Optimal Control of Switching Surfaces

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***Abstract*— This paper studies the problem of optimal switching surface design for hybrid systems. In particular, a formula is derived for computing the gradient of a given integral performance cost with respect to the switching surface parameters. The formula reflects the hybrid nature of the system in that it is based on a costate variable having a discrete element and a continuous element. A numerical example with a gradient descent algorithm suggests the potential viability of the formula in optimization.**

Keywords. Optimal Control, Hybrid Systems, Switching Surfaces, Gradient Descent, Numerical Algorithms

I. Introduction

This paper investigates an optimal-control approach to hybrid dynamical systems, where modal switching occurs whenever the state reaches a suitable switching surface. The switching surfaces are controlled by free variables (parameters), which have to be determined so as to optimize (minimize) a cost-performance functional defined on the state trajectory. Application domains of such optimal control problems include robotics [1], [9], manufacturing systems [3], [8], power converters [10], and scheduling of medical treatment [18]. The problem addressed here is how to characterize the gradient of the cost functional with respect to the switching-surface control parameters, and then use them in optimization algorithms. The special structure of the hybrid dynamical system lends itself to an especially simple computation of the gradient, and holds out promise of effective optimization in the aforementioned (as well as other) application areas.

The general framework for optimal control of hybrid dynamical systems, that has influenced many subsequent developments, had been defined in [6]. Following this work, Refs. [16], [17] derived variants of the maximum principle. At the same time, the question of numerical optimization algorithms has received a significant interest. In particular, the problem of computing optimal control laws given a partition of the state space [13], or a fixed set of switching surfaces [15], [16], [17], [19], has been investigated. Refs. [2], [12], [15] addressed a timing optimization problem in piecewise-linear systems with quadratic costs, and derived homogeneous regions in the state space that determine the optimal switching times. However, the problem of optimal design of switching surfaces has not yet been fully addressed.

In this paper, the switching surfaces are defined by solution sets of equations of the form $g(x,a)=0$, where $x \in R^n$ is the state variable and $a \in R^k$ is the control parameter; here $0 \in R^k$, and $g : R^n \times R^k \to R^k$ is a continuously differentiable function. In fact, we assume that there are a number of such switching points, with possibly different switching surfaces and control parameters. The main challenge is to develop a formula for the gradient of the cost functional with respect to these control parameters, that is computationally simple so that it can be deployed in an iterative optimization procedure. A first attempt resulted in a fairly complicated and time-consuming formula [4]. This paper derives a much simpler formula by defining an appropriate costate equation. We point out that the associated optimality condition is based on variational principles and hence may be derivable from classical results on optimal control (e.g., [7], Ch. 3), but here we provide a direct derivation and proof based on the problem's specific structure. The gradient formula will then be used in a descent algorithm to optimize an example problem.

The rest of the paper is organized as follows. Section 2 derives the formula for the gradient, Section 3 presents an example, and Section 4 concludes the paper.

II. Problem Formulation and Gradient Formula

Consider the following dynamical system defined on the interval $[0,T]$,

$$\dot{x}(t) \;=\; f(x(t),t), \qquad x(0)=x_0, \tag{1}$$

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where $x(t) \in R^n$, the initial condition x_0 and the final time $T > 0$ are given and fixed, and $f : R^n \times [0, T] \to R^n$ is a given function. Let $\tau_1, \ldots, \tau_N$, be a finite sequence of times, where $0 < \tau_1 < \ldots, < \tau_N < T$, and let $f_i : R^n \to R^n$, $i = 1, \ldots, N+1$, be a given set of functions. Suppose that,

$$f(x,t) \; = \; f_i(x) \quad \text{for every } t \in [\tau_{i-1}, \tau_i), \quad i = 1, \ldots, N, \tag{2}$$

where we define $\tau_0 = 0$ and $\tau_{N+1} = T$. The functions f_i are assumed to have the following properties.

Assumption 2.1. (i) The functions f_i, $i = 1, \ldots, N+1$, are continuously differentiable throughout R^n. (ii) There exists a constant $K > 0$ such that, for every $x \in R^n$ and for all $i = 1, \ldots, N+1$,

$$||f_i(x)|| \; \leq \; K(||x|| + 1). \tag{3}$$

This assumption guarantees the existence of unique solutions to equations of the form $\dot{x} = f_i(x)$, with a given initial condition x_i at a time τ_i, for any interval $[\tau_i, \tau_{i+1}]$.

Let $L : R^n \to R$ be a continuously differentiable function, and consider the cost functional, J, defined by

$$J \; = \; \int_0^T L(x(t))dt. \tag{4}$$

We will view J as a function of the switching times τ_i, $i = 1, \ldots, N$. These switching times are not independent variables, but they depend upon each other through controlled switching surfaces in the following manner. Let us define the switching surfaces by the solution sets of the equations

$$g_i(x(\tau_i), a_i) \; = \; 0, \tag{5}$$

where $a_i \in R^k$ is the control parameter of the ith switching surface, $g_i : R^n \times R^k \to R^k$ is a given continuously differentiable function (and hence the zero term in the right-hand side of (5) is $0 \in R^k$), and $k \leq n$. The switching time τ_i is defined by

$$\tau_i \; = \; \min\{t > \tau_{i-1} : g_i(x(t), a_i) \; = \; 0\}, \tag{6}$$

where the state trajectory $\{x(t)\}$ evolves according to Eq. (1) with the given initial condition x_0. The main issue concerning this section is a formula for the derivative dJ/da_i, for all $i = 1, \ldots, N$. The formula derived below makes use of the following defined notation:

$$x_i \; := \; x(\tau_i); \tag{7}$$

$$L_i \; := \; \frac{\partial g_i}{\partial x}(x_i, a_i) f_i(x_i), \tag{8}$$

where we recognize $L_i \in R^k$ as the Lie derivative of g_i along f_i; and

$$R_i \; := \; f_i(x_i) - f_{i+1}(x_i). \tag{9}$$

Furthermore, for every $i = 1, \ldots, N$, let us define the costate $p_i(\tau) \in R^n$ on the interval $[\tau_i, \tau_{i+1}]$ by the following differential equation,

$$\dot{p}_i(\tau) \; = \; -\big(\frac{\partial f_{i+1}}{\partial x}(x(\tau))\big)^T p_i(\tau) - \big(\frac{\partial L}{\partial x}(x(\tau))\big)^T \tag{10}$$

with the following recursively defined boundary conditions at the upper end-points τ_{i+1}: $p_N(T) \; = \; 0$ (recall that $\tau_{N+1} := T$), and for all $i = N-1, \ldots, 1$,

$$p_i(\tau_{i+1}) \; = \; \big(I \; - \; \frac{1}{||L_{i+1}||^2} R_{i+1} L_{i+1}^T \frac{\partial g_{i+1}}{\partial x}(x_{i+1})\big)^T p_{i+1}(\tau_{i+1}) \tag{11}$$

where I is the $n \times n$ identity matrix. We will make use of the values of these costate variables at their lower end points, τ_i, and correspondingly we define the term $p_i \in R^n$ by

$$p_i \; = \; p_i(\tau_i). \tag{12}$$

We will express the derivative term dJ/da_i in term of the total derivative $dJ/d\tau_i$. For the latter derivative we view J as a function of τ_i in the following way: a change in τ_i will cause a change in τ_{i+1} via Eq. (5) (with $i+1$), which in turn will cause changes in $\tau_{i+2}, \ldots, \tau_N$; and all of that will cause a change in J via Eq. (4).

Let us fix a_i, $i = 1, \ldots, N$, and to ensure that the derivatives mentioned below do indeed exist, we make the following assumption.

Assumption 2.2. For all $i = 1, \ldots, N$, $L_i \neq 0$.

The derivative terms dJ/da_i and $dJ/d\tau_i$ are related to each other by the following formula.

Proposition 2.1. The following equation is in force,

$$\frac{dJ}{da_i} \; = \; - \, \frac{dJ}{d\tau_i} \; L_i^T \; \frac{\partial g_i}{\partial a}(x_i, a_i) \; \frac{1}{||L_i||^2}. \tag{13}$$

Proof. Taking derivative with respect to a_i in Eq. (5) we obtain,

$$\frac{\partial g_i}{\partial x}(x_i, a_i) \; \frac{dx_i}{d\tau_i} \; \frac{d\tau_i}{da_i} \; + \; \frac{\partial g_i}{\partial a}(x_i, a_i) \; = \; 0. \tag{14}$$

By the definition of the term τ_i (see (6)) we have that $dx_i/d\tau_i = f_i(x_i)$. Multiplying all terms in (14) from the left by L_i^T, we have that

$$\frac{d\tau_i}{da_i} \; = \; - \, L_i^T \; \frac{\partial g_i}{\partial a}(x_i, a_i) \; \frac{1}{||L_i||^2}. \tag{15}$$

Finally, noting that $dJ/da_i = (dJ/d\tau_i)(d\tau_i/da_i)$, Eq. (13) follows from (15). □

Given a_i, $i = 1, \ldots, N$, all of the terms in the right-hand side of Eq. (13) but $dJ/d\tau_i$ can be directly computed from the state trajectory $x(t)$. Therefore, to complete the characterization of the derivative term dJ/da_i, all that remains is to compute the total derivative $dJ/d\tau_i$. This is the subject of the following proposition.

Proposition 2.2. The following equation is in force.

$$\frac{dJ}{d\tau_i} \; = \; p_i^T R_i. \tag{16}$$

We break down the proof into a number of steps. By (4), we have that

$$J = \int_0^T L(x(t))dt = \sum_{i=0}^{N} \int_{\tau_i}^{\tau_{i+1}} L(x(t))dt, \quad (17)$$

with $\tau_0 := 0$ and $\tau_{N+1} := T$. Therefore, and by the continuity of x and L, it follows that

$$\frac{dJ}{d\tau_i} = \sum_{j=i}^{N} \int_{\tau_j}^{\tau_{j+1}} \frac{\partial L}{\partial x}(x(t))\frac{dx(t)}{d\tau_i}dt. \quad (18)$$

Thus, we need to get an expression for the term $dx(t)/d\tau_i$ in (18). Note that in this term t is fixed while τ_i is the variable with respect to which we take the derivative.

Let us denote by $\Phi_{i+1}(t,\tau)$ the state transition matrix of the linearized system $\dot{z} = \frac{\partial f_{i+1}(x)}{\partial x}z$, and we mention the following well-known result concerning this state transition matrix.

Lemma 2.1. Let $z(\cdot) : [\tau_i, \tau_{i+1}] \to R^n$ be a differentiable function, and let $r \in R^n$ be a given vector. Suppose that for every $t \in [\tau_i, \tau_{i+1}]$, we have that

$$z(t) = \int_{\tau_i}^{t} \frac{\partial f_{i+1}}{\partial x}(x(\tau))z(\tau)d\tau + r. \quad (19)$$

Then, for every $t \in [\tau_i, \tau_{i+1}]$,

$$z(t) = \Phi_{i+1}(t, \tau_i)r. \quad (20)$$

Proof. Follows immediately by differentiating (19) with respect to t. □

Next, for all $i = 1, \ldots, N$, let us define the $n \times n$ matrices $\Theta_{j,i}$, $j = i, \ldots, N$, recursively in j, as follows.

$$\Theta_{i,i} = \Phi_{i+1}(\tau_{i+1}, \tau_i)^{-1}, \quad (21)$$

and for every $j = i, \ldots, N-1$,

$$\Theta_{j+1,i} = \big(I - \frac{1}{||L_{j+1}||^2} R_{j+1} L_{j+1}^T \frac{\partial g_{j+1}}{\partial x}(x_{j+1})\big) \times \Phi_{j+1}(\tau_{j+1}, \tau_j)\Theta_{j,i}, \quad (22)$$

where I denotes the $n \times n$ identity matrix.

Fix $i = 1, \ldots, N$. We have the following result.

Lemma 2.2. For every $j = i, \ldots, N$, and for every $t \in (\tau_j, \tau_{j+1})$,

$$\frac{dx(t)}{d\tau_i} = \Phi_{j+1}(t, \tau_j)\Theta_{j,i}\Phi_{i+1}(\tau_{i+1}, \tau_i)R_i. \quad (23)$$

Proof. We prove the statement by induction on $j = i, \ldots, N$. Consider first the case where $j = i$. For every $t \in (\tau_i, \tau_{i+1})$,

$$x(t) = x(\tau_i) + \int_{\tau_i}^{t} f_{i+1}(x(\tau))d\tau. \quad (24)$$

Taking derivatives with respect to τ_i, and using (9),

$$\begin{aligned} \frac{dx(t)}{d\tau_i} &= f_i(x_i) - f_{i+1}(x_i) \\ &+ \int_{\tau_i}^{t} \frac{\partial f_{i+1}}{\partial x}(x(\tau))\frac{dx(\tau)}{d\tau_i}d\tau \\ &= R_i + \int_{\tau_i}^{t} \frac{\partial f_{i+1}}{\partial x}(x(\tau))\frac{dx(\tau)}{d\tau_i}d\tau. \end{aligned} \quad (25)$$

By lemma 2.1 as applied to $dx(t)/d\tau_i$,

$$\frac{dx(t)}{d\tau_i} = \Phi_{i+1}(t, \tau_i)R_i. \quad (26)$$

Consequently, and by (21), (23) follows with $j = i$.

Suppose now that (23) holds for some $j \in \{i, \ldots, N-1\}$ and for all $t \in (\tau_j, \tau_{j+1})$. We next prove it for $j+1$. Note that for every $t \in [\tau_j, \tau_{j+1}]$,

$$x(t) = x(\tau_j) + \int_{\tau_j}^{t} f_{j+1}(x(\tau))d\tau, \quad (27)$$

and in particular, for $t = \tau_{j+1}$,

$$x(\tau_{j+1}) = x(\tau_j) + \int_{\tau_j}^{\tau_{j+1}} f_{j+1}(x(\tau))d\tau. \quad (28)$$

Now let us compare the derivatives with respect to τ_i in these two equations. The derivative of (27) yields $dx(t)/d\tau_i$, whose value is given by (23) by dint of the induction's hypothesis. The derivative of (28) yields the same expression as the derivative in (27) (with τ_{j+1} instead of t) plus the additional term $f_{j+1}(x_{j+1})\frac{d\tau_{j+1}}{d\tau_i}$. In other words, we have that

$$\begin{aligned} \frac{dx(\tau_{j+1})}{d\tau_i} & \\ &= \frac{dx(t)}{d\tau_i}|_{t=\tau_{j+1}} + f_{j+1}(x_{j+1})\frac{d\tau_{j+1}}{d\tau_i} \\ &= \Phi_{j+1}(\tau_{j+1}, \tau_j)\Theta_{j,i}\Phi_{i+1}(\tau_{i+1}, \tau_i)R_i \\ &+ f_{j+1}(x_{j+1})\frac{d\tau_{j+1}}{d\tau_i}. \end{aligned} \quad (29)$$

Consider next $t \in (\tau_{i+1}, \tau_{i+2})$ We have that

$$x(t) = x(\tau_{j+1}) + \int_{\tau_{j+1}}^{t} f_{j+2}(x(\tau))d\tau \quad (30)$$

and by taking derivatives with respect to τ_i, we obtain,

$$\begin{aligned} \frac{dx(t)}{d\tau_i} &= \frac{dx(\tau_{j+1})}{d\tau_i} - f_{j+2}(x_{j+1})\frac{d\tau_{j+1}}{d\tau_i} \\ &+ \int_{\tau_{j+1}}^{t} \frac{\partial f_{j+2}}{\partial x}(x(\tau))\frac{dx(\tau)}{d\tau_i}d\tau. \end{aligned} \quad (31)$$

Now plug Eq. (29) for the first term in the RHS of (31) to obtain,

$$\begin{aligned} \frac{dx(t)}{d\tau_i} &= \\ \Phi_{j+1}(\tau_{j+1}, \tau_j)\Theta_{j,i}\Phi_{i+1}(\tau_{i+1}, \tau_i)R_i &+ R_{j+1}\frac{d\tau_{j+1}}{d\tau_i} \\ &+ \int_{\tau_{j+1}}^{t} \frac{\partial f_{j+2}}{\partial x}(x(\tau))\frac{dx(\tau)}{d\tau_i}d\tau. \end{aligned} \quad (32)$$

By Lemma 2.1 we have, for all $t \in (\tau_{j+1}, \tau_{j+2})$,

$$\frac{dx(t)}{d\tau_i} = \Phi_{j+2}(t, \tau_{j+1}) \times \big(\Phi_{j+1}(\tau_{j+1}, \tau_j)\Theta_{j,i}\Phi_{i+1}(\tau_{i+1}, \tau_i)R_i + R_{j+1}\frac{d\tau_{j+1}}{d\tau_i}\big). \quad (33)$$

The last term in (33), $d\tau_{j+1}/d\tau_i$, can be computed from (29) as follows. By definition of τ_{j+1}, $g_{j+1}(x(\tau_{j+1})) = 0$. Taking derivative with respect to τ_i we get that

$$\frac{\partial g_{j+1}}{\partial x}(x(\tau_{j+1}))\frac{dx(\tau_{j+1})}{d\tau_i} = 0, \quad (34)$$

and accounting for (29),

$$\frac{\partial g_{j+1}}{\partial x}(x(\tau_{j+1})) \times \big(\Phi_{j+1}(\tau_{j+1}, \tau_j)\Theta_{j,i}\Phi_{i+1}(\tau_{i+1}, \tau_i)R_i + f_{j+1}(x_{j+1})\frac{d\tau_{j+1}}{d\tau_i}\big) = 0. \quad (35)$$

Multiplying (35) from the left by L_{j+1}^T and solving for $\frac{d\tau_{j+1}}{d\tau_i}$ we get,

$$\frac{d\tau_{j+1}}{d\tau_i} = -\frac{1}{||L_{j+1}||^2}L_{j+1}^T\frac{\partial g_{j+1}}{\partial x}(x_{j+1}) \times \Phi_{j+1}(\tau_{j+1}, \tau_j)\Theta_{j,i}\Phi_{i+1}(\tau_{i+1}, \tau_i)R_i. \quad (36)$$

Plugging this in (33) we obtain, after some algebra, for every $t \in (\tau_{j+1}, \tau_{j+2})$,

$$\frac{dx(t)}{d\tau_i} = \Phi_{j+2}(t, \tau_{j+1}) \times \big(I - \frac{1}{||L_{j+1}||^2}R_{j+1}L_{j+1}^T\frac{\partial g_{j+1}}{\partial x}(x_{j+1})\big) \times \Phi_{j+1}(\tau_{j+1}, \tau_j)\Theta_{j,i}\Phi_{i+1}(\tau_{i+1}, \tau_i)R_i. \quad (37)$$

It now follows from (22) that

$$\frac{dx(t)}{d\tau_i} = \Phi_{j+2}(t, \tau_{j+1})\Theta_{j+1,i}\Phi_{i+1}(\tau_{i+1}, \tau_i)R_i, \quad (38)$$

which verifies Eq. (23) for $j+1$ and for all $t \in (\tau_{j+1}, \tau_{j+2})$, and hence completes the proof. □

Recall that the matrices $\Theta_{j,i}$ were defined by a recursive relation in the first index, j; see (21) and (22). We need a recursive relation in the second index, i, and it is given by the following result.

Lemma 2.3. For every $i = 2, \ldots, N$, and for all $j = i, \ldots, N$,

$$\Theta_{j,i-1} = \Theta_{j,i}\Phi_{i+1}(\tau_{i+1}, \tau_i)\big(I - \frac{1}{||L_i||^2}R_iL_i^T\frac{\partial g_i}{\partial x}(x_i)\big). \quad (39)$$

Proof. Fix $i \in \{2, \ldots, N\}$. We will prove (39) by induction on $j = i, \ldots, N$.

First, consider the case where $j = i$. By (21) with $i-1$,

$$\Theta_{i-1,i-1} = \Phi_i(\tau_i, \tau_{i-1})^{-1}.$$

Therefore, and by (22), the left-hand side of (39) has the following form,

$$\Theta_{i,i-1} = I - \frac{1}{||L_i||^2}R_iL_i^T\frac{\partial g_i}{\partial x}(x_i).$$

By Eq. (21), the RHS of (39) (with $j = i$) has the same form. This proves (39) for $j = i$.

Next, suppose that (39) is in force for some $j \in \{i, \ldots, N-1\}$, and consider the case of $j+1$. An application of (22) yields,

$$\Theta_{j+1,i-1} = \big(I - \frac{1}{||L_{j+1}||^2}R_{j+1}L_{j+1}^T\frac{\partial g_{j+1}}{\partial x}(x_{j+1})\big) \times \Phi_{j+1}(\tau_{j+1}, \tau_j)\Theta_{j,i-1}, \quad (40)$$

and by using the induction's hypothesis (Eq. (39)) in the last term we obtain,

$$\Theta_{j+1,i-1} = \big(I - \frac{1}{||L_{j+1}||^2}R_{j+1}L_{j+1}^T\frac{\partial g_{j+1}}{\partial x}(x_{j+1})\big) \times \Phi_{j+1}(\tau_{j+1}, \tau_j)\Theta_{j,i}\Phi_{i+1}(\tau_{i+1}, \tau_i) \times \big(I - \frac{1}{||L_i||^2}R_iL_i^T\frac{\partial g_i}{\partial x}(x_i)\big). \quad (41)$$

Now we recognize the first three multiplicative terms in the RHS of (41) as the RHS of (22), and therefore, plugging in the LHS of (22), we obtain,

$$\Theta_{j+1,i-1} = \Theta_{j+1,i}\Phi_{i+1}(\tau_{i+1}, \tau_i)\big(I - \frac{1}{||L_i||^2}R_iL_i^T\frac{\partial g_i}{\partial x}(x_i)\big).$$

But this is Eq. (39) with $j+1$, thus completing the proof. □

We now are in a position to prve Proposition 2.2.

Proof of Proposition 2.2. Applying Lemma 2.2 to Eq. (18) we obtain,

$$\frac{dJ}{d\tau_i} = \sum_{j=i}^{N}\int_{\tau_j}^{\tau_{j+1}}\frac{\partial L}{\partial x}(x(t))\Phi_{j+1}(t, \tau_j)dt \times \Theta_{j,i}\Phi_{i+1}(\tau_{i+1}, \tau_i)R_i. \quad (42)$$

For every $\tau \in [\tau_i, \tau_{i+1}]$, let us define the vector $q_i(\tau) \in R^n$ by

$$q_i(\tau)^T = \int_{\tau}^{\tau_{i+1}}\frac{\partial L}{\partial x}(x(t))\Phi_{i+1}(t, \tau)dt + \sum_{j=i+1}^{N}\int_{\tau_j}^{\tau_{j+1}}\frac{\partial L}{\partial x}(x(t))\Phi_{j+1}(t, \tau_j)dt\Theta_{j,i}\Phi_{i+1}(\tau_{i+1}, \tau). \quad (43)$$

By (21) and (42), it is readily seen that

$$\frac{dJ}{d\tau_i} = q_i(\tau_i)^TR_i. \quad (44)$$

Therefore, it suffices to show that

$$p_i = q_i(\tau_i) \quad (45)$$

in order to complete the proof.

Recall the definition of $p_i(\tau)$ via Eqs. (10) and (11) with the boundary condition $p_N(T)=0$. Now Eq. (45) will be proved once we establish the following: (i) the differential equation (10) holds for $q_i(\tau)$ in the interval $[\tau_i,\tau_{i+1}]$; (ii) $q_N(T)=0$; and (iii) Eq. (11) is in force for $q_i(\tau_i)$ in lieu of p_i. This is what we now do.

(i). By taking derivatives with respect to τ in (43), Eq. (10) is satisfied for $q_i(\tau)$.

(ii). By (43) and the fact that $\tau_{N+1}=T$, it is evident that $q_N(T)=0$.

(iii). By Eq. (43),

$$q_i(\tau_{i+1})^T = \sum_{j=i+1}^{N}\int_{\tau_j}^{\tau_{j+1}}\frac{\partial L}{\partial x}(x(t))\Phi_{j+1}(t,\tau_j)dt\Theta_{j,i}. \quad (46)$$

Apply Lemma 2.3 with $i+1$ instead of i to the last term of (46) to obtain,

$$\begin{aligned} q_i(\tau_{i+1})^T &= \sum_{j=i+1}^{N}\int_{\tau_j}^{\tau_{j+1}}\frac{\partial L}{\partial x}(x(t))\Phi_{j+1}(t,\tau_j)dt \\ &\quad\times \Theta_{j,i+1}\Phi_{i+2}(\tau_{i+2},\tau_{i+1}) \\ &\quad\times \big(I-\frac{1}{||L_{i+1}||^2}R_{i+1}L_{i+1}^T\frac{\partial g_{i+1}}{\partial x}(x_{i+1})\big). \end{aligned} \quad (47)$$

By (43) with $i+1$, we recognize that

$$\begin{aligned} q_{i+1}(\tau_{i+1})^T &= \sum_{j=i+1}^{N}\int_{\tau_j}^{\tau_{j+1}}\frac{\partial L}{\partial x}(x(t))\Phi_{j+1}(t,\tau_j)dt \\ &\quad\times \Theta_{j,i+1}\Phi_{i+2}(\tau_{i+2},\tau_{i+1}), \end{aligned} \quad (48)$$

and hence, (47) implies that

$$\begin{aligned} q_i(\tau_{i+1})^T &= q_{i+1}(\tau_{i+1})^T \\ \times \big(I-\frac{1}{||L_{i+1}||^2}R_{i+1}L_{i+1}^T\frac{\partial g_{i+1}}{\partial x}(x_{i+1})\big). \end{aligned} \quad (49)$$

This shows that Eq. (11) is in force for $q_i(\tau_i)$ and hence establishes (45). This completes the proof of the Proposition. □

III. Example

As an example, consider the problem of letting the switched systems be composed from two second order, unstable linear systems, with switches between the different subsystems occurring on one-dimensional subspaces. Inspired by [5], we let the two subsystems be defined through $\dot{x}=A_ix,\ x\in\mathbb{R}^2,\ i=1,2$, where

$$A_1=\begin{pmatrix}-1 & -100\\ 10 & -1\end{pmatrix},\ A_2=\begin{pmatrix}1 & 10\\ -100 & 1\end{pmatrix}.$$

Starting at $x(0)=(-1,0)^T$, we let $x(t)$ evolve according to $\dot{x}=A_1x$ (mode 1) until the line $g_1(x,a)=x_1+a_1^2x_2=0$, at which point the dynamics change to $\dot{x}=A_2x$ (mode 2). The system returns to mode 1 when the line $g_2(x,a)=a_2^2x_1+x_2=0$, as seen in Figure 1.

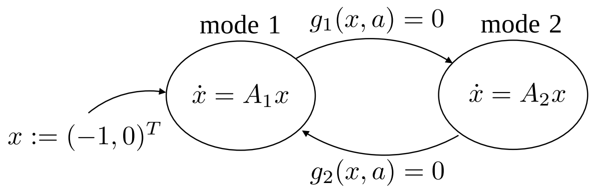

Fig. 1. A hybrid automaton showing the switching structure of the example problem.

Now, the problem that we will study is the problem of forcing the trajectories of this system to "look" circular through the selection of optimal switching surfaces. In other words, we let the cost functional be given by

$$J(a_1,a_2)=\int_0^T\left(\|x(t)\|^2-\xi^2\right)^2dt,$$

where T is the final time and ξ is the desired circle radius. In Figure 2, the solution is shown for the case when $T=0.5$ and $\xi=2$. The solution is obtained by computing the derivative in Eq. (9) and then adjusting the a-values using a gradient descent with Armijo stepsize [14]. From Figure 3, it can be seen that the algorithm terminated after 70 steps, with the initial a-values being $a=(1,0.8)^T$ and the final values being $a=(0.85,0.73)^T$.

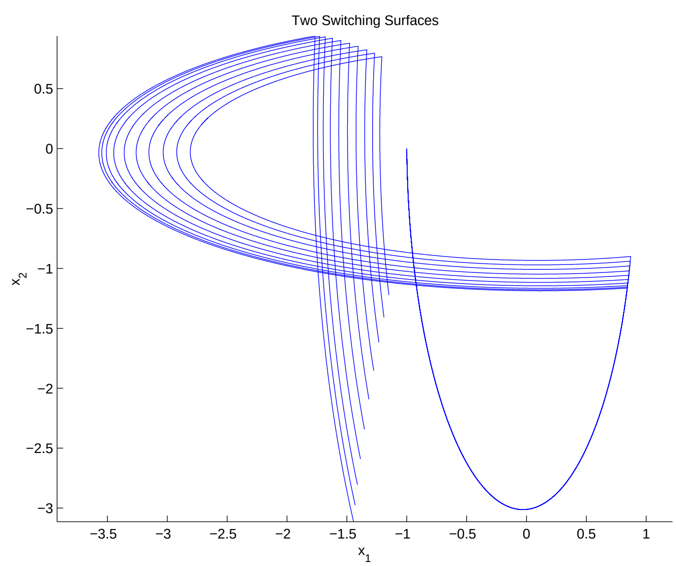

Fig. 2. The trajectories are shown as the switching parameters are varied.

A couple of comments should be made about this experiment. First, by switching between unstable systems, the resulting system is no longer unstable. Moreover, the computational burden of the proposed method is quite reasonable since only two forward (from 0 to T) differential equations must be solved in order to obtain $x(t)$ as well as ψ_i in Eq. (13). Moreover, only one backward differential equation (from T to 0) must be solved in order to obtain

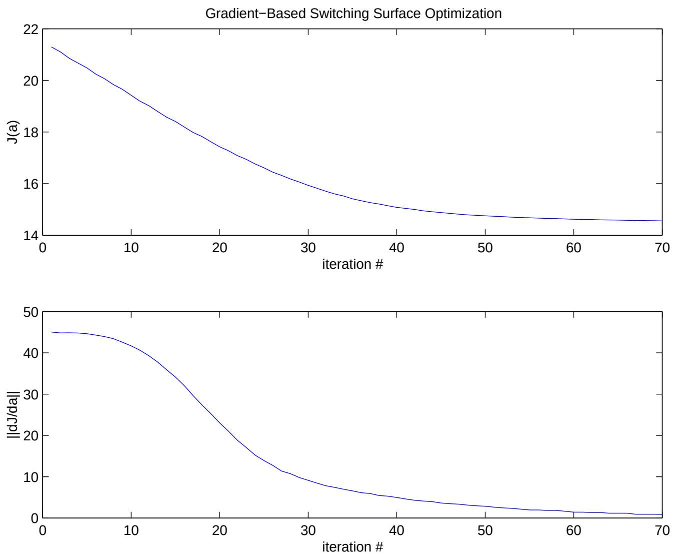

Fig. 3. J and $\|dJ/da\|$ are depicted as functions of the iteration index.

the continuous costate $p(t)$, which is a huge computational improvement over the result in [4].

IV. Conclusions

This paper concerns an optimal switching problem in hybrid dynamical systems in the setting of optimal control. The modal switching takes place whenever the state trajectory hits a certain switching surface, and the free variables of the optimization problem consist of control parameters of the switching surfaces. The paper investigates the structure of the gradient of the cost functional, and develops an algorithm for its computation. The algorithm is based on a hybrid costate having a discrete component and a continuous component. This structure of the gradient is amenable to efficient computation, as demonstrated by a numerical example.

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