Optimal Mode-Switching for Hybrid Systems with Varying Initial States*

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Abstract

This paper concerns a particular aspect of the optimal control problem for switched systems that change modes whenever the state intersects certain switching surfaces. These surfaces are assumed to be parameterized by a finite dimensional switching parameter, and the optimization problem we consider is that of minimizing a given cost-functional with respect to the switching parameter under the assumption that the initial state of the system is not a priori known. We approach this problem from two different vantage points by first minimizing the worst possible cost over the given set of initial states using results from min-max optimization. The second approach is based on a sensitivity analysis in which variational arguments give the derivative of the switching parameters with respect to the initial conditions.

1 Introduction

Over the last couple of decades, a lot of effort has been directed towards optimal control of hybrid systems, e.g. [9, 5, 16, 17, 18, 24, 12, 2]. Hybrid systems are complex systems that are characterized by discrete logical decision making at the highest level and continuous variable dynamics at the lowest level. Examples where such systems arise include situations where a control module has to switch its attention among a number of subsystems [19, 21, 23], or collect data sequentially from a number of sensory sources [10, 13, 18].

The type of hybrid systems under consideration in this paper belongs to the class of *switched autonomous systems*, where the continuous-time control variable is absent and the continuous-time dynamics change at discrete times (*switching-times*). For these, it is possible to derive gradient expressions for the cost functional with respect to the switching times when the initial state is fixed. In particular, [14] presented a gradient-based algorithm that finds optimal switching-times, dictating when to switch between a given set of modes, for the case when the switching-times are controlled directly. Furthermore, [6] considered the case when a switch between two different modes occurs when the state trajectory

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intersects a switching surface, defined by $g(x(t), a) = 0$, where $x(t) \in \mathbb{R}^n$ is the state of the system at time t, and a parameterizes the switching surface. Reference [6] can thus be thought of as the starting point for this paper, as we consider a similar problem, but instead of optimizing with respect to a given fixed initial condition $x_0 \in \mathbb{R}^n$, we will assume that the initial state can be anywhere within a given set $S \subset \mathbb{R}^n$. This problem arose, for example, in the context of a recent DARPA sponsored robotics competition (LAGR - Learning Applied to Ground Robots), where an autonomous mobile robot was to navigate an unknown, outdoor environment from an initial set (the robot could start anywhere in the set) to a given target destination [22].

In order to find a good value of the switching parameter a, independent of the starting point from S, we will use the gradient formula presented in [6] and find a locally optimal a such that we will minimize the worst possible cost for all trajectories starting in S. Hence, we have a min-max problem and the results presented in this paper are based on the initial study found in [4].

An alternative view, initiated in [8], that will also be pursued in this paper is to assume that a switching surface can be obtained by varying the initial conditions and then solving for the corresponding, varying optimal switching times (and consequently switching states). In this manner, a sensitivity-based approach can be exploited for obtaining suitable switching surfaces.

The outline of this paper is as follows: In Section 2, the problem at hand is introduced together with some previous results relating to the gradient formula. Section 3 presents our solution using a min-max strategy. This is followed by a sensitivity analysis in Section 4 together with a discussion about the transition from sensitivities to switching surfaces. The conclusions are given in Section 5.

2 Switching Parameter Optimization

The type of systems under consideration in this paper are of the form

$$\dot{x}(t) = f_i(x(t)),\ t \in [\tau_{i-1}, \tau_i),\ i \in \{1, \ldots, N+1\},\ x \in \mathbb{R}^n, \tag{1}$$

where we assume that the system switches N times between $N+1$ different dynamical regimes (or modal functions) at times $\tau_i,\ i = 1, \ldots, N$ over the time window $[0, T]$. (In the formulation above, we assume that $\tau_0 = 0$ and $\tau_{N+1} = T$, i.e. the final time.) We moreover assume that the switching times are not controlled directly. Instead, a switch occurs whenever the state trajectory intersects a switching surface and we assume that the geometry and dynamics of the system are such that the system does in fact undergo exactly N switches on the interval $[0, T]$ and that the intersections of the switching surfaces occur in a non-tangential manner. This problem was initially considered in [7] for a fixed initial state.

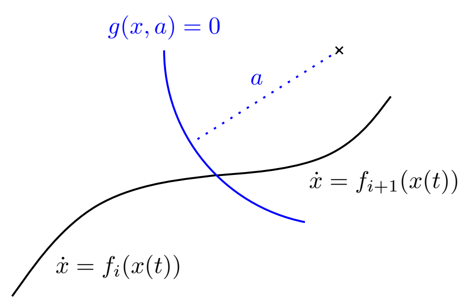


Figure 1: Mode switching occurs when the state trajectory intersects a switching surface. In this case, the switching surface is a circle parameterized by the radius a.

In this paper, we assume that the surfaces are defined by the solutions of parameterized equations from $\mathbb{R}^n$ to $\mathbb{R}$. We denote this parameter by a and suppose that $a \in \mathbb{R}^k$ for some integer $k \geq 1$, as shown in Figure 1. For every switching surface g_j (denoting the surface that correposnds to the j^{th} switch), we let $g_j : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ be a continuously differentiable function. For a given fixed value of $a \in \mathbb{R}^k$, denoted here by a_j, we thus let the switching surface be defined by the solution points x of the equation $g_j(x, a_j) = 0$ as illustrated in Figure 1. (Note that under mild assumption, the switching surface is a smooth $(n-1)$ dimensional manifold in $\mathbb{R}^n$, and a_j can be viewed as a control parameter of the surface.)

In order to minimize a cost function of the form

$$J = \int_0^T L(x(t))dt, \tag{2}$$

where $L : \mathbb{R}^n \to \mathbb{R}$, we need to determine the optimal switching surface parameters since the state trajectory depends on them. To this end, [7] presented an expression for the gradient of the cost functional with respect to the switching surface parameters. This gradient was presented under the assumption that the functions f_i, g_i, $i = 1, \ldots, N+1$, and L where continuously differentiable with respect to all its arguments. Furthermore, it was assumed that f_i $i = 1, \ldots, N+1$, was uniformly Lipschitz.

In order to compute the derivative of the cost with respect to the switching parameters, we define $x_i = x(\tau_i)$, and the terms $\mathcal{R}_i$ and $\mathcal{L}_i$ by

$$\mathcal{R}_i = f_i(x_i) - f_{i+1}(x_i), \tag{3}$$

and

$$\mathcal{L}_i = \frac{\partial g_i}{\partial x}(x_i, a_i) f_i(x_i), \tag{4}$$

where we recognize $\mathcal{L}_i$ as the Lie derivative of g_i along the direction of f_i, which we, in this paper, assume to be nonzero, i.e. that the switching surfaces are always intersected in a non-tangential manner.

Based on this notation, in reference [7] the following expression for the derivative dJ/da_i was derived.

Proposition 2.1 *[7] If, for all $i = 1, \ldots, N$, $\mathcal{L}_i \neq 0$, the following equation is in force,*

$$\frac{dJ}{da_i} = -\frac{1}{\mathcal{L}_i}\, p(\tau_i^+)\mathcal{R}_i\, \frac{\partial g_i}{\partial a_i}(x_i, a_i), \tag{5}$$

where the costate equation is given by

$$\begin{aligned} \dot{p}(t) \quad &= -\left(\tfrac{\partial f_{i+1}}{\partial x}(x(t))\right)^T p(t) - \left(\tfrac{\partial L}{\partial x}(x(t))\right)^T; \\ &\quad t \in [\tau_i, \tau_{i+1}),\ i = 1, \ldots, N, \end{aligned} \tag{6}$$

with terminal condition $p^T(\tau_{N+1}) = 0$ when the final time is fixed to $\tau_{N+1} = T$, and reset conditions

$$p(\tau_i^-) = (I - \frac{1}{\mathcal{L}_i}\mathcal{R}_i \frac{\partial g_i}{\partial x}(x_i, a_i))^T p(\tau_i^+), \quad i = 1, \ldots, N. \tag{7}$$

3 Min-Max Optimization

Given a set of possible initial states $S \subset \mathbb{R}^n$, a set of switching surfaces parameterized by some vector a, and an instantaneous cost L, the total cost, starting at $x_0 \in S$, is given by

$$J_{x_0}(a) = \int_0^T L(x(t))dt,\ a \in \mathbb{R}^k, \tag{8}$$

where, T is the fixed final time and subscript x_0 indicates the implicit dependence on the initial condition. With this notation, the optimization problem can be stated as follows: *Given a set of initial states S and a set of parameterized switching surfaces, find the surface parameter such that*

$$\max\{J_x(a) \mid x \in S\} \tag{9}$$

is minimized.

The theory of min-max optimization and consistent approximations [20] can be utilized in order to implement and solve this problem. In particular, we will achieve this by choosing a sequence of sets of initial states, $\{\mathbb{X}_i\}_{i=0}^{\infty}$. This sequence will satisfy the following three conditions: (1) $\mathbb{X}_i \subset S$ $i = 1, 2 \ldots$; (2) The cardinality of $\mathbb{X}_i$ is greater than the cardinality of $\mathbb{X}_{i-1}$; (3) Every point in S will be arbitrarily close to a point in $\mathbb{X}_i$, as i goes to infinity. Choosing $\{\mathbb{X}_i\}_{i=0}^{\infty}$ in this way enables us to find the solution to (9) by solving a sequence of optimization problems, each one with a different set of initial states.

For each $\mathbb{X}_i$, we will find the optimal switching parameter $a_i^\star$ that minimizes $\max\{J_x(a_i) \mid x \in \mathbb{X}_i\}$ through a gradient descent algorithm. After we have found the optimal $a_i^\star$, we will solve $\max\{J_x(a_{i+1}) \mid x \in \mathbb{X}_{i+1}\}$ by initializing a_{i+1} to $a_i^\star$. This gives a good starting point for the gradient descent algorithm. For each $\mathbb{X}_i$ we will moreover find the optimal $a_i^\star$ by executing a gradient descent algorithm with Armijo step size [1], under the assumption that $\mathbb{X}_i$ has $\mathcal{N}(i)$ elements, i.e. $\mathbb{X}_i = \{x_1, \ldots, x_{\mathcal{N}(i)}\}$ for some $x_1, \ldots, x_{\mathcal{N}(i)}$ in $S \subset \mathbb{R}^n$.

The resulting algorithm is given by:

Algorithm 3.1 *Gradient Projection Algorithm with Armijo Stepsize*

Given: Two constants $\delta > 0$, and $\epsilon > 0$, and the set of initial states $\mathbb{X}_k = \{x_1, \ldots, x_{\mathcal{N}(k)}\} \subset S$ for a given $k \geq 0$.

Initialize: Choose a feasible initial guess for the switching surface parameter a.

Step I: Calculate the maximum cost for the given set of initial states, denoted

$$F(\mathbb{X}_k, a) = \max_x \{J_x(a) \mid x \in \mathbb{X}_k\}, \tag{10}$$

where J_x is given by (8). Let $I(\mathbb{X}_k, a)$ denote the index set of *active constraints*, i.e.

$$I(\mathbb{X}_k, a) = \{j \in \{1, \ldots, \mathcal{N}(k)\} \mid F(\mathbb{X}_k, a) - J_{x_j}(a) < \epsilon\}. \tag{11}$$

Calculate the generalized gradient

$$\partial F(\mathbb{X}_k, a) = conv\{\nabla J_{x_j}(a) \mid j \in I(\mathbb{X}_k, a))\}, \tag{12}$$

where *conv* denotes the *convex hull*. Find the point in $\partial F(\mathbb{X}_k, a)$ closest to the origin and denote it by h. If $||h|| < \delta$ then STOP. Else, goto Step II.

Step II: Calculate the step-length λ according to Armijo's rule, i.e.

$$\begin{aligned} \lambda \;=\; & \max\{z = \beta^\ell \mid \ell = 0, 1, 2, \ldots\} \\ & F(\mathbb{X}_k, a - zh) - F(\mathbb{X}_k, a) \leq -\alpha z \|h\|^2\}, \end{aligned}$$

where $\alpha, \beta \in (0, 1)$ are the Armijo constants. Update a according to $a = a - \lambda h$ and go back to Step I. ∎

A few remarks concerning Algorithm 3.1 are due.

Remark 3.1 *The index set of active constraints, $I(\mathbb{X}_k, a)$, is introduced in order to determine what initial states in $\mathbb{X}_k$ we should take into consideration for a given a. If the index of an initial state is in the index set, then the gradient of the cost associated with that initial state is current in the calculation of the generalized gradient, $\partial F(\mathbb{X}_k, a)$. If $\epsilon = 0$ in (11), i.e., we only optimize with respect to the initial state corresponding to the maximal cost, it is conceivable that we can only take a very small descent step since the index set changes when a changes.*

Remark 3.2 *In order to find the optimal a for a given set of initial states, we would have to set the constants δ and ϵ to 0. However, doing this when we solve for a sequence of initial states, $\{\mathbb{X}_k\}_{k=0}^{\infty}$, would not give any additional benefit, instead we only require that for each consecutive problem we will solve, δ and ϵ will decrease, and in the limit when $i \to \infty$, they will be zero.*

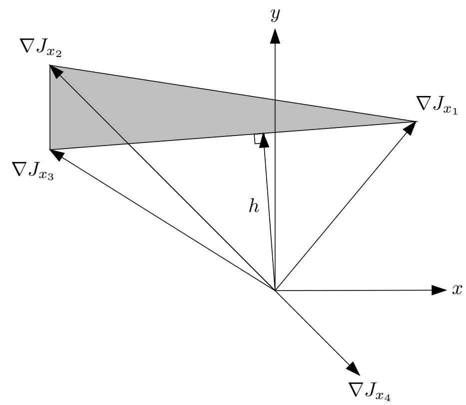


Figure 2: Calculation of h given four initial states and their respective gradients, where x_1 through x_3 are the active initial states. (Note here that the system evolves in $\mathbb{R}^2$, with components x and y.)

Remark 3.3 *Solving for h is a standard quadratic optimization problem over a convex set, and can be solved using a variety of optimization algorithms.*

In order to illustrate the calculation of h, a simple example is presented. Assume that we have four different initial states, x_1 through x_4 in $\mathbb{R}^2$. In Figure 2, their respective gradients are plotted and it is assumed that x_1 through x_3 are active initial states for the given switching surface parameter a. The shaded region in Figure 2 corresponds to the convex hull of the gradients of the active initial states, and h is the closest vector in this set from the origin.

Now, having presented Algorithm 3.1 and the subsequent remarks, we are now in the position to present the final Algorithm 3.2

Algorithm 3.2 *Min-Max Optimization for Unknown Initial States:*

Given: A sequence of initial sets $\{\mathbb{X}_k\}_{k=0}^{\infty} \in S \subset \mathbb{R}^n$, where $\mathbb{X}_k = \{x_1, \ldots, x_{\mathcal{N}(k)}\}$ and $\mathcal{N}(k) > \mathcal{N}(k-1)$. Two positive sequences $\{\epsilon_k\}_{k=0}^{\infty}$ and $\{\delta_k\}_{k=0}^{\infty}$ such that in the limit when $k \to \infty$, both are 0.

Initialize: Set $k = 0$, pick a feasible initial guess for k_0.

Step I: Use Algorithm 3.1 to optimize over a with $\delta = \delta_k$ and $\epsilon = \epsilon_k$. Initialize a with a_{k-1} if $k \neq 0$, and with a_0 if $k = 0$.

Step II: Set a_k to a given from Algorithm 3.1. Increase k by one, return to Step I.

We have thus obtained a descent algorithm that applies standard techniques from min-max optimization for solving the optimal switching time problem when the initial state is confined to a set S of possible initial conditions. The resulting switching surface parameters thus ensures that no matter where the initial state actually is located in S, the performance of the system is reasonable.

4 Optimal Switching Times Parameterized by Initial Conditions

In the previous section, we presented an algorithm that minimized the worst-case performance over a set of initial conditions. However, an additional view of this problem will be presented in this section, in which we instead assume that the problem is to find the optimal switching surface itself rather than the optimal surface parameterization. This problem can locally be addressed as an optimization problem involving multiple initial conditions. In fact, by allowing the free switching times to vary as the initial state varies, we obtain a sensitivity-like argument for characterizing a collection of locally optimal switching times, parameterized by $x_0 \in S$ rather than one single surface parameter, as was the case in this section. This collection implicitly defines a switching surface, as will be seen in the following paragraphs. It should be noted though that in this case, there is a direct correspondence between the initial conditions and the switching surfaces, and what we in fact are looking for is this correspondence. However, as the problem under consideration is that of a finite horizon problem, a potential problem with this approach that arises from the finite time horizon is that we will not get a surface but rather a set. This can be remedied in two distinctly different ways, namely *a)* assuming a particular shape on S such that it itself is a surface, or more interesting and practically relevant, *b)* making T large enough which essentially ensures that the a surface is obtained, as shown in [3].

It is well known that, under mild assumptions, executions of switched systems are continuous with respect to the initial conditions [11]. It is thus reasonable to expect that also the dependence of the optimal switching times on x_0 is continuous. This observation allows us to formulate an alternative view, as proposed in the previous paragraph, of the initial condition problem as a sensitivity problem. In this paper, we will assume that the system only undergoes a single transition, even though the derived results can easily be extended to situations with multiple switches.

Let

$$\Theta(\tau) = p^T(\tau)\mathcal{R}, \tag{13}$$

where τ is the switch time, p is the costate given in (6) (without the discontinuities defined in (7)), and where $\mathcal{R}$ is given in (3), where we have dropped the explicit dependence on i since we only have a single switching time τ.

Viewed as a free parameter over which we wish to minimize the performance index, we note that the optimality condition for τ becomes

$$\Theta(\tau) = 0. \tag{14}$$

But, since the optimal switching time depends on x_0, we can apply the chain rule to Θ and obtain

$$\frac{d\Theta}{dx_0} = \mathcal{R}^T \frac{dp(\tau)}{dx_0} + p^T(\tau)\frac{d\mathcal{R}}{dx_0}. \tag{15}$$

After some calculations, following the argument in [8], we get that

$$\frac{dp(\tau)}{dx_0} = \int_\tau^T \left[\Phi_2^T(s,\tau)L_{xx}(x(s))\frac{dx(s)}{dx_0} + \frac{d\Phi_2^T(s,\tau)}{d\tau}L_x^T(x_\tau)\frac{d\tau}{dx_0}\right] ds - \Phi_2^T(\tau,\tau)L_x^T(x_\tau)\frac{d\tau}{dx_0}. \tag{16}$$

Here $\Phi_i,\ i = 1,2$ are the transition matrices associated with the linearizations of the two systems, L_x is shorthand for $\partial L/\partial x$, and $x_\tau = x(\tau)$.

In order to compute $dx(s)/dx_0$, we again apply the chain rule as

$$\frac{dx(s)}{dx_0} = \frac{\partial x(s)}{\partial \tau}\frac{d\tau}{dx_0} + \frac{\partial x(s)}{\partial x_\tau}\frac{\partial x_\tau}{\partial \tau}\frac{d\tau}{dx_0} + \frac{\partial x(s)}{\partial x_\tau}\frac{\partial x_\tau}{\partial x_0}. \tag{17}$$

Now, $\partial x(s)/\partial\tau = -f_2(x(s))$, $\partial x(s)/\partial x_\tau = \Phi_2(s,\tau)$, $\partial x_\tau/\partial x_0 = \Phi_1(\tau,t_0)$, $\partial x_\tau/\partial\tau = f_1(x_\tau)$, $\Phi_2(\tau,\tau) = I$, $d\Phi_2(s,\tau)d\tau = -\Phi_2(s,\tau)\partial f_2(x_\tau)/\partial x$, i.e

$$\frac{d}{dx_0}p(\tau) = (I_1 - I_2 - I_3 - K)\frac{d\tau}{dx_0} + I_4, \tag{18}$$

where

$$
\begin{aligned}
I_1 &= \int_\tau^T \Phi_2^T(s,\tau)L_{xx}(x(s))\Phi_2(s,\tau)f_1(x_\tau)ds \\
I_2 &= \int_\tau^T \Phi_2^T(s,\tau)L_{xx}(x(s))f_2(x(s))ds \\
I_3 &= \int_\tau^T f_{2x}^T(x_\tau)\Phi_2^T(s,\tau)L_x^T(x(s))ds \\
I_4 &= \int_\tau^T \Phi_2^T(s,\tau)L_{xx}(x(s))\Phi_2(s,\tau)\Phi_1(\tau,t_0)ds \\
K &= L_x^T(x_\tau).
\end{aligned} \tag{19}
$$

After some straightforward calculations (as shown in [8]), we obtain

$$
\frac{d\Theta(\tau)}{dx_0} = \left[\mathcal{R}^T(Qf_1(x_\tau) - \Phi_2^T(T,\tau)L_x^T(x(T))) + p^T(\tau)\mathcal{R}_x f_1(x_\tau)\right]\frac{d\tau}{dx_0} + \left[\mathcal{R}^T Q + p^T(\tau)\mathcal{R}_x\right]\Phi_1(\tau,t_0), \tag{20}
$$

where

$$
Q = \int_\tau^T \Phi_2^T(s,\tau)L_{xx}(x(s))\Phi_2(s,\tau)ds, \tag{21}
$$

which can be interpreted as a quadratic costate. Hence, if we know that τ is a local optimum for an evolution starting from x_0, then, assuming that the system starts from $\tilde{x}_0 = x_0 + \delta x_0$, the new optimum is $\tau + \delta\tau + o(\delta x_0)$. And, according to (20),

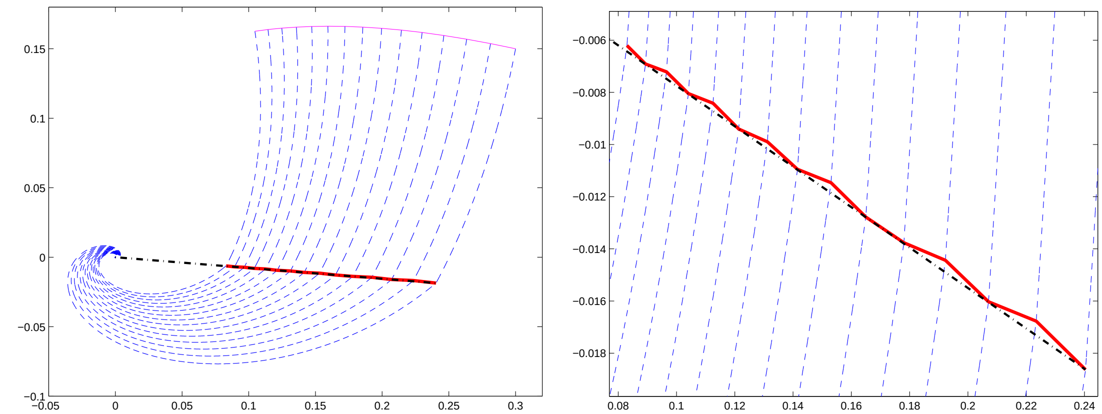
$$
\delta\tau = \frac{-[\mathcal{R}^T Q + p^T(\tau)\mathcal{R}_x]\Phi_1(\tau,t_0)\ \delta x_0}{\mathcal{R}^T(Qf_1(x_\tau) - \Phi_2^T(T,\tau)L_x^T(x(T))) + p^T(\tau)\mathcal{R}_x f_1(x_\tau)}. \tag{22}
$$

We now briefly discuss how to put (22) to use for the construction of optimal switching surfaces. The idea is to find a set of optimal switching states (which build up the optimal switching surface) through the knowledge of the variation in the optimal switching times as the initial conditions varies. And, drawing the optimal switching surface now amounts to finding the curve associated with those τ-parametrized states such that $\Theta(\tau) = 0$, which is a task that can be accomplished thanks to the sensitivity analysis described by (22).

As an example, consider a system described by $f_i(x) = A_i x$ with $A_1 = [-1\ 1; -2\ \ -1]$ and $A_2 = [-1\ 2; -1\ \ -1]$. Starting with an initial guess of $\tau = 0.5$, for the nominal $x_0 = [0.3, 0.15]^T$, a locally optimal switching time was found to be $\tau = 0.279$. However, a switching surface, depicted in Figures 3(a) and 3(b), can be obtained by varying the initial state, using $\tau = 0.279$ as the initial condition for the switching time, as the initial conditions were varied. In fact, the resulting switching surface matches quite well the true, optimal switching surface (for switched linear systems, the optimal switching surfaces are known to be homogeneous [15]) in the inifinite horizon case, as is indicated by Figure 3(a) and ven though our problem is a finite horizon problem, by choosing T large enough, the solutions to the finite horizon and the infinite horizon problems are qualitatively similar, as discussed in [3].

5 Conclusions

This paper presents two methods for getting rid of the dependence on the initial condition when optimizing over when to switch between different modes in a switched-mode system. The dependence on the initial condition was first dealt with by minimizing the switching parameter over the maximum cost for a given set of initial states. The only assumption made was that the initial state was confined to a given region in the state space. A second, alternative, sensitivity-based approach was also considered, and simulation results testify to the soundness of the two proposed methods.



(a) True optimal switching surface, dash-dotted; the designed one, bold.

(b) Close up to the two switching surface.

Figure 3: Executions of the system and the optimal switching surface

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