$x_3(t_i) \in [-20, 20], \quad x_4(t_i) \in [-250, 250],$$

where $t_i \in \{0.2, 0.5, 0.8, 1\}$ and $\mathscr{M}_\infty \in \Re^+$ is any large enough number. This gives a problem of the form discussed in Section 4.

It should be noted, however, that with this method we do not guarantee that the trajectory is feasible with respect to the saturation limit at all times. That type of constraint corresponds to an infinite dimensional constraint, making the problem much more complex. Instead, with our proposed method, we can only guarantee that we interpolate through the desired intervals at distinct times, but, as seen in Fig. 4(b), that is sometimes enough.

7. Conclusions

In this paper we develop a method for planning trajectories while minimizing a quadratic cost functional over the control input to a linear system. This is done while driving the system in such a way that the generated outputs are quadratically penalized for deviating from the desired points, at the same time as the outputs interpolate through prespecified intervals at the given interpolation times.

Furthermore, our approach clearly shows how a rich class of output curves, called smoothing splines, can be generated within a coherent, optimal control theoretical framework, based on ideas from both classical optimal control theory and mathematical programming. Our approach is also shown to have nice numerical features since it reduces the interval interpolation problem to a convex, quadratic programming problem that can be solved easily, resulting in piecewise entire output curves.

References

Brockett, R. W. (1970). *Finite dimensional linear systems*. New York: Wiley.

Bertsekas, D. P. (1995). *Dynamic programming and optimal control*. Cambridge, MA: Athena Scientific.

Chen, G. (1989). *Linear systems and optimal control.*. Berlin: Springer.

Egerstedt, M., & Martin, C.F. (1998). Trajectory Planning for Linear Control Systems with Generalized Splines. In *Proceedings of the mathematical theory of networks and systems*, Padova, Italy.

Luenberger, D. G. (1969). *Optimization by vector space methods*. New York: Wiley.

Luo, Z., & Wahba, G. (1997). Hybrid Adaptive Splines. *Journal of the American Statistical Association*, *92*(437), 107–116.

Martin, C. F., Tomlinson, J., & Zhang, Z. (1997). Splines and linear control theory. *Acta Applicandae Mathematicae*, *49*, 1–34.

Nash, S., & Sofer, A. (1996). *Linear and nonlinear programming*. New York: McGraw-Hill.

Rehbock, V., Teo, K.L., Jennings, L.S., & Lee, H.W.J. (1999). A survey of the control parameterization and control parameterization enhancing methods for constrained optimal control problems. In *Progress in optimization contributions from Australasia*, Dordrecht: Kluwer Academic Press.

Wahba, G. (1990). *Spline Models for Observational Data.* CBMS-NSF Regional Conference Series in Applied Mathematics 59. Philadelphia, PA: SIAM Books.

Wegman, E. J., & Wright, I. W. (1983). Splines in Statistics. *Journal of the American Statistical Association*, *78*(382), 351–365.

Magnus B. Egerstedt was born in Stockholm, Sweden, in 1971. He received the M.S. degree in Engineering Physics and the Ph.D. degree in Applied Mathematics from the Royal Institute of Technology, Stockholm, in 1996 and 2000, respectively. He also received a B.A. degree in philosophy from Stockholm University in 1996. During 1998 he was a visiting scholar at the Robotics Laboratory at the University of California, Berkeley, and he is currently a postdoctoral fellow at the Division of Engineering and Applied Sciences at Harvard University. Dr. Egerstedt's research interests include optimal control, as well as modeling and analysis of hybrid systems, with emphasis on motion planning and control of mobile robots.

Clyde F. Martin was born in Kansas. He received his bachelor's degree from Kansas State Teachers College in 1965 and his masters and Ph.D. from the University of Wyoming in 1967 and 1971, respectively. Both institutions have recognized his achievements through "outstanding alumni awards". Since 1983 he has been a member of the faculty of Texas Tech University where he holds the Paul Whitfield Horn Chair in Mathematics. He is the director of the Institute for the Mathematics of the Life Sciences and is assistant to the director for interdisciplinary research at the Institute for Environmental and Human Health at Texas Tech University. He has published numerous papers in mathematics, control theory, biosciences and medicine. He is a regular visitor to the Optimization and Systems Theory group at the Royal Institute of Technology in Stockholm, Sweden.