Monotone Smoothing Splines

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Abstract

The optimal solution to the problem of driving the output of a single-input, single-output linear control system close to given waypoints is analyzed. Furthermore, this smooth interpolation is done while the state space is constrained by an infinite dimensional, non-negativity constraint on the derivative of the output spline function. For the case when the acceleration is controlled directly this problem is completely solved. The solution is obtained by exploiting a finite parameterization of the problem, resulting in a dynamic programming formulation that can be solved analytically.

1 Introduction

When interpolating curves through given data points, a demand that arises naturally, when the data is noise contaminated, is that instead of doing exact interpolation we only require that the curve passes close to the interpolation points. This means that outliers will not be given too much importance, which could otherwise potentially corrupt the shape of the interpolation curve. In this paper, we investigate this type of interpolation problem from an optimal control point of view, where the task is to choose appropriate control signals in such a way that the output of a given, linear control system defines the desired interpolation curve. The curve is obtained by minimizing the energy of the control signal, and we, furthermore, deal with the outliers problem by adding quadratic penalties for deviating from the interpolation points to the energy cost functional in order to produce smooth output curves [3, 7]. The fact that we minimize the energy of the control input, while driving the output of the system close to the interpolation points, gives us curves that belong to a class that in the statistics literature is referred to as *smoothing splines* [7, 8].

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However, in many cases, this type of construction is not enough since one sometimes want the curve to have a certain structure, such as convexity or monotonicity properties. For instance, given a set of observations of how much an individual is growing during his first ten years. Any curve that interpolates through these points in such a way that the derivative is allowed to be negative is unsatisfactory and can not be of any use for future predictions.

This non-negative derivative constraint will be our main focus in this paper, and we will show how the corresponding infinite dimensional constraint (it has to hold for all times) can be reformulated and solved in a finite setting based on dynamic programming. The system used for generating these *monotone* smoothing splines will be a second order system, where we control the acceleration directly. The outline of this paper is as follows: In Section 2, we describe the problem and show some of the properties that the optimal solution has to exhibit. We then, in Section 3, solve the monotone interpolation problem for a second order system, followed by some concluding remarks in Section 4.

2 Problem Description

The problem, investigated in this paper, is how to produce a monotonously increasing curve that passes close to a given set of waypoints, $\alpha_1, \ldots, \alpha_m$, at times $t_1 < t_2 < \ldots < t_m$, while keeping the energy of the curve small.

In the following paragraphs, we will discuss some of the features that this problem exhibits. We will also show some preliminary results about the optimal control before we can proceed to actually solving the monotone interpolation problem for a double integrator system.

2.1 Unconstrained Optimization

Given a time-invariant, minimal, single-input, single-output linear control system

$$\dot{x} = Ax + bu, \quad y = c^T x, \tag{1}$$

where $x \in \mathbb{R}^n$. The convex cost functional that minimizes the energy of the control system, while interpolating close to the waypoints, is defined as

$$\min\left\{\frac{1}{2}\int_0^T u^2(t)dt + \frac{1}{2}\sum_{i=1}^{m}\tau_i(x(t_i)-\alpha_i)^2\right\}, \tag{2}$$

where τ_i is a factor that reflects how important it is that $y(t_i)$ is close to α_i.

If we now define a set of linearly independent[1] basis functions

$$g_i(t) = \begin{cases} c^T e^{A(t_i-t)}b & \text{if } t \le t_i \\ 0 & \text{if } t > t_i \end{cases} \quad i = 1, \dots, m, \tag{3}$$

and assume, without loss of generality, that $x(0) = 0$, we can rewrite (2) as

$$\min\left\{\frac{1}{2}\int_0^T u^2(t)dt + \frac{1}{2}\sum_{i=1}^{m}\tau_i(\int_0^T g_i(t)u(t)dt-\alpha_i)^2\right\}. \tag{4}$$

If no positivity constraints are imposed on the derivative of the output, the unique, optimal control, $u_0 \in L^2[0,T]$, that solves this convex problem is found in [3]. It is given by

$$u_0(t) = \xi^T g(t), \tag{5}$$
$$\xi = (I + \mathcal{T}G)^{-1}\mathcal{T}\alpha, \tag{6}$$

where $g(t) = (g_1(t), \dots, g_m(t))^T$, $\alpha = (\alpha_1, \dots, \alpha_m)^T$, $\mathcal{T} = \text{diag}(\tau_1, \dots, \tau_m)$, and the Grammian, G, is defined as

$$G = \int_0^T g(s)g(s)^T ds.$$

From this it is possible to derive that the optimal control is continuous and piecewise entire, with discontinuities only at the waypoints. A result from solving this type of interpolation problem can be seen in Figure 1.

2.2 Non-Negative Derivatives

If we now add the monotonicity constraint on the derivative to the problem formulation, we still want to minimize the cost functional (2) under the additional, infinite dimensional inequality constraint

$$\dot{y}(t) \ge 0. \tag{7}$$

Since we want $\dot{y}$ to be continuous, we set our *constraint space* to be $C[0,T]$ (the space of continuous functions), and we can thus form our associated *Lagrangian* as

$$\begin{aligned} L(u,\nu) &= \frac{1}{2}\int_0^T u^2(t)dt \\ &+ \frac{1}{2}\sum_{i=1}^{m}\tau_i(\int_0^T g_i(t)u(t)dt-\alpha_i)^2 \\ &- \int_0^T \dot{y}(t)d\nu(t), \end{aligned} \tag{8}$$

[1] Linear independence follows immediately from the fact that the different basis functions vanish at different times.

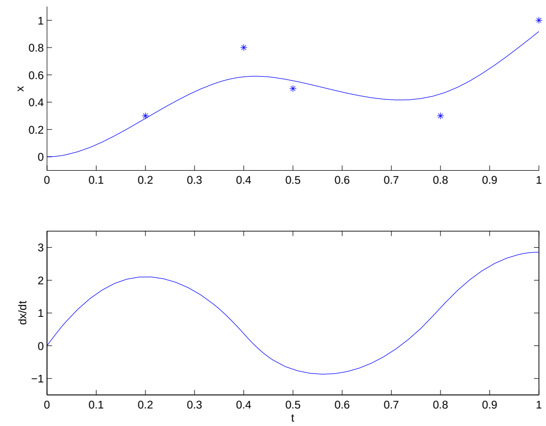

Figure 1: Smoothing splines with $\tau_i = 1000, i = 1, \dots, 5$. The depicted curve is the output of a second order system where the acceleration is controlled directly. In the upper graph, $y(t)$ is plotted, while the lower graph displays $\dot{y}(t)$. The stars correspond to the different waypoints.

where ν is in $BV[0,T]$ (the space of functions of bounded variation), which is the *dual space* of $C[0,T]$ [4]. The optimal solution is then found by solving the *dual problem*

$$\max_{\nu \ge 0} \inf_u L(u,\nu). \tag{9}$$

2.3 Second Order Systems

We now focus our attention to a system on the form used for producing the output curve in Figure 1, and in the sections to follow this is the control system that will be investigated.

We let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; c^T = \begin{pmatrix} 1 & 0 \end{pmatrix}, \tag{10}$$

which corresponds to controlling the acceleration directly while observing the position of the system. The reason for choosing to focus on this system is basically two-folded. First of all, it is a simple enough system which gives us some hope about actually being able to solve the hard, monotone interpolation problem in a way that suggests how further generalizations could be done. Secondly, in this specific, second order case we are minimizing the L^2-norm of the second derivative of the output. This is also the case when the widely used, standard cubic splines are produced [6], which indicates that this system has a rich enough dynamics to generate potentially useful, monotone output splines.

Furthermore, it is not always the case that the problem formulation explicitly involves any system dynamics. If we are just looking for a smooth interpolation function, $y(t)$, that satisfies

$$y \in C^1[0,T]$$
$$\dot{y}(t) \ge 0,$$

then is seems reasonable to model the underlying dynamics as a double integrator, since we then control the acceleration directly.

With this choice of system matrices, the g_i's reduce to the following simple, piece-wise linear form

$$g_i(t) = \begin{cases} t_i - t & \text{if } t \leq t_i \\ 0 & \text{if } t > t_i \end{cases} \qquad i = 1, \ldots, m. \tag{11}$$

With a slight abuse of notation, we let $x(t) \in \mathbb{R}$ denote the first of the state variables, and $\dot{x}(t) \in \mathbb{R}$ the second. Then the monotonicity constraint (7) can be rewritten as

$$\dot{x}(t) = \int_0^t u(s)ds \geq 0, \tag{12}$$

and thus the Lagrangian becomes

$$\begin{aligned} L(u, \nu) &= \frac{1}{2}\int_0^T u^2(t)dt \\ &+ \frac{1}{2}\sum_{i=1}^m \tau_i \left(\int_0^T g_i(t)u(t)dt - \alpha_i\right)^2 \\ &- \int_0^T \int_0^t u(s)ds d\nu(t). \end{aligned} \tag{13}$$

Integrating the Stieltjes integral in (13) by parts reduces the Lagrangian to the following:

$$\begin{aligned} L(u, \nu) &= \frac{1}{2}\int_0^T u^2(t)dt \\ &+ \frac{1}{2}\sum_{i=1}^m \tau_i \left(\int_0^T g_i(t)u(t)dt - \alpha_i\right)^2 \\ &- \int_0^T (\nu(T) - \nu(t))u(t)dt, \end{aligned} \tag{14}$$

which is a more easily manipulated expression.

2.4 Properties of the Solution

We now proceed by investigating what features the solution to the monotone interpolation problem has to have if the underlying dynamics is given by the system defined in (10). This is important since it allows us to narrow down the search for the optimal control to a search in the reduced set of controls that exhibit these necessary properties.

One fist observation from (13) is that, given our optimal solution u_0, in order for us to maximize $L(u_0, \nu)$ we need to let the optimal $\nu_0(t)$ vary only for those t where $\dot{x}_0(t) = 0$ since $\nu \geq 0$ and, by feasibility, $\dot{x}_0(t) \geq 0$ for all t. But this fact implies that $\nu_0(T) - \nu_0(t)$ is constant $(= C_t \geq 0)$ on intervals where $\dot{x}_0(t) > 0$. The fact that $C_t \geq 0$ follows from the positivity constraint on ν_0.

If we take our *optimization state space* to be the space of piecewise continuous functions $(PC[0,T])$, the set of such controls that satisfy the constraints is closed and convex, which is stated in the following lemma.

Lemma 2.1 *The set of controls in $PC[0,T]$ that satisfy the constraints is a closed and convex set.*

Proof. We first show convexity. Given two $u_i(t) \in PC[0,T],\ i = 1, 2$, such that

$$\int_0^t u_i(s)ds \geq 0, \quad \forall t \in [0,T], \quad i = 1, 2,$$

then for any $\lambda \in [0,1]$ we obviously have that

$$\int_0^t (\lambda u_1(s) + (1-\lambda)u_2(s))ds \geq 0, \quad \forall t \in [0,T]. \tag{15}$$

If we instead have a collection of controls, $\{u_i(t)\}_{i=0}^{\infty}$, where each individual control satisfies the constraint and where $u_i \to \hat{u}$ as $i \to \infty$, we can pass the limit through the integral due to the compactness of $[0,t]$ and still satisfy the constraint. The fact that $PC[0,T]$ is a Banach space gives us that the limit, $\hat{u}$, still remains in that space, which concludes the proof. □

Furthermore, as long as $\dot{x}(0) \geq 0$ we can always let $u \equiv 0$, and thus the set of admissible controls is non-empty.

This simplifies our optimization problem since Lemma 2.1 is a strong enough result to guarantee the existence of a unique optimal solution, and we can thus replace *inf* in (9) with *min* [4], which allows us to state the following key theorem about our optimal control.

Theorem 2.1 (Properties of the Solution) *The control in $PC[0,T]$ that minimizes (4), while keeping $\dot{x}(t) \geq 0$ for all $t \in [0,T]$, is piecewise linear. It furthermore only changes from different linear cases at the waypoints, or at times when $\dot{x}(t) = 0$.*

Proof. Due to the existence and uniqueness of the solution, we can obtain the optimal controller by calculating the *Fréchet differential* of L with respect to u, and setting this equal to zero for all increments $h \in PC[0,T]$.

By setting $L_\nu(u) = L(u, \nu)$ we get that

$$\begin{aligned} &\delta L_\nu(u,h) = \lim_{\epsilon \to 0} \tfrac{1}{\epsilon}(L_\nu(u + \epsilon h) - L_\nu(u)) = \\ &\int_0^T \Big[u(t) + \sum_{i=1}^m \tau_i \left(\int_0^T g_i(s)u(s)ds - \alpha_i\right)g_i(t) - \\ &(\nu(T) - \nu(t))\Big] h(t)dt. \end{aligned} \tag{16}$$

For (16) to be zero for all $h \in PC[0,T]$ we need to have that

$$\begin{aligned} &u_0(t) + \sum_{i=1}^m \tau_i \left(\int_0^T g_i(s)u_0(s)ds - \alpha_i\right)g_i(t) \\ &-(\nu(T) - \nu(t)) = 0. \end{aligned} \tag{17}$$

This especially has to be true for $\nu = \nu_0$, which gives that

$$u_0(t) + \sum_{i=1}^m \tau_i \left(\int_0^T g_i(s)u_0(s)ds - \alpha_i\right)g_i(t) - C_t = 0, \tag{18}$$

whenever $\dot{x}_0(t) > 0$.

In (18), the integral terms do not depend on t while $g_i(t)$ is linear in t for $i = 1, \ldots, m$. This, combined with the fact that $\nu_0(T) - \nu_0(t) = C_t$ if $\dot{x}(t) > 0$, directly gives us that the optimal control, $u_0(t)$, has to be *piecewise linear*. It obviously changes at the waypoints, due to the shape of the g_i's, but it also changes if C_t changes, i.e. it changes if $\dot{x}_0(t) = 0$. It should be noted that if $\dot{x}_0(t) \equiv 0$ on an interval, $\nu_0(t)$ may change on the entire interval, but since $\dot{x}_0(t) \equiv 0$ we also have that $u_0(t) \equiv 0$ on the interior of this interval. But a zero function is also, of course, a linear function. *Thus we know that our optimal control is a piecewise linear function for all* $t \in [0, T]$. □

This is as far as we will push this *Lagrange-duality approach* in this paper.

3 Dynamic Programming

Based on the general properties of the solution the idea now is to formulate the monotone interpolation problem as a finite-dimensional programming problem that can be dealt with efficiently, and in the following paragraphs we show how this can be done for the double integrator defined in (10).

Since the original cost functional (2) can be divided into one interpolation part and one smoothing part, it seems natural to define the following *optimal value function* as

$$\begin{cases} \hat{S}_i(x_i, \dot{x}_i) = \\ \quad \min_{x_{i+1} \geq x_i, \dot{x}_{i+1} \geq 0} \{V_i(x_i, \dot{x}_i, x_{i+1}, \dot{x}_{i+1}) + \\ \qquad \hat{S}_{i+1}(x_{i+1}, \dot{x}_{i+1})\} \\ \qquad + \tau_i (x_i - \alpha_i)^2, \quad i = 0, \ldots, m-1 \\ \\ \hat{S}_m(x_m, \dot{x}_m) = \tau_m (x_m - \alpha_m)^2, \end{cases} \tag{19}$$

where $V_i(x_i, \dot{x}_i, x_{i+1}, \dot{x}_{i+1})$ is the cost for driving the system between $(x_i, \dot{x}_i)$ and $(x_{i+1}, \dot{x}_{i+1})$ with minimum energy control, while keeping the derivative of the output non-negative.

The optimal control problem thus becomes that of finding $\hat{S}_0(0, 0)$, where we in (19) simply let $\tau_0 = 0$ while α_0 can be any arbitrary number. This problem is equivalent to the original problem, and if $V_i(x_i, \dot{x}_i, x_{i+1}, \dot{x}_{i+1})$ could be uniquely determined it would correspond to finding the $2 \times m$ variables $x_1, \ldots, x_m, \dot{x}_1, \ldots, \dot{x}_m$, which is a *finite dimensional* parameterization of the original, infinite dimensional programming problem.

For this dynamic programming approach to work, our next task becomes that of determining the function $V_i(x_i, \dot{x}_i, x_{i+1}, \dot{x}_{i+1})$.

3.1 Two-Points Interpolation

Given the times t_i, t_{i+1}, the positions x_i, x_{i+1}, and the derivatives $\dot{x}_i, \dot{x}_{i+1}$, the question is the following: *How do we drive the system between* $(x_i, \dot{x}_i)$ *and* $(x_{i+1}, \dot{x}_{i+1})$, *with a piecewise linear control input that changes between different linear regimes only when* $\dot{x}(t) = 0$, *in such a way that* $\dot{x}(t) \geq 0 \; \forall t \in [t_i, t_{i+1}]$, *while minimizing the integral over the square of the control input?* Without loss of generality, we, for notational purposes, translate the system and rename the variables so that we want to produce a curve, defined on the time interval $[0, t_F]$, between $(0, \dot{x}_0)$ and $(x_F, \dot{x}_F)$.

Assumption 3.1

$$\dot{x}_0, \dot{x}_F \geq 0, \; x_F > 0, \; t_F > 0.$$

It should be noted that if $x_F = 0$, and either $\dot{x}_0 > 0$ or $\dot{x}_F > 0$, then $\dot{x}(t)$ can never be continuous. This case has to be excluded since we already demanded that our constraint space was $C[0, T]$. If, furthermore, $x_F = \dot{x}_0 = \dot{x}_F = 0$ then the optimal control is obviously given by $u \equiv 0$ on the entire interval.

One first observation is that the optimal solution to this *two-points interpolation problem* is to use standard cubic splines if that is possible, i.e. if $\dot{x}(t) \geq 0$ for all $t \in [0, t_F]$. In this well-studied case [6] we would simply have that

$$x(t) = \frac{1}{6} a t^3 + \frac{1}{2} b t^2 + \dot{x}_0 t, \tag{20}$$

where

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{6}{t_F^3} \begin{pmatrix} t_F(\dot{x}_0 + \dot{x}_F) - 2x_F \\ t_F x_F - 1/3 t_F^2 (2\dot{x}_0 + \dot{x}_F) \end{pmatrix}. \tag{21}$$

This solution corresponds to having $\nu(t) = \nu(t_{i+1}) \; \forall t \in [t_i, t_{i+1})$ in (14), and it gives the total cost

$$\begin{aligned} \mathcal{I}_1 &= \int_0^{t_F} (at + b)^2 dt \qquad (22) \\ &= 4 \frac{(\dot{x}_0 t_F^2 - 3x_F t_F)(\dot{x}_0 + \dot{x}_F) + 3x_F^2 + t_F^2 \dot{x}_F^2}{t_F^3}, \end{aligned}$$

where the subscript 1 denotes the fact that only one linear segment was used to compose the second derivative.

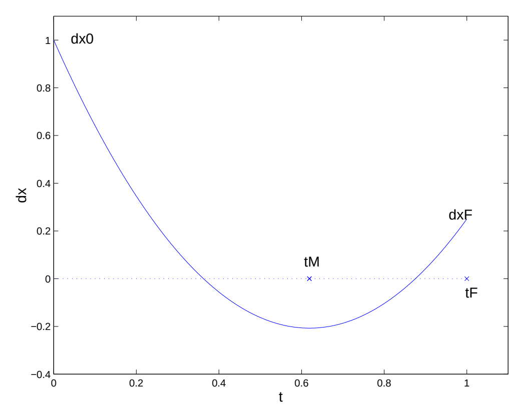

Figure 2: The case where a cubic spline can not be used if the derivative has to be non-negative. Plotted is the derivative that clearly intersects $\dot{x} = 0$.

However, not all curves can be produced by such a cubic spline if the curve has to be non-decreasing at all times. Given Assumption 3.1, the one case where we can not use a cubic spline can be seen in Figure 2, and we, from geometric considerations, get four different conditions that all need to hold for the derivative to be negative. These necessary and sufficient conditions for this are

$$\begin{array}{ll} (i) & a > 0 \\ (ii) & b < 0 \\ (iii) & \dot{x}(t_M) < 0 \\ (iv) & t_M < t_F, \end{array} \tag{23}$$

where a and b are defined in (20), and t_M is defined in Figure 2.

We can now state the following lemma.

Lemma 3.1 *Given Assumption 3.1, a standard cubic spline can be used to produce monotonously increasing curves if and only if*

$$x_F \geq \chi(t_F, \dot{x}_0, \dot{x}_F) = \frac{t_F}{3}(\dot{x}_0 + \dot{x}_F - \sqrt{\dot{x}_0 \dot{x}_F}). \tag{24}$$

Proof. The proof follows from simple algebraic manipulations. The definitions of a and b, together with Assumption 3.1, give that the first and the second conditions are equivalent to

$$x_F < \frac{1}{2} t_F(\dot{x}_0 + \dot{x}_F), \tag{25}$$

and

$$x_F < \frac{1}{3} t_F(2\dot{x}_0 + \dot{x}_F) \tag{26}$$

respectively.

The time at which the derivative of the curve obtains its extremum, as seen in Figure 2, is given by $at_M + b = 0$, which directly gives that the fourth condition is equivalent to

$$x_F < \frac{1}{3} t_F(\dot{x}_0 + 2\dot{x}_F). \tag{27}$$

Finally, the requirement that $\dot{x}(t_M) < 0$ gives us that

$$\dot{x}(t_M) = \dot{x}(-b/a) = \dot{x}_0 - \frac{b^2}{2a} < 0, \tag{28}$$

which can be shown to be equivalent to two different possible situations

$$x_F \quad < \quad \frac{1}{3} t_F(\dot{x}_0 + \dot{x}_F - \sqrt{\dot{x}_0 \dot{x}_F}) \tag{29}$$

$$x_F \quad > \quad \frac{1}{3} t_F(\dot{x}_0 + \dot{x}_F + \sqrt{\dot{x}_0 \dot{x}_F}). \tag{30}$$

The first of these two inequalities gives

$$x_F < \frac{1}{3} t_F(\dot{x}_0 + \dot{x}_F - \sqrt{\dot{x}_0 \dot{x}_F}) < \frac{1}{3} t_F(\dot{x}_0 + \dot{x}_F), \tag{31}$$

which is more restrictive than (25).

The other inequality gives that

$$x_F > \frac{1}{3} t_F(\dot{x}_F + \dot{x}_0 + \min\{\dot{x}_0, \dot{x}_F\}), \tag{32}$$

which either violates the second or the forth condition.

Thus (29) is the most restrictive constraint, and our lemma follows. Given Assumption 3.1, a monotone, cubic spline can be used if and only if $x_F \geq \chi(t_F, \dot{x}_0, \dot{x}_F)$. □

We now need to investigate what the optimal curve looks like in the case where we can not use standard, cubic splines.

3.2 Monotone Interpolation

Given two points such that $x_F < \chi(t_F, \dot{x}_0, \dot{x}_F)$, how should the interpolating curve be constructed so that the second derivative is piecewise linear, with switches only when $\dot{x}(t) = 0$? One first observation is that it is always possible to construct a path that consists of three segments that respects the interpolation constraint, and in what follows we will see that such a path also respects the monotonicity constraint.

The three interpolating segments are given by

$$u(t) = \begin{cases} a_1 t + b_1 & \text{if } 0 \leq t < t_1 \\ 0 & \text{if } t_1 \leq t < t_2 \\ a_2(t - t_2) + b_2 & \text{if } t_2 \leq t \leq t_F, \end{cases} \tag{33}$$

where

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \frac{6}{t_1^3} \begin{pmatrix} t_1 \dot{x}_0 - 2x_1 \\ t_1 x_1 - 2/3 t_1^2 \dot{x}_0 \end{pmatrix}$$

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \frac{6}{(t_F - t_2)^3} \begin{pmatrix} (t_F - t_2)\dot{x}_F - 2(x_F - x_1) \\ (t_F - t_2)(x_F - x_1) - 1/3(t_F - t_1)^2 \dot{x}_F \end{pmatrix}, \tag{34}$$

and where $x(t_1) = x(t_2) = x_1$ that, together with t_1 and t_2, is a parameter that needs to be determined.

Assumption 3.2

$$\dot{x}_0, \dot{x}_F, x_F, t_F > 0.$$

We need this assumption, which is stronger than Assumption 3.1, in the following paragraph but it should be noted that if $\dot{x}_0 = 0$ or $\dot{x}_F = 0$ we would then just let the first or the third segment of the curve be zero.

We now state the possibility of such a feasible three segment construction.

Lemma 3.2 *Given $(t_F, \dot{x}_0, x_F, \dot{x}_F)$ such that $x_F < \chi(t_F, \dot{x}_0, \dot{x}_F)$, then a feasible, monotone curve will be given by (33), given Assumption 3.2. Furthermore, the optimal t_1, t_2, and x_1 are given by*

$$\begin{cases} t_1 = 3\frac{x_1}{\dot{x}_0}, \\ t_2 = t_F - 3\frac{x_F - x_1}{\dot{x}_F}, \\ x_1 = \frac{\dot{x}_0^{3/2}}{(\dot{x}_0^{3/2} + \dot{x}_F^{3/2})} x_F. \end{cases} \tag{35}$$

Proof. The proof is constructive and is based on showing that with this type of construction (33), the optimal choice of t_1, t_2, x_1 gives a feasible curve.

If we let $u(t)$ be given by (33), while minimizing the corresponding cost integral $(= \mathcal{I}_3)$, where the subscript 3 denotes the fact that we use three linear segments to compose our second derivative, and solve

$$\frac{\partial \mathcal{I}_3}{\partial t_1} = \frac{\partial \mathcal{I}_3}{\partial t_2} = \frac{\partial \mathcal{I}_3}{\partial x_1} = 0, \tag{36}$$

we directly get that t_1, t_2 and x_1 are given by (35). If we insert these in $\mathcal{I}_3$ we get

$$\begin{aligned}\mathcal{I}_3 &= \int_0^{t_1} (a_1 t + b_1)^2 dt + \int_{t_2}^{t_F} (a_2(t - t_2) + b_2)^2 dt \\ &= \frac{4(\dot{x}_F^{3/2} + \dot{x}_0^{3/2})^2}{9x_F}. \qquad (37)\end{aligned}$$

What we now need to show is that with this choice of parameters, the resulting curve is always feasible with respect to the non-negativity constraint on the derivative. At the same time we also need to show that $0 \le t_1 \le t_2 \le t_F$, and $x_1 \in [0, x_F]$.

By using

$$x_1 = \frac{\dot{x}_0^{3/2}}{(\dot{x}_0^{3/2} + \dot{x}_F^{3/2})} x_F,$$

we directly get that $x_1 \in (0, x_F)$. Furthermore, $t_1 = 3x_1/\dot{x}_0$ gives that $t_1 > 0$ since $\dot{x}_0$ and x_1 are both positive. However, we also have

$$\begin{aligned}t_1 &= 3\frac{x_1}{\dot{x}_0} = 3\frac{\dot{x}_0^{1/2}}{(\dot{x}_0^{3/2} + \dot{x}_F^{3/2})} x_F \\ &< t_F \frac{\dot{x}_0^{3/2} - \dot{x}_0 \dot{x}_F^{1/2} + \dot{x}_0^{1/2} \dot{x}_F}{(\dot{x}_0^{3/2} + \dot{x}_F^{3/2})} \\ &= t_F \frac{\dot{x}_0^{3/2} + \dot{x}_F^{3/2} + \dot{x}_F^{1/2}(-\dot{x}_0 - \dot{x}_F + \dot{x}_0^{1/2}\dot{x}_F^{1/2})}{(\dot{x}_0^{3/2} + \dot{x}_F^{3/2})},\end{aligned}$$

where we used the fact that $x_F < \chi(t_F, \dot{x}_0, \dot{x}_F)$.

But since $x_F < t_F/3(\dot{x}_0 + \dot{x}_F - \dot{x}_0^{1/2}\dot{x}_F^{1/2})$, we have that $-\dot{x}_0 - \dot{x}_F + \dot{x}_0^{1/2}\dot{x}_F^{1/2} < 0$, and thus $t_1 < t_F$. We therefore conclude that $t_1 \in (0, t_F)$.

We now have that since $x_1 < x_F$, we directly from (33) get that $t_2 < t_F$, and we, furthermore, need to show that $t_2 - t_1 > 0$. We have

$$\begin{aligned}t_2 - t_1 &= t_F - 3\frac{x_F - x_1}{\dot{x}_F} - 3\frac{x_1}{\dot{x}_0} \\ &= t_F - s\frac{(\dot{x}_0^{1/2} + \dot{x}_F^{1/2})x_F}{(\dot{x}_0^{3/2} + \dot{x}_F^{3/2})} \\ &> t_F - \frac{(\dot{x}_0^{1/2} + \dot{x}_F^{1/2})(\dot{x}_0 + \dot{x}_F - \dot{x}_0^{1/2}\dot{x}_F^{1/2})}{(\dot{x}_0^{3/2} + \dot{x}_F^{3/2})} t_F \\ &= t_F - \frac{(\dot{x}_0^{3/2} + \dot{x}_F^{3/2})}{(\dot{x}_0^{3/2} + \dot{x}_F^{3/2})} t_F = 0.\end{aligned}$$

Thus we have shown that $t_2 \in (t_1, t_F)$.

What remains to be shown is that we also, with this choice of parameters, have positive derivatives at all times. Given Assumption 3.2, what we must show is that the left and right second derivative at the interpolation point $(x_1, 0)^T$, have the right sign, i.e.

$$\begin{aligned}(i) \quad & a_1 t_1 + b_1 \le 0 \\ (ii) \quad & b_2 \ge 0.\end{aligned} \qquad (38)$$

But straight forward calculations give that, in fact, we have $a_1 t_1 + b_1 = b_2 = 0$, and the lemma follows. □

We can thus construct a feasible path, as seen in Figure 3, by using three segments whose second derivatives are linear, that is feasible with respect to the non-negativity constraint on the derivative.

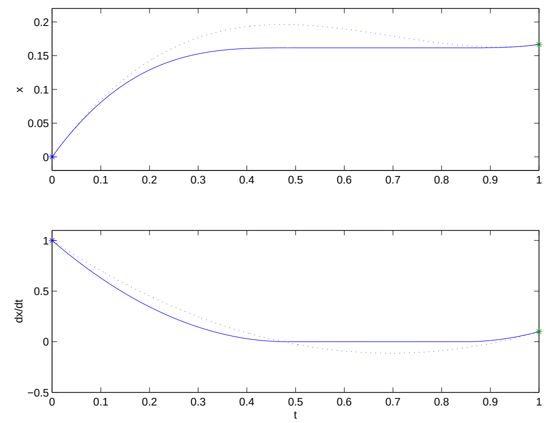

Figure 3: The dotted line corresponds to a standard, cubic spline, while the solid line shows the three segment construction from Lemma 3.2. Depicted is the position and the velocity.

Theorem 3.1 (Monotone Interpolation) *Given Assumption 3.1, the optimal control that drives the path between* $(0, \dot{x}_0)$ *and* $(x_F, \dot{x}_F)$ *is given by (20) if* $x_F \ge \chi(t_F, \dot{x}_0, \dot{x}_F)$ *and by (33) otherwise.*

Proof. The first part of the theorem is obviously true. If we can construct a standard, cubic spline, then this is optimal. However, what we need to show is that when $x_F < \chi(t_F, \dot{x}_0, \dot{x}_F)$ the path given by (33) is in fact optimal.

The cost for using a path given by (33) is

$$\begin{aligned}\mathcal{I}_3 &= \int_0^{t_1} (a_1 t + b_1)^2 dt + \int_{t_2}^{t_F} (a_2(t - t_2) + b_2)^2 dt \\ &= \frac{4(\dot{x}_F^{3/2} + \dot{x}_0^{3/2})^2}{9x_F},\end{aligned}$$

as seen in (37). We now add another, arbitrary segment, as seen in Figure 4, to the path as

$$u(t) = \begin{cases} a_1 t + b_1 & \text{if } 0 \le t < t_1 \\ 0 & \text{if } t_1 \le t < t_3 \\ a_3(t - t_3) + b_3 & \text{if } t_3 \le t < t_4 \\ 0 & \text{if } t_4 \le t < t_2 \\ a_2(t - t_2) + b_2 & \text{if } t_2 \le t \le t_F, \end{cases} \qquad (39)$$

where $0 < t_1 \le t_3 \le t_4 \le t_2 < t_F$. Furthermore, t_3, t_4, and $x_2 = x(t_4)$ (see Figure 4) are chosen arbitrarily while the old variables, t_1, t_2 and $x_1 = x(t_1)$, are defined to be optimal with respect to the new, translated end-conditions that the extra segments give rise to.

After some straight forward calculations, we get that the cost for this new path is

$$\mathcal{I}_5 = \frac{4(\dot{x}_F^{3/2} + \dot{x}_0^{3/2})^2}{9(x_F - x_2)} + \frac{12(x_2 - x_1)^2}{(t_4 - t_3)^3}, \qquad (40)$$

where the subscript 5 denotes the fact that we are now using five linear segments to compose our second derivate. It can be seen that we minimize $\mathcal{I}_5$ if we let $x_2 = x_1$ and make $t_4 - t_3$ as large as possible. This corresponds to letting $t_3 = t_1$ and $t_4 = t_2$, which gives us the old solution from Lemma 3.2, defined by (33). □

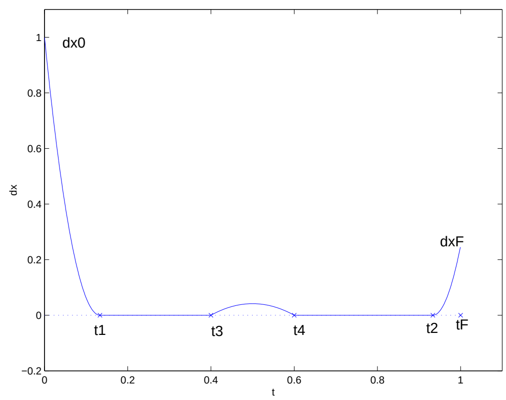

Figure 4: Two extra segments are added to the produced path. Depicted is the derivative of the curve.

3.3 Monotone Smoothing Splines

We now have a way of producing the optimal, monotone path between two points, while controlling the acceleration directly. We are thus ready to formulate the transition cost function in (19), $V_i(x_i, \dot{x}_i, x_{i+1}, \dot{x}_{i+1})$, that defines the cost for driving the system between $(x_i, \dot{x}_i)$ and $(x_{i+1}, \dot{x}_{i+1})$, with minimum energy, while keeping the derivative non-negative.

Based on Theorem 3.1 we, given Assumption 3.1, have that[2]

$$V_i(x_i, \dot{x}_i, x_{i+1}, \dot{x}_{i+1}) =$$

$$\begin{cases} 4\frac{\dot{x}_i(t_{i+1}-t_i)^2 - 3(x_{i+1}-x_i)(t_{i+1}-t_i)(\dot{x}_i+\dot{x}_{i+1})}{(t_{i+1}-t_i)^3} \\ +4\frac{3(x_{i+1}-x_i)^2+(t_{i+1}-t_i)^2\dot{x}_{i+1}^2}{(t_{i+1}-t_i)^3}, \\ \quad \text{if } x_{i+1} - x_i \geq \chi(t_{i+1} - t_i, \dot{x}_i, \dot{x}_{i+1}) \\ \frac{4(\dot{x}_{i+1}^{3/2}+\dot{x}_i^{3/2})^2}{9(x_{i+1}-x_i)}, \quad \text{if } x_{i+1} - x_i < \chi(t_{i+1} - t_i, \dot{x}_i, \dot{x}_{i+1}), \end{cases}$$

where $t_0 = x_0 = \dot{x}_0 = 0$.

[2] If $x_{i+1} - x_i = \dot{x}_i = \dot{x}_{i+1} = 0$ then the optimal control is obviously zero, meaning that $V_i(x_i, \dot{x}_i, x_{i+1}, \dot{x}_{i+1}) = 0$.

If we use this cost in the dynamic programming algorithm, formulated in (19), we get the results displayed in Figures 5–6, which shows that our approach does not only work in theory, but also in practice.

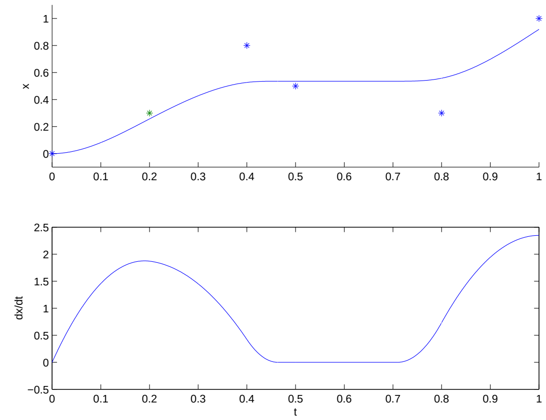

Figure 5: Monotone smoothing splines with $\tau_i = 1000, i = 1, \ldots, 5$.

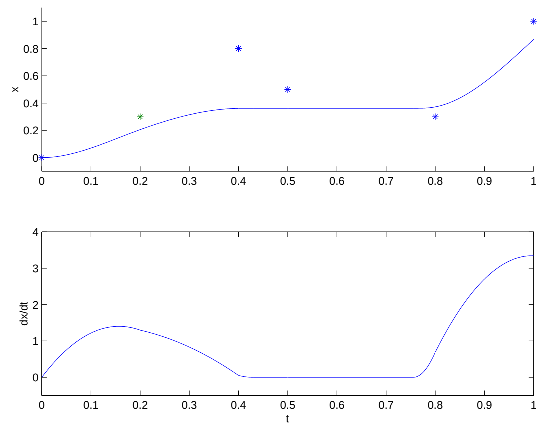

Figure 6: Monotone smoothing splines with $\tau_4 = 10\tau_i, i \neq 4$ (with $t_4 = 0.8$), resulting in a different curve compared to that in Figure 5 where equal importance is given to all of the waypoints.

4 Conclusions

In this paper we propose and analyze the optimal solution to the problem of driving the output of a linear control system close to given waypoints. This is done while the state space is constrained by an infinite dimensional non-negativity constraint on the derivative of the output spline function.

For the double integrator case, this problem is completely solved by exploiting a finite parameterization of the problem, resulting in a dynamic programming formulation that can be solved analytically.

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