First-Order, Networked Control Models of Swarming Silkworm Moths

Musad A. Haque, Magnus Egerstedt, and Clyde F. Martin

***Abstract*— Social insects have long served as inspiration to the multi-agent community. In this paper, we take the opposite approach and see if tools from decentralized, networked control can be used to predict observed, biological behaviors. In particular, we study the silkworm moth, the *Bombyx Mori*, and we model these moths as first-order networks in which the male-male interactions are defined through a proximity graph. The male-female interactions are given by a broadcast protocol in which the females that are releasing pheromones are visible to all the males. Using barrier certificates, the resulting, switched network is analyzed and it is shown that the males are attracted to and trapped in a region defined by the female moths, as is the case in actual silkworm moths as well.**

I. INTRODUCTION

The research on multi-agent robotics and decentralized, networked control has drawn significant inspiration from interaction-rules in social animals and insects [2], [5], [6]. In particular, the widely used nearest-neighbor-based interaction rules, used for example for formation control [4], [9], [12], consensus [8], [16], and coverage control [1], [11], has a direct biological counterpart, as pointed out in [2]. In this paper, we reverse this direction, i.e. we draw inspiration from recent results on common Lyapunov functions for switched systems, barrier certificates, and networked control to understand particular swarming phenomena observed in the silkworm moth *Bombyx Mori*.

Silkworm moths are known to swarm in tight geometrical configurations, such as vertical cylindrical structures. This is caused by the females' intermittent releasing of a pheromone - *bombykol* - to attract male moths. This pheromone in essence makes the females act as attractors to the males, but the intermittent nature of the release produces an inherently switched system. Moreover, the spatial distribution of the females imply that the males are attracted to a general area rather than to a particular point, which is what is believed to cause their characteristic swarming.

In this paper we try and produce a model that is as simple as possible yet expressive enough to capture the relevant biological phenomena under consideration. In particular, we need to be able to model the intermittent nature of the pheromone release and their effect on the male moths with sufficient fidelity that as a result, the males are provably attracted to the general area in which the females reside. At the same time, we want the model to be simple, and, as such, amenable to analysis. What we will propose is a first order model of the of the male moths' dynamics, in combination with a immediate broadcast protocol for pheromone propagation. As a result, we will arrive at a model that can predict the attraction to the domain around the females, without providing any clues as to what the males' behaviors might be once inside that region. These claims are supported both by the theoretical developments and by extensive simulations.

The outline of this paper is as follows: In Section II, we review some basic facts from the study of social insects, with particular focus on their communication strategies. We discuss the manner in which the silkworm moths use such strategies and, in Section III, we define the network characteristics that correspond to these interaction and communication strategies. In Section IV, the derivations are carried out and we show that the males are attracted to an area defined by the intermittently pheromone releasing female moths. Some simulation results are given in Section V, and the conclusions in Section VI.

II. SOCIAL INSECTS

Large-scale biological systems, i.e. systems that consist of a large number of interacting individuals, have provided guidance to the multi-agent community. This in particular true when studying networked, decentralized control systems in which one typically wants to model and infer global properties from the specifications of the individual components and their inter-connectivity [13]. One particular area where this guidance has proven useful is when trying to characterize the role of communications between agents. In fact, it has long been established that communications are vital for social insects, where tasks such as division of labor, foraging for food, and population control are crucial to their existence. (For a representative sample, see [2], [18].)

One instance in which communications are crucial for the survival of the social insects is during mating. Many species form swarm clouds to increase their chances of locating partners during their reproductive phase. These swarm clouds may contain up to thousands of insects at a time, and the three major methods of long-range communication to attract mates and initiate a swarm are acoustic, visual, and olfactory signaling, as discussed in [17].

Grasshoppers and crickets use acoustic signaling through chirps and songs while orchid bees and some butterflies use visual signaling by moving their bodies and performing a

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dance. These communication strategies have their disadvantages. For example, visual signals have "physical barriers" and along with acoustic signals, they expose the signaling insect to predators. Thus, initiating a swarm in such manners presents a risk.

In contrast to this, insects that use olfactory signaling through *pheromones* to attract partners avoid these problems since most predators do not have the specialized receptors required to identify the scents. Moreover, in the rare case that such predators do appear, the pheromones are usually exuded in such small quantities that predators cannot track down the origin of the scents. The particular social, swarming insect that we focus on in this paper is the silk moth *Bombyx Mori*, that uses an intermittent pheromone release strategy for long-range communications [7]. The particulars of this strategy, together with a mathematical model of the swarming silkwork moth, is the topic of the next section.

III. NETWORK CHARACTERISTICS

A. Network Topology

From the preceding paragraphs, it is clear that the network model has to be heterogeneous in the sense that the agents (network nodes) will be divided up into two classes, corresponding to male and female moths, respectively. For this, we assume that the swarm contains N^m male moths and N^f female moths, with the corresponding index sets $\mathbf{N}^m = \{1,\dots,N^m\},\ \mathbf{N}^f = \{1,\dots,N^f\}$. (Here we have used the convention that the superscripts m and f refer to "male" and "female" respectively, which we will continue to do throughout this paper). We moreover assume that the moths' states take on values in a d-dimensional space (in a kinematic moth model, d would typically be 3), i.e. that $x_i^m \in \mathbb{R}^d,\ \forall\ i \in \mathbf{N}^m$ and $x_i^f \in \mathbb{R}^d,\ \forall\ i \in \mathbf{N}^f$.

In order to produce a simple yet sufficiently expressive model of the inter-moth interactions, we first need to discuss some known facts about how communications occur in real silkworm moths. In general, a moth can determine the gender of another moth by looking at its abdomen, i.e. if the moths are close enough, they can determine if a neighboring moth is male or female. This visual communication strategy is, however, negligible at long ranges, especially during the mating process. In fact, when male moths detect pheromones in the air, they start following the pheromone trail and when they see other moths, they can immediately identify males without looking at their abdomen, since they will not "smell" right [17].

In the silkworm moths, only the females release pheromones as opposed to other insects that employ attractant-pheromones. For example, the male lovebugs release pheromones to create a swarm but once the swarm is initiated, they stop releasing pheromones and the males dart across the swarm in hopes of flying into females and carrying them to mate in vegetation below the swarm. In the process, males often fly into other males and this is avoided in moths since female moths keep releasing pheromones throughout the swarming process until they locate a mate.

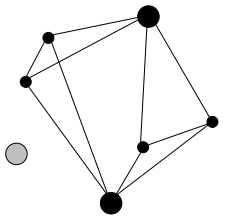


Fig. 1. Males, small circles, within the critical distance Δ of each other form an edge in the interaction graph. Moreover, all males form edges with females releasing pheromones (large, black), whereas the females not releasing pheromones (large, gray) are disconnected from the network.

Due to the limited interaction ranges over which the moths can detect each other in the pheromone-free case, we can define the instantaneous, male proximity graph $\mathbf{G}^m(t) = \mathbf{N}^m \times \mathbf{E}^m(t)$, where, for two distinct moths $(i,j) \in \mathbf{E}^m(t) \ \Leftrightarrow\ \|x_i^m(t) - x_j^m(t)\| \le \Delta$, for some critical interaction distance Δ. This construction ensures that the interaction graph is simple (no self-loops) and undirected. In fact, it is a so-called Δ-disk proximity graph, as defined for example in [1], [9].

Since it is costly for the females to be releasing pheromones, the females typically only release pheromones in small bursts. In our model, we let the females that are releasing pheromones at a given time t be denoted by $\mathbf{N}^f(t) \subseteq \mathbf{N}^f$ (with cardinality $N^f(t)$) as a subset of the total female moths in the swarm. This also brings us to our first assumption:

Assumption 1: $\mathbf{N}^f(t) \neq \emptyset,\ \forall\ t$.

Thus at any given time t, there is at least one female moth releasing pheromones.

Since the olfactory communication strategy acts at much greater ranges and at fairly high-speeds, we assume that the pheromones act as a broadcast strategy that enables each male to immediately detect the relative displacement of a female moth in $\mathbf{N}^f(t)$. As a result, we define the total interaction graph as $\mathbf{G}(t) = \mathbf{N}(t) \times \mathbf{E}(t)$, where the node set is given by $\mathbf{N}(t) = \mathbf{N}^m \cup (N^m + \mathbf{N}^f(t))$, and the edge set is defined through $(i,j) \in \mathbf{E}(t)$ if and only if one of the following three conditions hold

$$\begin{aligned} &(i,j) \in \mathbf{E}^m(t) \\ &i \in \mathbf{N}^m,\ j \in N^m + \mathbf{N}^f(t) \\ &j \in \mathbf{N}^m,\ i \in N^m + \mathbf{N}^f(t). \end{aligned}$$

The interpretation behind this somewhat dense formulation is that an edge exists between nodes in the total interaction graph if and only if the nodes are both corresponding to male moths within a distance Δ of each other or exactly one of the nodes corresponds to a female moth that is releasing pheromones.

The main idea now is to apply a decentralized control strategy over the set of male moths. This should moreover be done in such a way that the individual moths are governed by a control law that is only allowed to contain references to the relative displacements between the moth and its neighboring moths in the interaction graph, as seen in Figure 1.

B. Moth Dynamics

One explicit aim with this paper is to tap in to the large multi-agent literature on decentralized coordination in order to understand biological phenomena. Even though there are scant biological evidence that the moths execute linear control strategies, we, for the sake of analytical simplicity, assume that this is the case. And, we leave the biological reasonableness of this model to a further endeavor and take the point-of-view that the model should be judged based on the results it generates. We have already established that the male moths will only detect other male moths when they are within a certain critical distance of each other while female moths releasing pheromones are always visible to them. Thus, we will assume that the nowadays highly widespread linear consensus equation is used for defining the motion of the male moths as

Assumption 2:

$$\dot{x}_i = -\sum_{j \mid (i,j)\in \mathbf{E}(t)} (x_i - x_j),\ \forall i \in \mathbf{N}^m,$$

where we use the convention that $x_i = x_i^m,\ \forall\ i \in \mathbf{N}^m$ and $x_i = x_{i-N^m}^f,\ \forall\, i \in N^m + \mathbf{N}^f(t)$. Under this allocation of indices, we have that the first N^m x_i's correspond to male moths, and the remaining N^f to female moths. And, to make matters a little bit less complicated, we will also assume that the female moths are stationary during the swarming process.

Assumption 3:

$$\dot{x}_i = 0,\ \forall\ i \in N^m + \mathbf{N}^f(t).$$

As a result of this assumption, it is expected that the male moths will end up close to the stationary, female moths by following their (intermittent) pheromone trails. This is the topic of the next section.

IV. ANALYSIS OF SILKWORM MOTH SWARMS

A. Barrier Certificates and Attraction Functions

Since the aim of this paper is, in part, to show that the simple, first-order network model of the silkworm moths is in fact rich enough to predict the known behavior that the moths exhibits, we must show that they do in fact converge to an appropriate, geometric shape. For this we first need to establish convergence to a specific region, and then show that this region in fact acts as a trapping region, i.e. once inside, they never leave. For this, we first recall some results involving barrier certificates.

Barrier certificates are typically used to show that a certain region is a trapping region [14], but we also need to establish attraction. For this, we need to slightly modify the results in [14] and we do this for a general system whose state evolves in $\mathbb{R}^n$. In fact, if there exists a continuously differentiable, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$ such that $V(0) = 0$ and $V > 0\ \forall\ x \in \mathbb{R}^n \backslash \{0\}$ and $\dot{V} < 0,\ \forall x \in C(c)$, where $C(c) = \{x \in \mathbb{R}^n \mid V(x) \geq c\}$ then

(i) $x(0) \in C(c)\ \Rightarrow\ x(T) \notin C(c)$ for some $T > 0$
(ii) $x(0) \notin C(c)\ \Rightarrow\ x(t) \notin C(c),\ \forall t > 0.$

To see that state trajectories, starting in set $C(c)$, enter $C(c)$'s complement (denoted by $C'(c)$) after some finite time, one just have to follow Lyapunov's stability theorem, e.g. [10]. Let us choose a level set $\partial C(c_1)$, with $c_1 \geq c$, as $\partial C(c_1) = \{x \in \mathbb{R}^n | V(x) = c_1\}$. Hence, $\partial C(c_1) \subset C(c)$ and consequently, $\dot{V} < 0$ in the entire set $\partial C(c_1)$. As a result, a state trajectory that crosses the level surface $V(x) = c_1$ advances to a set $\partial C(c_2)$ for some $c_2 < c_1$.

As long as $\dot{V} < 0$ the trajectory moves from one level set to another. Therefore, if a state $x(t) \in C(c)$, there exists a $T > 0$ such that $x(t+T) \notin C(c)$, thus establishing the attraction.

In [14], [15] the existence of a so-called barrier certificate guarantees that when the state space is separated into "safe" and "unsafe" regions, then trajectories originating in the set of safe states never enter the unsafe regions. We can now apply exactly the same thinking, separating the state space into the disjoint regions $C(c)$ and $C'(c)$, and defining the barrier certificate $B(x)$ as

$$B(x) = V(x) - c$$

The fact that $B(x)$ is indeed a barrier certificate since it satisfies the following conditions:

$$\begin{aligned} &B(x) \geq 0\ \forall\ x \in C(c) \\ &B(x) < 0\ \forall\ x \in C'(c) \\ &\tfrac{\partial B}{\partial x}(x) f(x) < 0,\ \forall x \in \partial C(c), \end{aligned}$$

where $\dot{x} = f(x)$. And, according to Theorem(1) in [14], $C'(c)$ will thus act as a trapping region, thus establishing the two properties needed to characterize the behavior of the silkworm moths.

However, what remains to be done is of course to find such a function $V(x)$ together with the corresponding c for the actual moth dynamics. And, as the network topology is changing, we need to find V and c that are common to all possible network topologies [3].

B. 1D-Swarms

For the sake of notational clarity, we first start with the situation in which the moths are all evolving in $\mathbb{R}$, i.e. they are one-dimensional. We then proceed to the general case in which they evolve in $\mathbb{R}^d$.

The first problem is to find a suitable candidate function for V in the previous section. And, since there is no reason to believe that the male moths will end up close to the origin, we instead focus our attention on the centroid of *all* the female moths. We let ρ^f denote this (one-dimensional) centroid, i.e.

$$\rho^f = \frac{1}{N^f} \sum_{i=1}^{N^f} x_i^f.$$

It should be noted that the system dynamics will undergo discrete transitions when male moths enter or leave each others proximity disks, or when female moths initiate or terminate a pheromone release phase. As such, we obtain a switched dynamical system. And, one technique for proving the stability of such systems is to try and find a common Lyapunov function. In fact, as shown in [3], a switched system is asymptotically stable for any possible switching sequence if and only if there exists a common Lyapunov

function. In our case, asymptotic stability is not what we are after, but nonetheless, we need to find the appropriate V that acts as an attraction-barrier certificate for all network topologies.

For this, we define the continuously differentiable function $W : \mathbb{R}^{N^m} \to \mathbb{R}$ as

$$W(x^m(t)) = \frac{1}{2}\|x^m(t) - \mathbf{1}\rho^f\|^2,$$

then a promising candidate attraction-barrier certificate is

$$V(x) = \frac{1}{2} \parallel W(x) - \frac{1}{2}N^m(\rho^f)^2 \parallel^2 .$$

where $x^m = (x_1^m, \ldots, x_{N^m}^m)^T \in \mathbb{R}^{N^m}$, and where $\mathbf{1} = (1, \ldots, 1)^T$ is the vector with ones along each components.

We now get that

$$\dot{W}(x^m(t)) = \langle \dot{x}^m(t), x^m(t) - \mathbf{1}\rho^f \rangle,$$

but before we can tackle this expression, some comments about the dynamics must be made. In fact, in the one-dimensional case, the dynamics in Assumption 2 can be rewritten as

$$\dot{x}^m(t) = -\mathbf{L}^m(t)x^m(t) + N^f(t)\mathbf{1}\rho^f(t),$$

where $\rho^f(t)$ is the centroid associated with the female moths currently releasing pheromones. Moreover, this expression comes from the graph Laplacian $\mathbf{L}(t)$ associated with the total interaction graph $\mathbf{G}(t)$ as

$$\mathbf{L}(t) = \begin{bmatrix} \mathbf{L}^m(t) & -\mathbf{1}\mathbf{1}^T \\ -\mathbf{1}\mathbf{1}^T & N^m\mathbf{I} \end{bmatrix}.$$

The reason that the graph Laplacian takes on this specific form is that each male moth is assumed to be able to interact with all female moths currently releasing pheromones. We have furthermore made the assumption that the females do not interact with each other (hence the $N^m\mathbf{I}$ in the Laplacian) but this assumption does not matter for the developments in this paper. Moreover, under Assumption 1, we get that $\mathbf{L}(t)$ is positive semidefinite, and, as shown in [9], $\mathbf{L}^m(t)$ (which is not a graph Laplacian) is positive definite for all t.

This leads us to a formulation of the derivative of W as

$$\dot{W} = \langle x^m(t) - \mathbf{1}\rho^f, -\mathbf{L}^m(t)x^m(t) + N^f(t)\mathbf{1}\rho^f(t) \rangle.$$

A further observation to make is that under the assumption that each male moth can interact with every female moth releasing pheromones, we observe that

$$\mathbf{L}^m(t)\mathbf{1} = N^f(t)\mathbf{1},$$

and hence

$$\dot{W} = -\langle x^m(t) - \mathbf{1}\rho^f, x^m(t) - \mathbf{1}\rho^f(t) \rangle_{\mathbf{L}^m(t)},$$

where we have used the fact that $\mathbf{L}^m(t)$ is positive definite to let it induce a norm. Now

$$\begin{aligned}
\dot{W} &= -\langle x^m(t) - \mathbf{1}\rho^f + (\mathbf{1}\rho^f - \mathbf{1}\rho^f(t)), x^m(t) - \mathbf{1}\rho^f \rangle_{\mathbf{L}^m(t)} \\
&= -\|x^m(t) - \mathbf{1}\rho^f\|^2_{\mathbf{L}^m(t)} - \langle x^m(t) - \mathbf{1}\rho^f, \\
&\quad \mathbf{1}\rho^f - \mathbf{1}\rho^f(t) \rangle_{\mathbf{L}^m(t)} \\
&\leq -\|x^m(t) - \mathbf{1}\rho^f\|^2_{\mathbf{L}^m(t)} + \|x^m(t) - \mathbf{1}\rho^f\|_{\mathbf{L}^m(t)} \\
&\quad \|\mathbf{1}\rho^f - \mathbf{1}\rho^f(t)\|_{\mathbf{L}^m(t)} \\
&= -\|x^m(t) - \mathbf{1}\rho^f\|_{\mathbf{L}^m(t)}(\|x^m(t) - \mathbf{1}\rho^f\|_{\mathbf{L}^m(t)} \\
&\quad - \|\mathbf{1}\rho^f - \mathbf{1}\rho^f(t)\|_{\mathbf{L}^m(t)}).
\end{aligned}$$

In other words, $\dot{W} < 0$ if

$$\|x^m(t) - \mathbf{1}\rho^f\|_{\mathbf{L}^m(t)} > \|\mathbf{1}\rho^f - \mathbf{1}\rho^f(t)\|_{\mathbf{L}^m(t)}$$

As a final step, we note that

$$\|\mathbf{1}\rho^f - \mathbf{1}\rho^f(t)\|^2_{\mathbf{L}^m(t)} = (\rho^f{}^2 + \rho^f(t)^2 - 2\rho^f\rho^f(t))\mathbf{1}^T\mathbf{L}^m(t)\mathbf{1}.$$

which in turn is equal to

$$N^mN^f(t)(\rho^f{}^2 + \rho^f(t)^2 - 2\rho^f\rho^f(t)) = N^mN^f(t)(\rho^f - \rho^f(t))^2.$$

Notice that $N^f(t) \leq N^f$, and that the distance from the centroid to any centroid associated with a subset of female moth positions is maximized by a single moth position, denoted by $f^\star$, where

$$f^\star = \text{argmax}_{x_i^f}\{(\rho^f - x_i^f)^2\}.$$

Summarizing these observations, we get that $\dot{W}$ is decreasing as long as

$$\|x^m(t) - \mathbf{1}\rho^f\|^2_{\mathbf{L}^m(t)} > N^mN^f(\rho^f - f^\star)^2.$$

This result involves the $\mathbf{L}^m(t)$ norm and to obtain a result that holds for all topologies, we need to expand $\|x^m(t) - \mathbf{1}\rho^f\|^2_{\mathbf{L}^m(t)}$. It is in fact straightforward to show that

$$\begin{aligned}
\|x^m(t) - \mathbf{1}\rho^f\|^2_{\mathbf{L}^m(t)} &\geq N^mN^f(t)((x_1^m - \rho^f)^2 + (x_2^m - \rho^f)^2 \\
&+ \ldots + (x_{Nm}^m - \rho^f)^2).
\end{aligned}$$

If we now let x_{max}^m maximize $(x_i^m - \rho^f)^2, \quad \forall i \in \mathbf{N}^m$, i.e. if x_{max}^m is the male moth furthest away from the centroid of the females, then,

$$\begin{aligned}
N^mN^f(t)((x_1^m - \rho^f)^2 &+ (x_2^m - \rho^f)^2 + \ldots + (x_{Nm}^m - \rho^f)^2) \\
&\geq N^mN^f(t)(x_{max}^m - \rho^f)^2 \\
&\geq N^m(x_{max}^m - \rho^f)^2.
\end{aligned}$$

Therefore, $\dot{W} < 0$ and subsequently $\dot{V} < 0$ when $(x_{max}^m - \rho^f)^2 > N^f(\rho^f - f^\star)^2$.

Now, let us define the set S such that

$$S = \{x \in \mathbb{R}^{Nm} | (x_{max}^m - \rho^f)^2 > N^f(\rho^f - f^\star)^2\},$$

where we recall that x_{max}^m is the male moth furthest away from the centroid of the females, while $f^\star$ is defined as $f^\star = \text{argmax}_{x_i^f}\{(\rho^f - x_i^f)^2\}$. It is clear that $\dot{V} < 0$ if $x \in S$, i.e. S obeys the property:

$$S' \subseteq C'(d)$$

for some choices of d. We can thus find the c required in the previous section as

$$c = min\{d \in \mathbb{R} | S' \subseteq C'(d)\}.$$

In other words, with this choice of V and c, we have exactly the required properties from the previous section needed to establish attraction as well as trapping. (For a special case of these results, see [19],[20]).

C. In Higher Dimensions

For higher dimensions, we assume that the motion of the male moths is still governed by the linear consensus equation in each dimension, i.e.

$$\dot{x}_{i,k} = - \sum_{j \mid (i,j) \in \mathbf{E}(t)} (x_{i,k} - x_{j,k}), \ \forall \ i \in N^m, \ k \in \{1, \ldots, d\}.$$

If we define the state $z \in \mathbb{R}^{N^m d}$ as $z_i = (x_{1,i}, \ldots, x_{N^m,i})^T, \ \forall i \in \{1, \ldots, d\}$, i.e. z_i is a column vector that contains all moth locations in the $i-th$ dimension, then the consensus equation can be rewritten as

$$\dot{z}_i = -\mathbf{L}(t) z_i, \ \forall \ i \in \{1, \ldots, d\}.$$

Moreover, if we define the state $x = (z_1, \ldots, z_d)^T$ then the consensus equation can be expressed as

$$\begin{aligned} \dot{x}^m(t) &= -I_d \otimes \mathbf{L}^m(t) x^m(t) + N^f(t) \rho^f(t) \otimes \mathbf{1}_{N^m} \\ &= (I_d \otimes \mathbf{L}^m(t))(-x^m(t) + (I_d \otimes \mathbf{1}_{N^m}) \rho^f(t)). \end{aligned}$$

The centroid is given by

$$\rho^f = \frac{1}{N^f} \sum_{i=1}^{N^f} x_i^f,$$

and we now proceed in the same way as for the 1D case in the previous section by letting W be given by

$$W = \frac{1}{2} \| x^m(t) - \rho^f \otimes \mathbf{1}_{N^m} \|^2,$$

with the corresponding candidate attraction-barrier certificate V being $V(x) = \frac{1}{2} \parallel W(x) - \frac{1}{2} N^m (\rho^f)^2 \parallel^2$.

Now, $\dot{W}$ in the higher dimensions can be expressed as

$$\dot{W}(x^m(t)) = \langle \dot{x}^m(t), x^m(t) - \rho^f \otimes \mathbf{1}_{N^m} \rangle,$$

or, equivalently

$$\begin{aligned} \dot{W} = \quad & \langle (I_d \otimes \mathbf{L}^m(t))(-x^m(t) + (I_d \otimes \mathbf{1}_{N^m}) \rho^f(t)), \\ & x^m(t) - \rho^f \otimes \mathbf{1}_{N^m} \rangle. \end{aligned}$$

As per the previous section, we again let a positive definite matrix, in this case $I_d \otimes \mathbf{L}^m(t)$, induce a norm

$$\begin{aligned} \dot{W} &= \langle -x^m(t) + (I_d \otimes \mathbf{1}_{N^m}) \rho^f(t), x^m(t) - \rho^f \otimes \mathbf{1}_{N^m} \rangle_{I_d \otimes \mathbf{L}^m(t)} \\ &= -\| x^m(t) - (I_d \otimes \mathbf{1}_{N^m}) \rho^f \|_{I_d \otimes \mathbf{L}^m(t)} (\| x^m(t) \\ &\quad - (I_d \otimes \mathbf{1}_{N^m}) \rho^f \|_{I_d \otimes \mathbf{L}^m(t)} - \| (I_d \otimes \mathbf{1}_{N^m}) \rho^f \\ &\quad - I_d \otimes \mathbf{1}_{N^m} \rho^f(t) \|_{I_d \otimes \mathbf{L}^m(t)}. \end{aligned}$$

We thus note that $\dot{W} < 0$ if $\| x^m(t) - (I_d \otimes \mathbf{1}_{N^m}) \rho^f \|_{I_d \otimes \mathbf{L}^m(t)} > \| (I_d \otimes \mathbf{1}_{N^m}) \rho^f - I_d \otimes \mathbf{1}_{N^m} \rho^f(t) \|_{I_d \otimes \mathbf{L}^m(t)}$. Furthermore, $\dot{W}$ is still decreasing as long as $\| x^m(t) - (I_d \otimes \mathbf{1}_{N^m}) \rho^f \|^2_{I_d \otimes \mathbf{L}^m(t)} > \| (I_d \otimes \mathbf{1}_{N^m}) \rho^f - I_d \otimes \mathbf{1}_{N^m} \rho^f(t) \|^2_{I_d \otimes \mathbf{L}^m(t)}$.

Notice now that the term $\| (I_d \otimes \mathbf{1}_{N^m}) \rho^f - I_d \otimes \mathbf{1}_{N^m} \rho^f(t) \|^2_{I_d \otimes \mathbf{L}^m(t)}$ can be written as $N^m N^f(t) \| \rho^f - \rho^f(t) \|^2$ and that $N^m N^f(t) \| \rho^f - \rho^f(t) \|^2$ is bounded above by $N^m N^f \| \rho^f - \rho^f(t) \|^2$. Hence, if we let $f^\star = \text{argmax}_{x_i^f} \| \rho^f - \rho^f(t) \|^2$, we have

$$N^m N^f(t) \| \rho^f - \rho^f(t) \|^2 \leq N^m N^f \| \rho^f - f^\star \|^2.$$

We note that $\| x^m(t) - (I_d \otimes \mathbf{1}_{N^m}) \rho^f \|^2$ is equal to $N^f(t) \| x^m - \mathbf{1} \rho^f \|^2$, and since $N^m N^f(t) \| x^m - \mathbf{1} \rho^f \|$ is bounded below by $N^m \| x^m_{max} - \mathbf{1} \rho^f \|^2$, we can guarantee attraction and trapping in arbitrary dimensions since $\dot{W} < 0$ (and consequently $\dot{V} < 0$) if

$$\| x^m_{max} - \mathbf{1} \rho^f \|^2 > N^f \| \rho^f - f^\star \|^2,$$

where x^{max} is the male position that maximizes $\| x_i - \rho^f \|^2$.

V. SIMULATIONS

In Figure 3, a simulation is shown that illustrates how the males moths eventually get trapped in a region defined by the female moths. There are 30 males (small dot) and 4 females, where the females that are currently releasing pheromones have a ring around the dot. Moreover, the centroid of visible females is denoted by an 'x'.

To illustrate the fact that we do not have asymptotic stability to a point, we plot W as a function of time in in Figure 2. From that figure, it is clear that W serves as an attraction-barrier certificate in that it has a negative time derivative only initially. In fact, from the figure we observe that $\dot{W} > 0$ around $t = 200$.

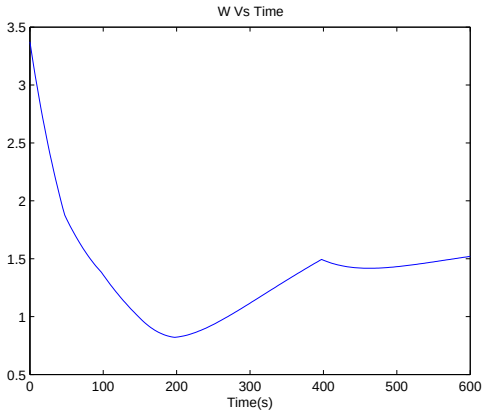
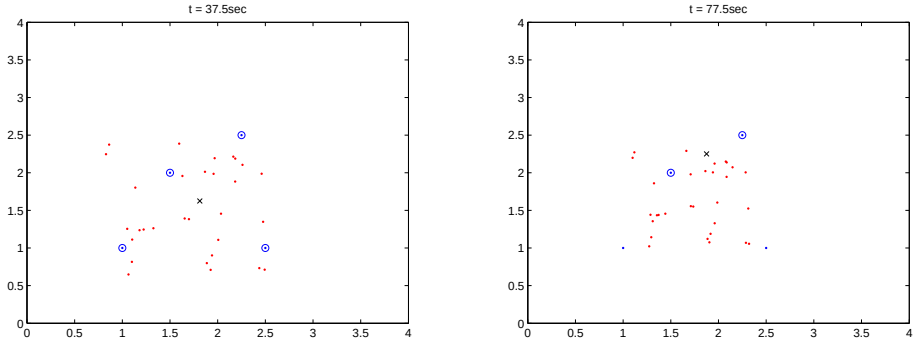


Fig. 2. Plot of the function $W(x)$ against time.

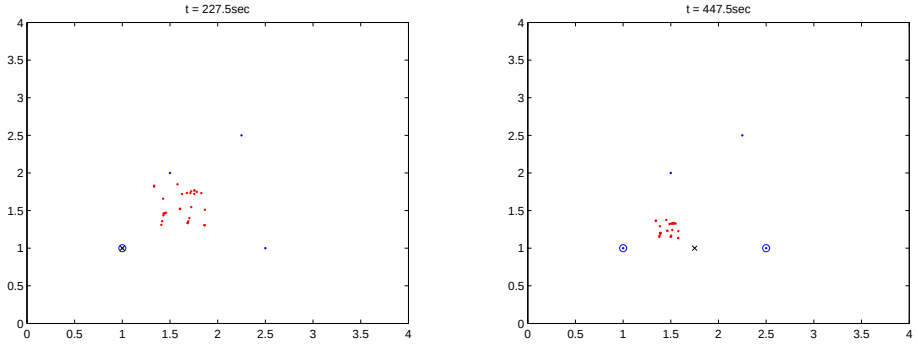
VI. CONCLUSIONS

In this paper, we model the silkworm moth, the *Bombyx Mori*, as a first-order network in which the male-male interactions are defined through a proximity graph. The male-female interactions are given by a broadcast protocol in which the females that are currently releasing pheromones



(a) All four females are releasing pheromones.

(b) Two females are releasing pheromones.



(c) A single female is releasing pheromones.

(d) Two females are releasing pheromones.

Fig. 3. A simulation is shown, where male moths are shown in as small dots, while females releasing pheromones are denoted with a circle around them, and their centroid is denoted by a cross.

instantaneously are visible to the males. The resulting, switched network is then analyzed using barrier certificate tools. In fact, our aim was to show that with such a simple model, the observed swarming phenomenon in which the male moths end up around the females moths can in fact be predicted. Simulation results illustrate these results further.

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