Topology-induced connectivity bounds in leader-follower networks

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Abstract

In this paper we derive a set of constraints that are sufficient to guarantee maintained connectivity in a leader-follower multi-agent network with proximity based communication topology. In the scenario we consider, only the leaders are aware of the global mission, which is to converge to a known destination point. Thus, the followers need to stay in contact with the group of leaders in order to reach the goal. In the paper we show that we can maintain the initial network structure, and thereby connectivity, by setting up bounds on the ratio of leaders-to-followers and on the magnitude of the goal attraction force experienced by the leaders. The results are first established for an initially complete communication graph and then extended to an incomplete graph. The results are illustrated by computer simulations.

1 Introduction

Distributed control of networked systems is an important issue in a number of applications, including multi-agent robotics (Jr. et al. (2004), Loizou and Kyriakopoulos (2008), Dimarogonas and Kyriakopoulos (2008), Arsie and Frazzoli (2007)), networked sensor and health maintenance (Mehyar et al. (2007), Xiao and Boyd (2003), Martinez et al. (2007)) and formation control (Jadbabaie et al. (2003), Olfati-Saber (2006), Ji and Egerstedt (2007), Arcak (2007)) just to name a few. One way in which the user can interact with such systems is through so-called leader (or anchor) agents, whose dynamics need not conform to those of the non-leader (follower) agents. In this paper we study such systems, i.e., systems where a selected subset of the agents are following a task-level controller encoding the transport of the network from one location to another. The rest of the agents are executing a simple local interaction-based control strategy for keeping the team together. The reason why such a heterogeneous network configuration is desirable is that it frees up resources by only insisting on a select subset of agents being able to tell global positions and/or positions relative to particular landmarks, thus limiting the required sensor load of the remaining agents. This was for instance the case in Smith et al. (2009), in which a collection of mobile sensor nodes where to traverse long distances before assembling the desired sensing configuration.

While many issues regarding controllability and stability of leader-follower networks have been addressed recently in, for instance, Tanner (2004), Rahmani et al. (2008), the issue of how the ratio of leaders-to-followers affect connectivity is a novel topic. In Couzin et al. (2005), the authors presented extended numerical results on the subject, but in this paper we treat the problem from an analytical standpoint. Moreover, a lot of results have appeared recently regarding maintaining connectivity in networks of homogeneous agents *directly* through the control law. These include Ji and Egerstedt (2007), Schuresko and Cortes (2007), DeGennaro and Jadbabaie (2006), Zavlanos and Pappas (2007), Notarstefano et al. (2006). On the contrary, in this paper we provide indirect metrics to treat the problem in a leader-follower network. These metrics are the leader to follower ratio and the parameters of the goal attraction function. It is well known that a general connectivity analysis of an arbitrary network is extremely complex, but in the current paper we show how network structure can be used to obtain interesting results for some special classes of networks. The approach is demonstrated on two classes of networks and for these networks inter-

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esting theoretical results are obtained. For instance, we show the somewhat counterintuitive result that, from a stability point of view, more leaders are not always better. The work presented in this paper is an extension of the work presented in Dimarogonas et al. (2008).

The paper is organized as follows: in Section 2 we introduce the considered framework and present the approach. In Sections 3 and 4 we demonstrate the proposed approach on two examples of connected networks and in Section 5 the theoretical results from Sections 3 and 4 are illustrated in simulations. Finally, in Section 6, the results of the paper are summarized.

2 System and Problem Statement

Consider N agents evolving in $\mathbb{R}^n$. We use single integrator agents whose motions obey the model:

$$\dot{x}_i = u_i, i \in \mathcal{N} = [1, \ldots, N]$$

We assume that the agents belong either to the subset of leaders, $\mathcal{N}^l$, or to the subset of followers, $\mathcal{N}^f$, where $\mathcal{N}^l \cup \mathcal{N}^f = \mathcal{N}$, $\mathcal{N}^l \cap \mathcal{N}^f = \emptyset$ and the number of agents in each set is given by $|\mathcal{N}^f| = N_f$ and $|\mathcal{N}^l| = N_l$ respectively. Due to shortcomings of the sensors, each agent has a limited sensing zone of radius $\Delta > 0$. At any given time, the set of agents located within the sensing zone of agent $i \in \mathcal{N}$ are referred to as the *neighbors* of agent i, $\mathcal{N}_i = \{j \in \mathcal{N} : |x_i - x_j| \leq \Delta\}$. Each agent has knowledge of the relative coordinates of all its neighbors, but can not detect or communicate with agents outside its sensing zone. To keep track of the active communication links we introduce a *communication graph*, $\mathcal{G} = \{V, E\}$, that describes the group topology. $\mathcal{G} = \{V, E\}$ consists of a set of vertices, $V = \{1, ..., N\}$, representing the team members, and a set of edges, $E = \{(i, j) \in V \times V | i \in \mathcal{N}_j\}$ representing the active inter-agent communication links. Note that both $\mathcal{N}_i$ and $\mathcal{G}$ are time-varying.

In the application considered in the current paper, the objective of the agents is to reach a common goal defined by a set of position coordinates, d. Unlike the leaders, the follower agents are not aware of the position of the goal, so in order to reach d they must stay connected, either directly or indirectly, with the set of leaders. In this paper we focus on the case where the communication graph has some well-defined structure. We present a framework for connectivity analysis and show how it can be used to guarantee connectivity of the time-varying communication graph $\mathcal{G}(t)$.

The key to the connectivity analysis is the observation that, under some assumptions on differentiability and boundedness of motion, two initially connected agents are guaranteed to remain connected if the time derivative of the distance between the agents is negative in the critical case where the inter-agent distance is equal to Δ. We start by introducing a notation for the distance between two arbitrary agents i and j. Let

$$\delta_{ij} = \delta_{ji} = |x_i - x_j| = \sqrt{(x_i - x_j)^T (x_i - x_j)} \geq 0.$$

Since we are considering a physical system we can assume that $x_k(t)$ is a continuous function for any agent $k \in \mathcal{N}$. The time derivative $\dot{\delta}_{ij}$ is not directly defined when $\delta_{ij} = 0$ so here we shall instead consider the time derivative of δ_{ij}^2,

$$\frac{d\delta_{ij}^2}{dt} = 2\delta_{ij}\dot{\delta}_{ij} = 2(x_i - x_j)^T (\dot{x}_i - \dot{x}_j), \qquad (1)$$

which has the same sign as $\dot{\delta}_{ij}$ for all $\delta_{ij} > 0$ but is defined on all of $\mathbb{R}^n$. Hence, a sufficient condition for agent i and j to remain connected is $\frac{d\delta_{ij}^2}{dt} \leq 0$ when $\delta_{ij} = \Delta$. Obviously, $\dot{\delta}_{ij}$ depends on the motion of agents i and j.The dynamics of the followers are given by the standard Laplacian consensus equation, meaning that each agent moves in the direction of the average position of its neighbors. For an arbitrary follower agent $i \in \mathcal{N}^f$ we have:

$$\dot{x}_i = -\sum_{k \in \mathcal{N}_i} (x_i - x_k) = -N_i x_i + \sum_{k \in \mathcal{N}_i} x_k. \qquad (2)$$

The dynamics of the leaders are also based on the Laplacian consensus equation, but include an additional goal attraction term which aims at dragging the team to the pre-defined goal position $x = d$. For agent i, define $\delta_i = |d - x_i|$. The dynamics for an arbitrary leader agent $i \in \mathcal{N}^l$ are thus given by

$$\dot{x}_i = -N_i x_i + \sum_{k \in \mathcal{N}_i} x_k + F(x_i, d), \qquad (3)$$

where $F(x_i, d)$ is the goal attraction function

$$F(x_i, d) = \begin{cases} f(\delta_i) \frac{d - x_i}{\delta_i} & \delta_i > 0 \\ 0 & \delta_i = 0. \end{cases} \qquad (4)$$

At any given position $x_i \neq d$, the direction of $F(x_i, d)$ is towards the goal and the magnitude is decided by the continuous scalar function $f(\delta_i) \geq 0$. $f(\delta_i)$ is depending only on agent i's distance to d and can be designed to suit the application. The continuity of $F(x_i, d)$ is guaranteed by requiring $f(0) = 0$ and $\lim_{\delta^+ \to 0} \frac{f(\delta)}{\delta} < \infty$, such that $\lim_{x^+ \to d} F(x, d) = F(d, d) = 0$. A simple example of a possible goal attraction function that satisfies these requirements is $f(\delta) = \delta$.

Let us now state a result that is later needed to prove connectivity. The following Lemma guarantees the boundedness of solutions of the closed-loop system:

Lemma 1 *Let $\mathcal{G}$ be a nonempty graph consisting of followers and leaders with dynamics decided by (2) and (3).*

Define Ω to be the convex hull of the agents in $\mathcal{G}$ and the goal position d, $\Omega = \text{Co}(\mathcal{G} \cup d)$, and let Ω_0 denote the convex hull at time $t = 0$. Now assume that none of the agents in $\mathcal{G}$ are connected to any agents $x_k, k \notin \mathcal{G}$, and that there exists a constant f_{max} such that $f(|d - x|) \leq f_{max} < \infty$ for all $x \in \Omega_0$. Then the trajectories of all agents in $\mathcal{G}$ will remain within Ω_0 as $t \to \infty$.

Proof: We will show that for an arbitrary agent $i \in \mathcal{G}$, positioned on the boundary of Ω, the motion is either on the boundary of Ω or pointing inside the polytope Ω. To do this we follow the outline in Ji et al. (2008), where a similar problem is considered. If $i \in \mathcal{N}^f$ the motion is given by (2). If $N_i = 0$ the agent will not move at all and the proof is trivial. Now consider the case $N_i > 0$. By setting $\alpha = N_i^{-1}$ and rearranging the terms we can show:

$$\alpha \dot{x}_i = -x_i + \sum_{k \in \mathcal{N}_i} \frac{x_k}{N_i}.$$

Apparently the motion of follower i is directed towards the mass center of the subgraph $\mathcal{N}_i \subseteq \mathcal{G}$, which, thanks to convexity, is known to lie either on the boundary or in the interior of Ω. From the definition of convexity we can also conclude that the motion of follower i must lie within Ω. Now assume that $i \in \mathcal{N}^l$. The motion is now given by (3) and (4). If $x_i = d$ the analysis is equivalent to the case where $i \in \mathcal{N}^f$. If $N_i = 0$ the agent will, depending on the magnitude of $f(\delta_i)$, either not move at all, or move directly towards the goal located at d. Since d is in Ω and since Ω is convex, we see that in either case agent i will remain in Ω. If $N_i > 0$ and $x_i \neq d$ we define $\beta = (N_i + f(\delta_i)/\delta_i)^{-1}$. Then we get:

$$\beta \dot{x}_i = -x_i + \beta\big(N_i \sum_{k \in \mathcal{N}_i} \frac{x_k}{N_i} + \frac{f(\delta_i)}{\delta_i} d\big).$$

The motion of agent i is directed towards a convex combination of the mass center of the subgraph $\mathcal{N}_i \subseteq \mathcal{G}$ and the goal d. By definition, this convex combination lies within the convex hull of $\mathcal{G} \cup d$, and therefore, by the convexity of Ω, the motion of agent i is within Ω. Since the motion of any agent on the boundary of Ω is either on the boundary of Ω or directed into the interior of Ω, we can conclude that no agent will ever enter outside the convex hull defined by the initial positions of the agents and the goal d. Hence, Ω_0 is an invariant set. $\diamondsuit$

The next Theorem states that the closed-loop system converges to the goal position d if the communication graph remains connected:

Theorem 2 *Let the closed loop dynamics be given by (2) and (3). Let $x(0) \in \Omega$ and assume that the communication graph $\mathcal{G}(t)$ remains connected. Then, $\lim_{t \to \infty} x_i(t) = d$ for all $i \in \mathcal{N}$.*

Proof: Equations (2) and (3) are written in stack vector form as $\dot{x} = -(L \otimes I_2)x + \bar{F}(x, d)$, where $x = [x_1, \ldots, x_N]^T$, L is the Laplacian of $\mathcal{G}(t)$ and $\bar{F}(x, d) = [\bar{F}_1(x, d), \ldots, \bar{F}_N(x, d)]^T$ is a vector in which $\bar{F}_i(x, d) = 0$, if $i \in \mathcal{N}^f$ and $\bar{F}_i(x, d) = F(x_i, d)$, if $i \in \mathcal{N}^l$. Let $\bar{d} = [d, ..., d]^T$ and define a new set of coordinates $z = x - \bar{d}$. It follows that $\dot{z} = -(L \otimes I_2)x + \bar{F}(x, d) = -(L \otimes I_2)(z + \bar{d}) + \bar{F}(x, d) = -(L \otimes I_2)z + \bar{F}(z, 0)$. By taking $V = 0.5 z^T z$ as a candidate Lyapunov function we get $\dot{V} = z^T \dot{z} = -z^T (L \otimes I_2) z + z^T \bar{F}(z, 0) = -z^T (L \otimes I_2) z - \sum_{i \in \mathcal{N}^l \& x_i \neq d} \delta_i f(\delta_i)$. Both terms are negative semidefinite. Due to the eigen-properties of L (see for instance Godsil and Royle (2001)), the first term, $-z^T (L \otimes I_2) z$, is strictly negative unless $z = 0$ or $x_1 = x_2 = \cdots = x_N$. The second term is strictly negative unless $x_i = d, \forall i \in \mathcal{N}^l$. Thus, $\dot{V} \leq 0$, and $\dot{V} = 0$ only for $z = 0$. It follows from Lyapunovs results on stability that $z_i \to 0$, or equivalently $x_i \to d$, for all $i \in \mathcal{N}$. $\diamondsuit$

By virtue of the above Theorem, agents converge to the desired goal configuration as long as the communication graph remains connected over time. This result will be used in the following sections.

3 Complete Graph Case

In this section, we assume that all agents are initially within the sensing zone of one another. Hence, the initial graph $\mathcal{G}(t)$ is complete and of course, connected. The dynamics for followers and leader agents are given by (2) and (3) respectively, where the neighboring set for both leader and follower agents is the complete set of all agents, $\mathcal{N}$. In the sequel, we derive sufficient conditions for the graph to remain complete as the leaders drag all followers towards the desired target point d.

Follower-follower connections: Consider the distance between two arbitrary followers $i, j \in \mathcal{N}^f$. (2) inserted in (1) give

$$\frac{d\delta_{ij}^2}{dt} = 2(x_i - x_j)^T (\dot{x}_i - \dot{x}_j) = -2N\delta_{ij}^2.$$

We see that δ_{ij}^2 is exponentially decreasing, $\delta_{ij}^2 \to 0$, so we can conclude that if the the two followers are initially within each others sensing zones, they will remain connected as $t \to \infty$ given that all other inter-agent links hold.

Leader-leader connections: Now consider instead the distance between two arbitrary leaders $i, j \in \mathcal{N}^l$. It can easily be shown that in the special case where one of the considered leaders is located at d (where $F(x, d) = 0$), δ_{ij} is decreasing. In the general case where $x_i \neq d, x_j \neq d$, (1) and (3) give after some simplification,

$$\frac{d\delta_{ij}^2}{dt} \leq -2N\delta_{ij}^2 - 2\big(f(\delta_i) - f(\delta_j)\big)(\delta_i - \delta_j). \quad (5)$$

The inequality in (5) is derived by considering the worst-case scenario where $(d - x_i)^T (d - x_j) = \delta_i \delta_j$. The first term on the right hand side of (5) is always negative. The second term is negative regardless of δ_i and δ_j if we require $f(\delta)$ to be a monotonically increasing function,

$$f'(\delta) \geq 0, \qquad \forall \delta \geq 0. \tag{6}$$

In other words, assuming that all other inter-agent links hold, any two leaders that are initially connected will remain so if condition (6) is satisfied. Note that this is a conservative bound. (The case $x_i = x_j = d$ is not considered since we are primarily interested in the limit case $\delta_{ij} = \Delta$.)

Leader-follower connections: For the case $i \in \mathcal{N}^f, j \in \mathcal{N}^l$, we have

$$\frac{d\delta_{ij}^2}{dt} = -2N\delta_{ij}^2 - 2(x_i - x_j)^T F(x_j, d).$$

From the equation above it is easy to see that if the goal attraction term is zero, as is the case when $x_j = d$, δ_{ij} will be decreasing. If $x_j \neq d$ we can use $-\delta_{ij}\delta_j \leq (x_i - x_j)^T(d - x_j) \leq \delta_{ij}\delta_j$ to obtain

$$\frac{d\delta_{ij}^2}{dt} \leq -2N\delta_{ij}^2 + 2f(\delta_j)\delta_{ij}.$$

Apparently $f(\delta)$ must be bounded above such that in the critical case, $\delta_{ij} = \Delta$, the attraction to goal can not be stronger than the inter-agent attraction. We will now make use of Lemma 1. Let f_{max} be the largest value that $f(\delta_j)$ assumes within the convex hull defined by the agents initial positions and the goal position d. Then $|f(\delta_j)| \leq f_{max}$ for all $t \geq 0$ and

$$f_{max} \leq N\Delta \tag{7}$$

is a sufficient condition to guarantee that $\frac{d\delta_{ij}^2}{dt} \leq 0$ if $\delta_{ij} = \Delta$, *i.e.* that any initially connected pair consisting of one leader and one follower will remain connected at all times. Possible choices of $f(\delta)$ are discussed in Section 5.

Theorem 3 *Let the closed loop dynamics of the system be given by (2) and (3), where d represents the coordinates of the goal. Assume that the communication graph $\mathcal{G}(t)$ is initially complete and that both (6) and (7) hold. Then, $\mathcal{G}(t)$ remains complete for all $t \geq 0$ and, by Lemma 1, all agents will converge to d.*

4 The incomplete graph case

In this section, we analyze a special case of incomplete graphs. It is still assumed that both the subset of leaders and the subset of followers initially make up complete graphs, but it is no longer assumed that all followers are connected to all leaders. In order to describe this scenario we introduce some additional notation. Let $\mathcal{N}_i^f = \mathcal{N}_i \cap \mathcal{N}^f$ and $\mathcal{N}_i^l = \mathcal{N}_i \cap \mathcal{N}^l$ be the subsets of agent i's neighbors that belong to the group of followers and the group of leaders respectively, $|\mathcal{N}_i^f| = N_{fi}$, $|\mathcal{N}_i^l| = N_{li}$. Thus, for an arbitrary follower $i \in \mathcal{N}^f$ in the incomplete graph, $\mathcal{N}_i^f = \mathcal{N}^f$ while $\mathcal{N}_i^l \subseteq \mathcal{N}^l$. For an arbitrary leader $j \in \mathcal{N}^l$ we instead have $\mathcal{N}_j^l = \mathcal{N}^l$, $\mathcal{N}_j^f \subseteq \mathcal{N}^f$. Using this notation, the dynamics of follower i are described by

$$\dot{x}_i = -\sum_{k \in \mathcal{N}^f}(x_i - x_k) - \sum_{k \in \mathcal{N}_i^l}(x_i - x_k), \tag{8}$$

while the dynamics of leader j are given by

$$\dot{x}_j = -\sum_{k \in \mathcal{N}^l}(x_j - x_k) - \sum_{k \in \mathcal{N}_j^f}(x_j - x_k) + F(x_j, d). \tag{9}$$

The computations in this section follow the outline in Section 3.

Follower-follower connections: Similar to the previous section, we start by considering the connection between two arbitrary followers $i, j \in \mathcal{N}^f$. Both followers have links to all other follower agents and possibly, but not necessarily, links to some or all of the leader agents. The sets $\mathcal{N}_i^l$ and $\mathcal{N}_j^l$ may be disjunct or overlapping. Define $\mathcal{N}_c^l = \mathcal{N}_i^l \cap \mathcal{N}_j^l \subseteq \mathcal{N}^l$ to be the set of leaders that followers i and j have in common, $|\mathcal{N}_c^l| = N_{lc}$. Note that if $k \in \mathcal{N}_i^l$ then $|x_i - x_k| \leq \Delta$. For the two followers we get

$$\frac{d\delta_{ij}^2}{dt} \leq -2(N_f + N_{lc})\delta_{ij}^2 + 2(N_{li} + N_{lj} - 2N_{lc})\Delta\delta_{ij}.$$

The inequality above is derived from mathematical constraints and is valid even though physical constraints may sometimes make it impossible to obtain strict equality. Note also that $(N_{li} + N_{lj} - 2N_{lc}) \geq 0$. Requiring $\frac{d\delta_{ij}^2}{dt} \leq 0$ when $\delta_{ij} = \Delta$ leads to the constraint $N_{li} + N_{lj} \leq N_f + 3N_{lc}$, which is satisfied for all follower-follower connections, regardless of the topology, for every graph that has

$$N_f \geq N_l. \tag{10}$$

Leader-leader connections: Let us now consider the connection between two arbitrary leaders $i, j \in \mathcal{N}^l$. Following the notation in the follower-follower case, we let $\mathcal{N}_c^f = \mathcal{N}_i^f \cap \mathcal{N}_j^f \subseteq \mathcal{N}^f$ be the set of followers that i and j have in common, $|\mathcal{N}_c^f| = N_{fc}$. We get

$$\begin{aligned}\frac{d\delta_{ij}^2}{dt} \leq & -2(N_l + N_{fc})\delta_{ij}^2 + 2(N_{fi} + N_{fj} - 2N_{fc})\Delta\delta_{ij} \\ & +2(x_i - x_j)^T\Big[F(x_i, d) - F(x_j, d)\Big]. \end{aligned} \tag{11}$$

In a network where the follower agents are in majority, the attraction exerted by the group of followers may be large enough to cause a link between two leader agents to break. For this reason, the function $f(\delta)$ must be designed such that the net effect of the goal attraction on a pair of leader agents is a term that aims at bringing the two agents closer together. We shall investigate the case $\delta_{ij} = \Delta$ closer. Define α to be the angle between j and d, as seen by i (see Figure 1). Then, for

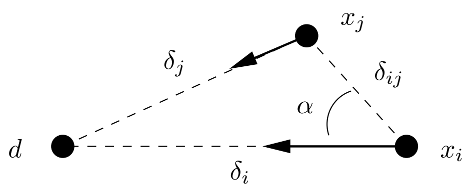


Fig. 1. Analysis of leader-leader stability in an incomplete network.

an arbitrary choice of x_i we have $x_j = x_i + \Delta\bar{e}$, where $\bar{e} = \frac{x_j - x_i}{\Delta}$. The dependence between δ_i and δ_j follows from the law of cosines, such that for a given α we get $\delta_j = \sqrt{\Delta^2 + \delta_i^2 - 2\Delta\delta_i \cos\alpha}$. Without loss of generality we can assume that $\delta_i \geq \delta_j$, and that consequently $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ and $\cos\alpha \geq 0$. For the moment we also assume that $\delta_j > 0$. Combining $\delta_{ij} = \Delta$ with (11) gives

$$\frac{d\delta_{ij}^2}{dt} \leq 2(N_f - N_l)\Delta^2 + \underbrace{2(x_i - x_j)^T \Big[F(x_i, d) - F(x_j, d)\Big]}_{\gamma}$$

and after inserting $x_j = x_i + \Delta\bar{e}$, γ can be simplified as

$$\gamma = -2\Delta\Big(\frac{f(\delta_i)}{\delta_i} - \frac{f(\delta_j)}{\delta_j}\Big)\delta_i \cos\alpha - 2\frac{f(\delta_j)}{\delta_j}\Delta^2.$$

Requiring $\frac{d\delta_{ij}^2}{dt} \leq 0$ leads to the following condition for connectivity

$$\underbrace{\Big(\frac{f(\delta_i)}{\delta_i} - \frac{f(\delta_j)}{\delta_j}\Big)}_{(*)}\delta_i \cos\alpha + \frac{f(\delta_j)}{\delta_j}\Delta \geq (N_f - N_l)\Delta.$$

From condition (10) it follows that the right hand side of the equation is positive. Since $f(\delta) \geq 0$, the only term in the above expression that may shift sign is $(*)$. The sign and magnitude of $(*)$ depends both on the characteristics of $f(\delta)$ and on the values of δ_i and δ_j in a way that complicates analysis. Recall that, by assumption, $\delta_i \geq \delta_j$. Also it is known that $f(\delta) \geq 0$ and that $f(0) = 0$. Under these conditions $(*) \geq 0, \forall \delta_i, \delta_j$, is equivalent to $f(\delta)$ being a convex function. Due to page limitations we here confine the analysis to the case $f''(\delta) \geq 0$, which is independent of the initial positions of the agents. Then the connectivity condition is satisfied if

$$\frac{f(\delta_j)}{\delta_j} \geq N_f - N_l. \tag{12}$$

By using l'Hôpitals rule, a lower bound for $\frac{f(\delta)}{\delta}$ can be computed as $\lim_{\delta + \to 0} \frac{f(\delta)}{\delta} = f'(0)$. Apparently (12) will be satisfied as long as

$$f'(0) \geq N_f - N_l. \tag{13}$$

Note that $N_f - N_l \geq 0$. In the special case where we have $\delta_j = 0$, the condition corresponding to (12) becomes $\frac{f(\Delta)}{\Delta} \geq N_f - N_l$. This is obviously satisfied for all convex goal-attraction functions that satisfy (13).

Remark 1 If $f(\delta)$ is not convex, it is difficult to analytically obtain constraints on $f(\delta)$ that guarantee that any leader-leader connection is maintained. However, for a given function $f(\delta) \geq 0$ that is locally monotonically increasing around d, it may be possible to find a circle, centered in d, within which it is possible to show that $\frac{d\delta_{ij}^2}{dt} < 0$. If all agents are initially placed within that circle it is still possible to state connectivity for $t \geq 0$.

Leader-follower connections: Finally we derive what it takes to keep the leader and the follower subgraphs connected. Consider follower $i \in \mathcal{N}^f$ and leader $j \in \mathcal{N}^l$. Let f_{max} as before be defined as the largest value the goal attraction function $f(\delta_j)$ can assume within the convex hull of the initial positions of all agents in the network and the goal position d. When $x_j \neq d$ we can obtain

$$\begin{aligned}\frac{d\delta_{ij}^2}{dt} \leq & -2(N_{fj} + N_{li})\delta_{ij}^2 + 2f_{max}\delta_{ij} \\ & +2(N_f - N_{fj})\Delta\delta_{ij} + 2(N_l - N_{li})\Delta\delta_{ij}.\end{aligned}$$

Setting $\frac{d\delta_{ij}^2}{dt} \leq 0$ when $\delta_{ij} = \Delta$ leads to

$$\frac{N}{2} + \frac{f_{max}}{2\Delta} \leq N_{fj} + N_{li}, \tag{14}$$

which can be seen as a lower bound on the number of links connecting the leader and follower subgraphs. It is easy to see that if $x_j = d$, (14) will still be a sufficient condition for the two agents to remain connected since the effect of setting $F(x_j, d) = 0$ will be a relaxation of the inequality. Thus, if the leader and follower subgraphs remain complete and if (14) holds for all initial links between leaders and followers, all connections in the graph will be maintained.

The derivations of this section are summarized in the following theorem:

Theorem 4 *Assume that the communication graph $\mathcal{G}(t)$ is initially connected and constituted of two complete subgraphs, the subgraph of leaders and the subgraph of followers. The dynamics of the follower and leader agents are given by (8) and (9) and the magnitude of the maximum goal attraction force that can be experienced by the leader agents is $f_{max} = \max_{x \in Co(\mathcal{G}(0) \cup d)} f(|d - x|)$. Assume also that*

(A1) *$\mathcal{G}(t)$ satisfies condition (10),*

(A2) *$f(\delta)$ is a convex function on $Co(\mathcal{G}(0) \cup d)$ such that (13) is true,*

(A3) *(14) holds for all initial links (i, j) such that $i \in \mathcal{N}^l$, $j \in \mathcal{N}^f$.*

Then all links in $\mathcal{G}(t)$ will be maintained for $t \geq 0$ and , by Lemma 1, all agents will converge to d.

It is not always be necessary that *all* connections between leaders and followers are maintained. In many applications one may simply want to ensure that the two subgroups remain connected. In those cases the following result is useful.

Lemma 5 *Let $\mathcal{G}(t)$ be the communication graph in Theorem 4 and let E^* be a subset of the initial links between the group of leaders and the group of followers. Re-define the neighbor sets of the graph such that $i \in \mathcal{N}^f$ is considered a neighbor of $j \in \mathcal{N}^l$, and vice versa, if and only if they are initially connected and the link $(i, j) \in E^*$. Then, if (A1) and (A2) are satisfied and it is possible to find a subset E^*, $|E^*| \geq 1$, such that condition (14) is satisfied for all links in E^*, then all links in E^* will remain intact and the group of leaders and the group of followers will remain connected with each other.*

Proof: The proof follows directly from Theorem (4) and from constraint (14). By definition, the links included in E^* are invariant, *i.e.* for each robot there exist a well defined lower bound on the number of neighbors. If the inequality (14) is satisfied for all robots for the lower bounds on their number of neighbors, then the inequality will hold even if additional non-invariant links are added and removed. ◇

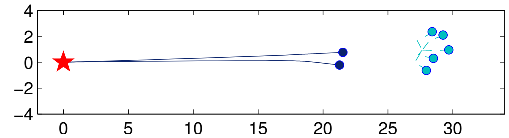
5 Simulations

In this section, the theoretical results of Section 3 and 4 are illustrated in a series of computer simulations. The agents are assumed to have a sensor range of $\Delta = 10$ and the coordinates of the goal are set to $d = [0\ 0]$. In the figures, leader agents are represented by dark dots, follower agents are represented by lighter dots and the goal is marked with an asterisk.

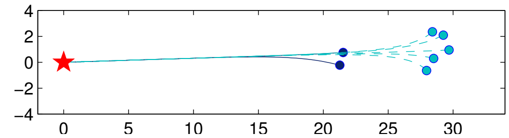
Complete graph

To start with, we investigate how the choice of goal attraction function, $f(\delta)$, affects the behavior of a group of agents that start off with a complete communication graph at time $t = 0$. The graph used in this example consists of seven agents, two leaders and five followers.

Recall from Section 3, Eq. (6), that the goal attraction function must be chosen as a monotonically increasing function. In accordance with this constraint we can let $f(\delta)$ be a linear function, $f(\delta) = \beta\delta$. With a proper choice of weight β the connectivity graph will remain complete and the agents will all converge to the goal.



(a) Simulation with condition (7) not satisfied.



(b) Simulation with condition (7) satisfied.

Fig. 2. Initial positions and robot trajectories from simulations with two leaders, five followers and initially complete communication graphs.

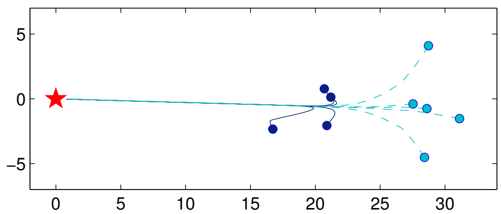
For the agent configuration shown in Fig. 2(a) and 2(b), β can safely be chosen as, for instance, $\beta = 1$, but if the weight is chosen too large, the attraction to goal experienced by the agents becomes larger than the inter-agent attraction and the formation may be forced to break. Fig. 2(a) shows the trajectories of the agents in a simulation with $\beta = 5$. The trajectories clearly show how the initial graph split into two separate subgraphs of which only one eventually converges to d. This outcome can be avoided by making sure that β is chosen such that condition (7) is satisfied.

With a linear function, the allowed choices of β are implicitly depending on the initial positions of the agents, but with a clever choice of goal attraction function, this dependence can be avoided. One function that is both monotonically increasing, thereby satisfying (6), and bounded above to satisfy (7) is $f(\delta) = \frac{2N\Delta}{\pi} \arctan(\delta)$. If a simulation is run with the same setup as before, but with the new, bounded, goal attraction function, all agents will eventually converge to goal. The trajectories of the robots are shown in Fig. 2(b).

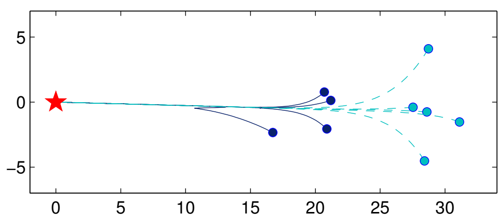
Incomplete graph

Let us now consider the case where the subgraph of leaders and the subgraph of followers are complete but where the full graph is not. In the remaining simulations, a linear goal attraction function $f(\delta) = \beta\delta$ is used. The sensor range and the coordinates of the goal remain the same as in previous simulations.

We start with an example with nine agents, four leaders and five followers. The initial configuration used in the simulations can be seen in Fig. 3(a) and 3(b). In this configuration, the rightmost of the followers is outside the sensing range of all the leaders while the reverse is true for the leftmost of the leaders. The remaining seven agents form a complete subgraph. With a goal attraction function defined by $\beta = 1$, both conditions (10) and (13) are satisfied for the graph and condition (14) is true for all existing links between the leader and the follower



(a) Robot trajectories with condition (14) satisfied.



(b) Robot trajectories with condition (14) not satisfied.

Fig. 3. Initial positions and robot trajectories from simulations with four leaders, five followers and initially incomplete communication graphs.

group. According to Theorem 4, this is sufficient to guarantee that all links in the initial graph should be maintained. Simulations confirm the result. The trajectories of the converging robots are shown in Fig. 3(a).

We now run the same simulation but with $\beta = 2$. Conditions (10) and (13) are still satisfied but because of the increased goal attraction force, the inequality in condition (14) is no longer true for any of the existing leader-follower links. The simulation shows that all agents still converge towards the goal (see Fig. 3(b)), but as the leaders now move faster, one of the leader-follower links is temporarily broken. The link is later re-formed, but in order to predict this fortunate outcome of the simulation a thorough analysis of the system would have been needed. Making sure that condition (14) is satisfied presents an easy way of avoiding the undesirable situation where the leader and the follower agents loose contact with each other.

If instead β is decreased rather than increased, the condition for inter-leader connectivity (13) will eventually be violated. Even so, simulations show that all existing inter-agent links will remain intact. To understand why, remember that the bounds derived in Section 4 are critical only in some extreme situations. Consider for instance the graph in Fig. 4. With this initial setup it is possible to choose β such that the violation of (13) causes the leftmost of the leaders to lose contact with the rest of the agents. Similarly, it is possible to design a setup such that the violation of (10) causes one of the follower agents to lose contact with its group. One example of this is illustrated in Fig. 5. With five leaders and four followers, the ratio of leaders-to-followers in this graph clearly violates condition (10), but with a proper choice

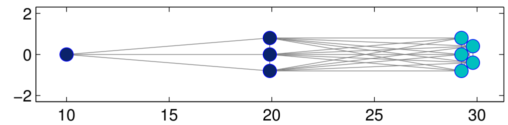


Fig. 4. Example of setup where (13) is violated. The edges represent existing inter-agent communication links.

of β, for instance $\beta = 1$, it is possible to satisfy both the condition for inter-leader connectivity (13) and condition (14) for all leader-follower links. If the simulation is run with this choice of β, the rightmost of the followers is immediately disconnected. While a disconnected leader can still find its way to the goal, a disconnected follower has no way of localizing neither the other agents nor the goal. In this particular example, the lost follower never regains contact with the group and is left behind.

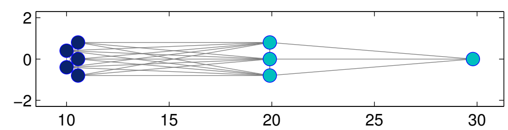


Fig. 5. Example of setup where (10) is violated. The edges represent existing inter-agent communication links.

6 Conclusions

In this paper we have examined how information on network structure can be used in combination with geometric constraints to guarantee connectivity and convergence to a common goal for a group of agents in a leader-follower multi-agent network with proximity based communication topology. The geometric approach is demonstrated on two networks with special structure on the initial topology. For these networks we derived bounds on the ratio of leaders-to-followers and constraints on the goal attraction function, $f(\delta)$, that were sufficient to secure connectivity. The theoretical results were supported by illustrating computer simulations. Future work includes applying the suggested framework to networks with other communication topologies. The use of different control laws and other control objectives also remain to be examined.

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References

Arcak, M., 2007. Passivity as a design tool for group coordination. IEEE Transactions on Automatic Control 52 (8), 1380–1390.

Arsie, A., Frazzoli, E., 2007. Efficient routing of multiple vehicles with no communications. International Journal of Robust and Nonlinear Control 18 (2), 154–164.

Couzin, I., Krause, J., Franks, N., Levin, S., 2005. Effective leadership and decision making in animal groups on the move. Nature 433, 513–516.

DeGennaro, M., Jadbabaie, A., 2006. Decentralized control of connectivity for multi-agent systems. 45th IEEE Conf. Decision and Control , 3628–3633.

Dimarogonas, D., Gustavi, T., Egerstedt, M., Hu, X., 2008. On the number of leaders needed to ensure network connectivity. 47th IEEE Conf. Decision and Control To appear.

Dimarogonas, D., Kyriakopoulos, K., 2008. A connection between formation infeasibility and velocity alignment in kinematic multi-agent systems. Automatica 44 (10), 2648–2654.

Godsil, C., Royle, G., 2001. Algebraic Graph Theory. Springer Graduate Texts in Mathematics # 207.

Jadbabaie, A., Lin, J., Morse, A., 2003. Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Transactions on Automatic Control 48 (6), 988–1001.

Ji, M., , Ferrari-Trecate, G., Egerstedt, M., Buffa, A., 2008. Containment control in mobile networks. IEEE Transactions on Automatic Control 53 (8), 1972–1975.

Ji, M., Egerstedt, M., 2007. Distributed coordination control of multi-agent systems while preserving connectedness. IEEE Transactions on Robotics 23 (4), 693–703.

Jr., M., A.Speranzon, Johansson, K. H., X.Hu, 2004. Multi-robot tracking of a moving object using directional sensors. 2004 IEEE International Conference on Robotics and Automation .

Loizou, S., Kyriakopoulos, K., 2008. Navigation of multiple kinematically constrained robots. IEEE Transactions on Robotics 24 (1), 221–231.

Martinez, S., Bullo, F., Cortes, J., Frazzoli, E., 2007. On synchronous robotic networks - Part I: Models, tasks and complexity. IEEE Transactions on Automatic Control 52 (12), 2199–2213.

Mehyar, M., Spanos, D., Pongsajapan, J., Low, S., Murray, R., 2007. Asynchronous distributed averaging on communication networks. IEEE Transactions on Networking 15 (3), 512–520.

Notarstefano, G., Savla, K., Bullo, F., Jadbabaie, A., 2006. Maintaining limited-range connectivity among second-order agents. 2006 American Control Conference , 2124–2129.

Olfati-Saber, R., 2006. Flocking for multi-agent dynamic systems: Algorithms and theory. IEEE Transactions on Automatic Control 51 (3), 401–420.

Rahmani, A., Ji, M., Mesbahi, M., Egerstedt, M., 2008. Controllability of multi-agent systems from a graph-theoretic perspective. SIAM Journal on Control and Optimization To appear.

Schuresko, M., Cortes, J., 2007. Safe graph rearrangements for distributed connectivity of robotic networks. 46rd IEEE Conference on Decision and Control , 4602–4607.

Smith, B., Wang, J., Egerstedt, M., Howard, A., 2009. Automatic formation deployment of decentralized heterogeneous multiple-robot networks with limited sensing capabilities. IEEE International Conference on Robotics and Automation Submitted.

Tanner, H., 2004. On the controllability of nearest neighbor interconnections. 43rd IEEE Conference on Decision and Control , 2467–2472.

Xiao, L., Boyd, S., 2003. Fast linear iterations for distributed averaging. 42nd IEEE Conf. Decision and Control , 4997–5002.

Zavlanos, M., Pappas, G., 2007. Distributed connectivity control of mobile networks. 46th IEEE Conf. Decision and Control , 3591–3596.