Transition-Time Optimization for Switched-Mode Dynamical Systems

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Abstract. This paper considers the problem of determining optimal switching times at which mode transitions should occur in multi-modal, hybrid systems. It derives a simple formula for the gradient of the cost functional with respect to the switching times, and uses it in a gradient-descent algorithm. Much of the analysis is carried out in the setting of optimization problems involving fixed switching-mode sequences, but a possible extension is pointed out for the case where the switching-mode sequence is a part of the variable. Numerical examples testify to the viability of the proposed approach.

Keywords. Switched dynamical systems, switching-time control, optimal control, hybrid systems, gradient-descent algorithms.

I. Introduction

Switched dynamical systems are often described by differential inclusions of the form

$$\dot{x}(t) \;\in\; \{g_\alpha(x(t), u(t))\}_{\alpha \in A}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^k$, and $\{g_\alpha : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n\}_{\alpha \in A}$ is a collection of continuously differentiable functions, parameterized by α belonging to some given set A. The time t is confined to a given finite-length interval $[0, T]$. Such systems arise in a variety of applications, including situations where a control module has to switch its attention among a number of subsystems [12], [15], [19], or collect data sequentially from a number of sensory sources [4], [6], [11]. A supervisory controller is normally engaged for dictating the switching law, i.e. the rule for switching among the functions g_α in the right-hand side of Eq. (1).

Recently, there has been a growing interest in optimal switching time control of such hybrid systems, where the control variable consists of a proper switching law as well as the input function $u(t)$ (see [3], [5], [9], [10], [16], [17], [18], [20]). Ref. [3] establishes a framework for optimal control, and [16], [17], [18] present suitable variants of the maximum principle to the setting of hybrid systems. Refs. [1], [2], [9], [14] consider piecewise-linear or affine systems. The special case of autonomous systems, where the term $u(t)$ is absent and the control variable consists solely of the switching times, is considered in [9], [11], [21], [22]. In particular, Xu and Antsaklis [21], [22] consider general nonlinear systems, and they have developed nonlinear-programming algorithms that compute the gradient and second-order derivatives of the cost functional. This paper, whose preliminary version has appeared in [8], also falls in this category.

Ref. [21] provides the starting point for the results presented in this paper, as we initially consider a similar problem, where the sequence of switching functions as well as the number of switching times are fixed. We develop a simpler formula than the one in [21] for the gradient of the cost functional, and use it in a gradient-descent algorithm. Finally, we suggest a possible extension of the algorithm to a class of problems where the number of switching times as well as the switching-mode sequence is a part of the variable parameter.

Section II formulates the problem and derives a formula for the gradient of its cost function. Section III derives an optimality condition having an intuitive appeal, and uses it to define a gradient-descent algorithm. Section IV points out a possible extension to more general scheduling problems, and Section V presents numerical experiments. Finally, Section VI concludes the paper.

II. Problem Formulation and Gradient Formula

Consider an autonomous switched-mode dynamical system where the initial state $x_0 \in \mathbb{R}^n$ and the final time $T > 0$ are given. The functions g_α in the right-hand side of (1) correspond to the modes of the system, and hence will be referred to as the *modal functions*. Suppose that the system switches between the modes (and their corresponding modal functions) a finite number of times, N, in the time-interval $[0,T]$. Let us denote the switching times by τ_i, $i = 1,\ldots,N$, in nondecreasing order, and further define $\tau_0 := 0$ and $\tau_{N+1} := T$. Then according to Eq. (1) and since the system is autonomous, for every $i \in \{1,\ldots,N+1\}$ there is an associated index term $\alpha(i) \in A$ such that

$$\dot{x} \;=\; g_{\alpha(i)}(x), \qquad \text{for all } t \in [\tau_{i-1}, \tau_i], \qquad i = 1,\ldots,N+1, \tag{2}$$

where at the boundary points τ_{i-1} and τ_i the derivative term $\dot{x}(t)$ is replaced by the appropriate one-sided derivative. Note that the state trajectory $x(t)$ is thus well defined and continuous throughout the interval $[0,T]$. Furthermore, we call the index-sequence $\{\alpha(i)\}_{i=1}^{N+1}$ the *modal sequence*, and denote it by σ. Let $L:\mathbb{R}^n\rightarrow\mathbb{R}$ be a given cost function, and define the total cost J by

$$J \ = \ \int_0^T L(x(t))dt. \tag{3}$$

We make the following assumption concerning the modal functions g_α and the cost function L.

Assumption 2.1. (i). The functions g_α and L are twice continuously differentiable on $\mathbb{R}^n$.

(ii) There exists a constant $K_0>0$ such that, for every $x\in\mathbb{R}^n$, and for all $\alpha\in A$,

$$||g_\alpha(x)|| \ \leq \ K_0(||x||+1). \tag{4}$$

Observe that J is a function of the modal sequence $\sigma=\{\alpha(i)\}_{i=1}^{N+1}$ as well as the switching times $\tau_1,\ldots,\tau_N$. In this and the next sections we assume a fixed modal sequence σ and consider J as a function of the switching times. To simplify the notation, let us define the functions f_i, $i=1,\ldots,N+1$, by $f_i=g_{\alpha(i)}$. Then, Eq. (2) assumes the following form,

$$\dot{x}(t) \ = \ f_i(x(t)), \qquad \text{for all } t\in[\tau_{i-1},\tau_i], \qquad i=1,\ldots,N+1, \tag{5}$$

with the given initial condition $x(0)=x_0$. Furthermore, let us denote the set of switching times by $\bar{\tau}$ in a vector form, i.e., $\bar{\tau}:=(\tau_1,\ldots,\tau_N)^T\in\mathbb{R}^N$. Then J is a function of $\bar{\tau}$ via Eqs. (5) and (3), and hence it is denoted by $J(\bar{\tau})$. We consider the following optimization problem, denoted by P_σ.

P_σ: *Minimize* $J(\bar{\tau})$ *subject to the inequality constraints* $0=\tau_0\leq\tau_1\leq\ldots\leq\tau_N\leq\tau_{N+1}=T$.

This section derives a formula for the gradient $\nabla J(\bar{\tau})$, which will be used later in a gradient-descent algorithm. We first need a technical, preliminary result, Lemma 2.1, whose description and statement follow. Recall that the final time, T, is fixed. Given constants $C>0$, $K_1>0$, $K_2>0$, and a convex compact set $\Gamma\subset\mathbb{R}^n$, we denote by $H[C;K_1;K_2;\Gamma]$ the set of Lebesgue measurable functions $h:\mathbb{R}^n\times[0,T]\rightarrow\mathbb{R}^n$ having the following four properties:

1) $||h(x,t)||\leq C$ for every $(x,t)\in\Gamma\times[0,T]$,
2) $h(x,t)$ is twice continuously differentiable in $x\in\mathbb{R}^n$ for all $t\in[0,T]$,
3) $||h(x_2,t)-h(x_1,t)||\leq K_1||x_2-x_1||$ for every $x_1\in\Gamma$, $x_2\in\Gamma$, and $t\in[0,T]$,

4) $||\frac{\partial h}{\partial x}(x_2,t)-\frac{\partial h}{\partial x}(x_1,t)|| \le K_2||x_2-x_1||$ for every $x_1 \in \Gamma$, $x_2 \in \Gamma$, and $t \in [0,T]$.[1]

We remark that the definition of $H[C;K_1;K_2;\Gamma]$ does not require continuity of $h(x,t)$ in its second variable, t.

Now fix constants $C>0$, $K_1>0$, and $K_2>0$, and a convex compact set $\Gamma \subset \mathbb{R}^n$, and let $h_1 \in H[C;K_1;K_2;\Gamma]$ and $h_2 \in H[C;K_1;K_2;\Gamma]$ be two given functions. Let $x_1(t)$ and $x_2(t)$ be defined by the respective differential equations, $\dot{x}_1(t)=h_1(x_1(t),t)$ and $\dot{x}_2=h_2(x_2(t),t)$, $t\in[0,T]$, with a common initial condition, $x_1(0)=x_2(0)=x_0\in\Gamma$. Define $\Delta h(x,t):=h_2(x,t)-h_1(x,t)$ and $\Delta x(t):=x_2(t)-x_1(t)$. Let $\Phi(t,\tau)\in\mathbb{R}^{n\times n}$ denote the state transition matrix of the linearized system $\dot{z}=\frac{\partial h_1}{\partial x}(x_1(t),t)z$.

The following lemma essentially has been proved in [13], Lemma 5.6.7 and in the proof of Theorem 5.6.8. The setting there is slightly different from ours, but the underlying arguments are identical. A detailed proof can be found in [7].

Lemma 2.1. There exist constants $K>0$ and $\bar{K}>0$, depending only on C, K_1, K_2, and Γ, such that, for all $h_1 \in H[C;K_1;K_2;\Gamma]$ and $h_2 \in H[C;K_1;K_2;\Gamma]$ with the property that $x_1(t)\in\Gamma$ and $x_2(t)\in\Gamma$ for all $t\in[0,T]$, the following two inequalities are in effect:

$$||\Delta x(t)|| \;\le\; K\int_0^T ||\Delta h(x_1(t),t)||dt, \tag{6}$$

and

$$\begin{aligned} &||\Delta x(t)-\int_0^t \Phi(t,\tau)\Delta h(x_1(\tau),\tau)d\tau|| \le \\ &\le \bar{K}\Big(\int_0^T ||\Delta h(x_1(t),t)||dt\Big)\Big(\int_0^T ||\Delta h(x_1(t),t)||dt+\int_0^T ||\frac{\partial \Delta h}{\partial x}(x_1(t),t)||dt\Big). \end{aligned} \tag{7}$$

Proof. See [13], Ch. 5.6.2. ■

As an application of this lemma, consider a family of functions, $h_\lambda \in H(C;K_1;K_2;\Gamma)$, parameterized by $\lambda\in[0,\bar{\lambda})$ for some $\bar{\lambda}>0$, for given $C>0$, $K_1>0$, $K_2>0$, and a compact set $\Gamma\subset\mathbb{R}^n$. Let $x_\lambda(t)$ be defined by the differential equation $\dot{x}_\lambda=h_\lambda(x_\lambda,t)$, $t\in[0,T]$, with a common initial condition $x_0\in\Gamma$. For the special case where $\lambda=0$ we will use the notation $h(x,t)=h_0(x,t)$ and $x(t)=x_0(t)$, and we define $\Delta h_\lambda(x,t)=h_\lambda(x,t)-h(x,t)$. Fix $\tau_0\in(0,T)$ such that $\tau_0+\bar{\lambda}\le T$, and let $g:\mathbb{R}^n\to\mathbb{R}^n$ be a function satisfying Assumption 2.1. Suppose

[1] The norm in the left-hand side of the inequality is the induced matrix norm.

that $\Delta h_\lambda(x,t)$ has the following form,

$$\Delta h_\lambda(x,t) \;=\; \begin{cases} g(x), & \text{if } \tau_0 \le t \le \tau_0+\lambda \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Let $L : \mathbb{R}^n \to \mathbb{R}$ be a function satisfying Assumption 2.1(i) (i.e., it is twice continuously differentiable), and define the function $J : [0,\bar{\lambda}] \to \mathbb{R}$ by

$$J(\lambda) \;=\; \int_0^T L(x_\lambda(t))dt. \tag{9}$$

Proposition 2.1. If $x_\lambda(t) \in \Gamma$ for every $t \in [0,T]$ and for all $\lambda \in [0,\bar{\lambda})$, then J has the following right derivative at 0,

$$\frac{dJ}{d\lambda^+}(0) \;=\; p(\tau_0)^T g(x(\tau_0)), \tag{10}$$

where the costate $p(t)$ satisfies the differential equation

$$\dot{p}(t) \;=\; -\Big(\frac{\partial h}{\partial x}(x(t),t)\Big)^T p(t) - \Big(\frac{\partial L}{\partial x}(x(t))\Big)^T, \tag{11}$$

with the boundary condition $p(T) = 0$.

Proof. Fix $\lambda \in [0,\bar{\lambda})$, and define $\Delta J_\lambda = J(\lambda) - J(0)$. By Eq. (9), $\Delta J_\lambda = \int_0^T \big(L(x_\lambda(t)) - L(x(t))\big)dt$, and by the mean value theorem,

$$\Delta J_\lambda \;=\; \int_0^T \frac{\partial L}{\partial x}(x(t)+s(t)\Delta x_\lambda(t))\Delta x_\lambda(t)dt \tag{12}$$

for some $s(t) \in [0,1]$. By Assumption 2.1, there exists $K_3 > 0$ such that

$$||\frac{\partial L}{\partial x}(x(t)+s(t)\Delta x_\lambda(t)) - \frac{\partial L}{\partial x}(x(t))|| \le K_3||\Delta x_\lambda(t)||. \tag{13}$$

Next, by lemma 2.1 (Eq. (6)), there exists $K_4 > 0$ such that, for all $t \in [0,T]$,

$$||\Delta x_\lambda(t)|| \;\le\; K_4 \int_0^T ||\Delta h_\lambda(x(t),t)||dt. \tag{14}$$

By the definition of Δh_λ (Eq. (8), there exists $K_5 > 0$ such that

$$\int_0^T ||\Delta h_\lambda(x(t),t)||dt \;\le\; K_5\lambda, \tag{15}$$

and

$$\int_0^T ||\frac{\partial \Delta h_\lambda}{\partial x}(x(t),t)||dt \;\le\; K_5\lambda. \tag{16}$$

Combining (13), (14) and (15) we obtain, $||\big(\frac{\partial L}{\partial x}(x(t)+s(t)\Delta x_\lambda(t)) - \frac{\partial L}{\partial x}(x(t))\big)\Delta x_\lambda(t)|| \leq K_3||\Delta x_\lambda(t)||^2 \leq K_3K_4^2K_5^2\lambda^2$. By defining $K_6 = K_3K_4^2K_5^2T$, we have that,

$$\int_0^T ||\big(\frac{\partial L}{\partial x}(x(t)+s(t)\Delta x_\lambda(t)) - \frac{\partial L}{\partial x}(x(t))\big)\Delta x_\lambda(t)||dt \ \leq \ K_6\lambda^2. \tag{17}$$

Consequently, and by (12), we have that

$$\Delta J_\lambda \ = \ \int_0^T \frac{\partial L}{\partial x}(x(t))\Delta x_\lambda(t)dt + o(\lambda), \tag{18}$$

where $o(\lambda)/\lambda \to 0$ as $\lambda \to 0$. Next, applying Lemma 2.1 (Eq. (7)) with $h_1 = h$, $x_1 = x$, $h_2 = h_\lambda$, and $x_2 = x_\lambda$, it follows (by Eqs. (7), (15) and (16)) that $\Delta x_\lambda(t) - \int_0^t \Phi(t,\tau)\Delta h_\lambda(x(\tau),\tau)d\tau = o(\lambda)$, where the function $o(\lambda)$ is independent of $t \in [0,T]$ or of $\lambda \in [0,\bar{\lambda})$. Consequently, and by Eq. (18), we have that

$$\Delta J_\lambda \ = \ \int_0^T \frac{\partial L}{\partial x}(x(t))\int_0^t \Phi(t,\tau)\Delta h_\lambda(x(\tau),\tau)d\tau dt + o(\lambda). \tag{19}$$

Changing the order of integration in (19) we obtain,

$$\Delta J_\lambda \ = \ \int_0^T\int_\tau^T \frac{\partial L}{\partial x}(x(t))\Phi(t,\tau)dt\Delta h_\lambda(x(\tau),\tau)d\tau + o(\lambda). \tag{20}$$

Define the costate $p(\tau) \in \mathbb{R}^n$ by $p(\tau)^T = \int_\tau^T \frac{\partial L}{\partial x}(x(t))\Phi(t,\tau)dt$. Taking derivative, it is apparent that $\dot{p}(\tau)^T = -p(\tau)^T\frac{\partial h}{\partial x}(x(\tau),\tau) - \frac{\partial L}{\partial x}(x(\tau))$, and hence Eq. (11) is in effect; and also $p(T)^T = 0$. It now follows from Eq. (20) that $\Delta J_\lambda = \int_0^T p(\tau)^T\Delta h_\lambda(x(\tau),\tau)d\tau + o(\lambda)$. Hence, by Eq. (8), $\Delta J_\lambda = \int_{\tau_0}^{\tau_0+\lambda} p(\tau)^T g(x(\tau))d\tau + o(\lambda)$. Dividing by λ and taking the limit $\lambda \to 0$, and noting that $p(\tau)^T g(x(\tau))$ is a continuous function of τ, we obtain that $\frac{dJ}{d\lambda^+}(0) = p(\tau_0)^T g(x(\tau_0))$. This completes the proof. ■

We remark that the left derivative has the same formula, as can be seen by repeating the arguments of the proof of Proposition 2.1 with minor modifications.

Consider the function $J(\bar{\tau})$ as defined by Eqs. (5) and (3). Define the feasible set, denoted by Λ, by $\Lambda := \{\bar{\tau} = (\tau_1,\ldots,\tau_N)^T : 0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_N \leq \tau_{N+1} = T\}$. For every $\bar{\tau} \in \Lambda$, define the costate $p(t) \in \mathbb{R}^n$ by the differential equation

$$\dot{p} = -\left(\frac{\partial f_{i+1}}{\partial x}(x,t)\right)^T p - \big(\frac{\partial L}{\partial x}(x)\big)^T, \qquad t \in [\tau_i,\tau_{i+1}], \qquad i = N, N-1,\ldots,0, \tag{21}$$

with the boundary condition $p(T) = 0$.

Proposition 2.2. Suppose that Assumption 2.1 is in effect. For every point $\bar{\tau}$ in the interior of

Λ, and for all $i = 1, \ldots, N$, the derivative $\frac{dJ}{d\tau_i}(\bar{\tau})$ has the following form

$$\frac{dJ}{d\tau_i}(\bar{\tau}) \;=\; p(\tau_i)^T\big(f_i(x(\tau_i)) - f_{i+1}(x(\tau_i))\big). \tag{22}$$

Proof. Define the function $h(x,t) : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ by $h(x,t) = f_i(x)$ for all $t \in [\tau_{i-1}, \tau_i)$. Then $\dot{x} = h(x,t)$ with the initial condition $x(0) = x_0$. By Assumption 2.1, there exists a convex compact set $\Gamma \subset \mathbb{R}^n$ such that $x(t) \in \Gamma$ for every feasible $\bar{\tau} = (\tau_1, \ldots, \tau_N)^T$ and for all $t \in [0,T]$. Moreover, by the same assumption there exist constants $C > 0$, $K_1 > 0$, and $K_2 > 0$, such that $h(x,t) \in H[C; K_1; K_2; \Gamma]$ for all $\bar{\tau}$. Given $i \in \{1, \ldots, N\}$ and $\lambda \in [0, \tau_{i+1} - \tau_i)$, define $\Delta h_\lambda(x,t)$ as in (8), with τ_i instead of τ_0, and with $g(x) = f_i(x) - f_{i+1}(x)$. An application of Proposition 2.1 and the remark that follows it now yields (22). ∎

We observe that the derivative $dJ/d\tau_i$ may not be well defined on the boundary of Λ. The reason is that, if $\tau_{i+1} = \tau_i$, then changing these variables in a way that swaps their order leaves unclear the identity of the modal function between them and hence the right-hand side of Eq. (22). However, the expression in the right-hand side of (22) is defined on the boundary of Λ where, in the event that $\tau_{i+1} = \tau_i$, the domain of the modal function f_{i+1} is the single point $\tau_{i+1} = \tau_i$. Let us define, for every $\bar{\tau} \in \Lambda$, by $q_i(\bar{\tau})$ the right-hand side of Eq. (22), and define $\bar{q}(\bar{\tau}) := (q_1(\bar{\tau}), \ldots, q_N(\bar{\tau}))^T \in \mathbb{R}^N$. Then the function $\bar{\tau} \to \bar{q}(\bar{\tau})$ is well defined throughout Λ. Note that $\bar{q}(\bar{\tau}) = \nabla J(\bar{\tau})$ in the interior of Λ. Furthermore, it is evident that the directional derivative of J at $\bar{\tau} \in \Lambda$ in a feasible direction h is given by the inner product $< \bar{q}(\bar{\tau}), h >$. This fact will be used in the analysis carried out in the next section.

III. Optimality Condition and an Algorithm

This section derives a special form of the Kuhn-Tucker optimality condition that is based on the structure of the constraint set Λ, and uses it to compute a descent direction. The analysis requires the following result concerning continuity of $\nabla J(\bar{\tau})$.

Proposition 3.1. The function $\bar{q}(\bar{\tau}) : \Lambda \to \mathbb{R}^N$ is Lipschitz continuous throughout Λ.

Proof. Given $\bar{\tau} \in \Lambda$, denote by $x(t;\bar{\tau})$ and $p(t;\bar{\tau})$ the state and costate variables defined by Eqs. (5) and (21), respectively, with the switching-time vector $\bar{\tau}$. By Assumption 2.1, there exist compact sets $\Gamma_x \subset \mathbb{R}^n$ and $\Gamma_p \subset \mathbb{R}^n$ such that $x(t,\bar{\tau}) \in \Gamma_x$ and $p(t;\bar{\tau}) \in \Gamma_p$ for every $t \in [0,T]$ and for every $\bar{\tau} \in \Lambda$. Consider two points $\bar{\tau}(1) = (\tau_1(1), \ldots, \tau_N(1))^T \in \Lambda$ and $\bar{\tau}(2) = (\tau_1(2), \ldots, \tau_N(2))^T \in \Lambda$. By Lemma 2.1 (Eq. (6)) applied first to x (Eq. (5)) and then to p (Eq. (21)), there exists a constant $K_1 > 0$ such that, for every $\bar{\tau}(1)$ and $\bar{\tau}(2)$, and for all $t \in [0,T]$, $||x(t;\bar{\tau}(1)) - x(t;\bar{\tau}(2))|| \leq K_1||\bar{\tau}(1) - \bar{\tau}(2)||$, and $||p(t;\bar{\tau}(1)) - p(t;\bar{\tau}(2))|| \leq K_1||\bar{\tau}(1) - \bar{\tau}(2)||$. Next, by (21) and Assumption 2.1, there exists $K_2 > 0$ such that, for every $\bar{\tau} \in \Lambda$, $||\dot{p}(t;\bar{\tau})|| \leq K_2$ for every $t \in [0,T]$, and hence, for every $t_1 \in [0,T]$ and $t_2 \in [0,T]$, $||p(t_1;\bar{\tau}) - p(t_2;\bar{\tau})|| \leq K_2|t_1 - t_2|$. Consequently, for every $i = 1,\ldots,N$, we get, after some algebra,

$$||p(\tau_i(1),\bar{\tau}(1)) - p(\tau_i(2),\bar{\tau}(2))|| \ \leq \ (K_1 + K_2)||\bar{\tau}(1) - \bar{\tau}(2)||. \tag{23}$$

This establishes that the mapping $\tau_i \rightarrow p(\tau_i)$ is Lipschitz continuous in τ_i. A similar (and actually, simpler) argument applies to the Lipschitz continuity of the function $\tau_i \rightarrow x(\tau_i)$. Consequently, and by (22), $q_i(\bar{\tau}) : \Lambda \rightarrow R^N$ is a Lipschitz-continuous function. This completes the proof. ∎

We next derive a special form of the Kuhn-Tucker optimality condition. Fix a point $\bar{\tau} = (\tau_1,\ldots,\tau_N)^T \in \Lambda$, and recall that we defined $\tau_0 := 0$ and $\tau_{N+1} = T$. If $\bar{\tau}$ is on the boundary of Λ then $\tau_i = \tau_{i+1}$ for some $i = 0,\ldots,N$. To account for this case we define, for all $i \in \{0,\ldots,N+1\}$, the integer-quantities $k(i)$ and $n(i)$ as follows: $k(i) := \min\{k \leq i : \tau_k = \tau_i\}$, and $n(i) := \max\{n \geq i : \tau_n = \tau_i\}$. In other words, $\tau_j = \tau_i$ for all $j \in \{k(i),\ldots,n(i)\}$; if $\tau_i > 0$ then $\tau_{k(i)-1} < \tau_{k(i)}$; and if $\tau_i < T$ then $\tau_{n(i)} < \tau_{n(i)+1}$. Furthermore, define $r_i(\bar{\tau}) := \sum_{j=k(i)}^{i} q_j(\bar{\tau})$ and $R_i(\bar{\tau}) := \sum_{j=i}^{n(i)} q_j(\bar{\tau})$. The following result characterizes Kuhn-Tucker points.

Proposition 3.2. Let $\bar{\tau} = (\tau_1,\ldots,\tau_N)^T$ be a local minimum for P_σ. Then, for every $i \in \{1,\ldots,N\}$, $r_i(\bar{\tau}) \leq 0$ unless $\tau_i = 0$, and $R_i(\bar{\tau}) \geq 0$ unless $\tau_i = T$.

Proof. Let $\bar{\tau} = (\tau_1,\ldots,\tau_N)^T$ be a local minimum for P_σ. Consider $k \in \{1,\ldots,N\}$ and $n \in \{k,\ldots,N\}$ such that: (i) $\tau_k = \tau_n$; (ii) either $\tau_k = 0$ or $\tau_{k-1} < \tau_k$; and (iii) $\tau_n < \tau_{n+1}$. We will prove that $R_i(\bar{\tau}) \geq 0$ for all $i = k,\ldots,n$; since similar arguments apply to proving the reverse inequality regarding r_i, this will complete the proposition's proof.

If $k = n$ then certainly $q_k(\bar{\tau}) = 0$ if $\tau_k > 0$ and $q_k(\bar{\tau}) \geq 0$ if $\tau_k = 0$, and hence $R_k(\bar{\tau}) = q_k(\bar{\tau}) \geq 0$ in either case. Next, consider the case where $k < n$. For all $j = k,\ldots,n-1$, since $\tau_j = \tau_{j+1}$, there exists a Lagrange multiplier $\lambda_j \geq 0$ for the constraint $\tau_j - \tau_{j+1} \leq 0$. Moreover, if $\tau_k = 0$ then there exists a Lagrange multiplier $\mu_k \geq 0$ for the constraint $-\tau_k \leq 0$. From the Kuhn-Tucker optimality condition, it follows that (i) $q_k(\bar{\tau}) + \lambda_k = 0$ if $\tau_k > 0$, and $q_k(\bar{\tau}) + \lambda_k - \mu_k = 0$ if $\tau_k = 0$; (ii) $q_j(\bar{\tau}) - \lambda_{j-1} + \lambda_j = 0$ for all $j = k+1,\ldots,n-1$; and (iii) $q_n(\bar{\tau}) - \lambda_{n-1}$=0. Fix $i \in \{k,\ldots,n\}$. Summing up these equations for $j = i,\ldots,n$, we obtain:

(i) for $i > k$, $R_i(\bar{\tau}) = \lambda_{i-1}$; and (ii) for $i = k$, $R_k(\bar{\tau}) = 0$ if $\tau_k > 0$, and $R_k(\bar{\tau}) = \mu_k$ if $\tau_k = 0$. In any event, $R_i(\bar{\tau}) \geq 0$. ■

Corollary 3.1. In the setting of Proposition 3.2, if $\tau_{k(i)-1} < \tau_{k(i)}$ and $\tau_{n(i)} < \tau_{n(i)+1}$, then $R_{k(i)}(\bar{\tau}) = r_{n(i)}(\bar{\tau}) = 0$.

Proof. Follows immediately from Proposition 3.2, since $R_{k(i)}(\bar{\tau}) = r_{n(i)}(\bar{\tau})$. ■

In order to solve the problem P_σ, to the extent of computing a point satisfying the above optimality condition, we use a gradient-projection algorithm with Armijo step sizes. Given a point $\bar{\tau} \in \Lambda$, let $\Psi(\bar{\tau})$ denote the set of feasible directions from the point $\bar{\tau}$, namely,

$$\Psi(\bar{\tau}) := \{\bar{h} \in \mathbb{R}^N \ \mid \ \text{for some } \tilde{\zeta} > 0, \text{ and for all } \zeta \in [0, \tilde{\zeta}), \quad \bar{\tau} + \zeta\bar{h} \in \Lambda\}.$$

Let $\bar{h}(\bar{\tau})$ denote the projection of the vector $-\bar{q}(\bar{\tau})$ onto $\Psi(\bar{\tau})$. The following algorithm uses the Armijo step sizes in this direction.

Algorithm 3.1 Gradient-Projection Algorithm with Armijo Step Sizes.

Given: Constants $\alpha \in (0,1)$, $\beta \in (0,1)$, and $\bar{z} > 0$.

Initialize. Choose an initial point $\bar{\tau}_0 \in \Lambda$. Set $i = 0$.

Step 1. If $\bar{\tau}_i + \bar{z}\bar{h}(\bar{\tau}_i) \notin \Lambda$ then compute $z_{\max} := \max\{z \geq 0 \mid \bar{\tau}_i + z\bar{h}(\bar{\tau}_i) \in \Lambda\}$; otherwise set $z_{max} := \bar{z}$.

Step 2. Compute the step size ζ_i by

$$\zeta_i \ = \ \max\{z = z_{max} \cdot \beta^k; \ k \geq 0 \ \mid \quad J(\bar{\tau}_i + z\bar{h}(\bar{\tau}_i)) - J(\bar{\tau}_i) \ \leq \alpha z < \bar{h}(\bar{\tau}_i), \bar{q}(\bar{\tau}_i) >\}. \quad (24)$$

Step 3. Set $\bar{\tau}_{i+1} := \bar{\tau}_i + \zeta_i\bar{h}(\bar{\tau}_i)$, set $i = i + 1$, and go to Step 1. ■

Note that, if $\bar{\tau}_i + \bar{z}\bar{h}(\bar{\tau}_i) \in \Lambda$ then $z_{max} = \bar{z}$, and if $\bar{\tau}_i + \bar{z}\bar{h}(\bar{\tau}_i) \notin \Lambda$ then z_{max} is the maximum step size z for which $\bar{\tau}_i + z\bar{h}(\bar{\tau}_i) \in \Lambda$. Moreover, the step size computed in Step 2 is $\zeta_i := z_{max} \cdot \beta^k$ for some integer $k \geq 0$. Ref. [13] contains an analysis of this algorithm and various alternative versions thereof. In particular, it proves that (i) $\bar{h}(\bar{\tau}_i)$ indeed is a descent direction from $\bar{\tau}_i$, i.e., $J(\bar{\tau}_{i+1}) \leq J(\bar{\tau}_i)$; (ii) the step size ζ_i is nonzero as long as $\bar{\tau}_i$ does not satisfy the Kuhn-Tucker optimality condition (and hence the optimality condition established in Proposition 3.2), and (iii) every accumulation point of a sequence $\{\bar{\tau}_i\}_{i=0}^{\infty}$, computed by the algorithm, satisfies the optimality condition. Therefore, a practical stopping rule is to end the algorithm's run at a point $\bar{\tau}_i$ whenever $||\bar{h}(\bar{\tau}_i)|| < \epsilon$ for an a-priori chosen value of $\epsilon > 0$. Moreover, the algorithm is stable in the sense that it will converge from every starting point, and it has a linear asymptotic convergence rate. Ref. [13] also gives recommendations for the choices of α, β and $\bar{z}$. For details, please see pp. 30-31 of [13].

Finally, a word must be said about the computation of $\bar{h}(\bar{\tau})$ for a given $\bar{\tau} := (\tau_1, \ldots, \tau_N)^T \in \Lambda$. Let us define a *block* to be a contiguous integer-set $\{k, \ldots, n\} \subset \{1, \ldots, N\}$ such that $\tau_n = \tau_k$ (and hence $\tau_i = \tau_k$ for all $i \in \{k, \ldots, n\}$). Observe that every set of contiguous integers that is a subset of a block is also a block. Furthermore, we say that a block is *maximal* if no superset thereof is a block. Obviously, the set $\{1, \ldots, N\}$ is partitioned into disjoint maximal blocks in a way that depends on $\bar{\tau}$.

The following computation of $\bar{h}(\bar{\tau}) := (h_1(\bar{\tau}), \ldots, h_N(\bar{\tau}))^T$ is done one-block-at-a-time in the following manner. Let $\{k, \ldots, n\}$ be a maximal block associated with $\bar{\tau}$.

Algorithm 3.2. Procedure for computing $h_i(\bar{\tau})$, $i = k, \ldots, n$.

Step 0. Set $\ell = k$.

Step 1. Compute r_{max} defined by

$$r_{max} \ := \ \max\{\frac{r_i(\bar{\tau})}{i-k+1} \ \ | \ \ i = \ell, \ldots, n\}.$$

Define $m := \max\{i = k, \ldots, n \ : \ \frac{r_i}{i-k+1} = r_{max}\}$.

Step 2. For all $i \in \{\ell, \ldots, m\}$, define $h_i(\bar{\tau})$ by $-r_{max}$ unless either (i) $\tau_m = 0$ and $r_{max} > 0$, or (ii) $\tau_m = T$ and $r_{max} < 0$. In either case (i) or (ii), set $h_i(\bar{\tau}) = 0$.

Step 3. If $m = n$, exit. If $m < n$, set $\ell := m + 1$ and go to Step 1. ■

Proving that the resulting vector $\bar{h}(\bar{\tau})$ indeed is the projection of $-\bar{q}(\bar{\tau})$ onto $\Psi(\bar{\tau})$ is fairly straightforward once we realize that that projection is the vector in $\Psi(\bar{\tau})$ of least distance from $-\bar{q}(\bar{\tau})$. The proof will lead us astray from the main topic of this paper, and hence will not be presented here. We point out, however, that the optimality condition established in Proposition 3.2 has the following associated intuitive geometric appeal. If the optimality condition is satisfied, then obviously $\bar{h}(\bar{\tau}) = 0$. If it is not satisfied, then Algorithm 3.2 indicates which variables τ_i, $i \in \{k, \ldots, n\}$, should be increased and which ones should be decreased; in other words, a descent direction for J clearly emerges.

IV. Inserting Switching Modes to a given Schedule

The optimization problem that we consider is greatly complicated if the modal schedule becomes part of the variable parameter. In this case we have a mixed integer problem which may be NP hard. However, it appears to be possible to overlay a continuous structure over the parameter space, and hence compute sensitivity measures (gradients) and use them in algorithms that seek local minima. This section describes such a continuous structure as applied to the insertion of a new mode into an existing schedule. For the sake of clarity we present only a simple case and numerical results therewith, while deferring the general case and its analysis to a future publication.

Fix a modal sequence $\sigma = \{\alpha(i)\}_{i=1}^{N+1}$ and the associated switching-time vector $\bar{\tau} = (\tau_1, \ldots, \tau_N)^T$. Recall that the state trajectory $\{x(t)\}$ evolves according to Eq. (2), and by defining $f_i = g_{\alpha(i)}$, Eq. (2) is transformed into Eq. (5). Let $\{p(t)\}$ be the costate trajectory defined by Eq. (21). Now fix $\alpha \in A$, $\tau \in (0,T)$, and $\lambda > 0$ such that $\tau + \lambda < T$, and consider inserting the modal function g_α in the time-interval $[\tau, \tau+\lambda]$. This insertion will result in a modification of the modal sequence σ by adding to it the index α. Recall the cost functional J as defined by Eq. (3), and consider it as a function of λ, hence to be denoted by $J(\lambda)$. Then the following is an immediate corollary of Proposition 2.1.

Proposition 4.1. Let $\tau \in [\tau_{i-1}, \tau_i)$ for some $i \in \{1, \ldots, N+1\}$. Then, the one-sided derivative $\frac{dJ}{d\lambda^+}(0)$ has the following form,

$$\frac{dJ}{d\lambda^+}(0) \;=\; p(\tau)^T\big(g_\alpha(x(\tau)) - g_{\alpha(i)}(x(\tau))\big). \tag{25}$$

Proof. Follows directly from Proposition 2.1. ■

If the above insertion takes place at a point $\tau \in (\tau_{i-1}, \tau_i)$ then, for $\lambda < \tau_i - \tau$, the switching-time vector becomes $(\tau_1, \ldots, \tau_{i-1}, \tau, \tau+\lambda, \tau_i, \ldots, \tau_N)^T \in \mathbb{R}^{N+2}$, and the associated, modified modal sequence becomes $\{\alpha(1), \ldots, \alpha(i), \alpha, \alpha(i), \alpha(i+1), \ldots, \alpha(N+1)\}$. If $\tau = \tau_{i-1}$, then only one switching time is appended at time $\tau_{i-1} + \lambda$, and the modal function g_α is inserted in the interval $[\tau_{i-1}, \tau_{i-1} + \lambda)$, but Eq. (25) holds true. We later will denote the term $\frac{dJ}{d\lambda^+}(0)$ by $\frac{dJ}{d\lambda^+}(0;\tau)$ in order to emphasize its dependence on the insertion point τ. We point out that when the above term has to be computed for a number of insertion points τ, the costate trajectory need be computed only once. Proposition 4.1 and the sensitivity formula (25) will be used in the next section for computing insertion points.

It is possible to consider inserting multiple switching modes at a point $\tau \in (\tau_{i-1}, \tau_i)$ and extend Proposition 4.1 in the following way. Fix $\beta(j) \in A$, $j = 1, \ldots, m$, for some $m > 0$; fix $\tau \in (0,T)$ and $\lambda > 0$ such that $\tau + \lambda < T$; and fix $a_j > 0$ such that $\sum_{j=1}^{N} a_j = 1$. Now insert the modal functions $g_{\beta(1)},\ldots,g_{\beta(m)}$, in the indicated order, in successive time-intervals of respective durations λa_j, $j=1,\ldots,m$, starting with $g_{\beta(1)}$ at time τ. Consider the cost functional J as a function of $\lambda\geq 0$.

Proposition 4.2. The one-sided derivative $\frac{dJ}{d\lambda^+}(0)$ has the following form,

$$\frac{dJ}{d\lambda^+}(0) \;=\; p(\tau)^T\Big(a_1g_{\beta(1)}(x(\tau))+\sum_{j=2}^{m}(a_j-a_{j-1})\big(g_{\beta(j)}(x(\tau))-g_{\beta(j-1)}(x(\tau))\big)-a_mg_{\alpha(i)}(x(\tau))\Big). \tag{26}$$

Proof. Follows directly from Proposition 2.1 and the chain rule. ∎

V. Numerical Example

To test the viability of the gradient formula, we apply Algorithm 3.1 to a simple problem. Admittedly, the problem is but of an academic nature, but it highlights the salient features of our approach.

The system in question alternates between two modes, *mode 1* and *mode 2*. Mode 1 has the dynamic representation by the equation $\dot{x}=A_1x$, and mode 2 is represented by the equation $\dot{x}=A_2x$; here $x\in\mathbb{R}^2$, and A_1 and A_2 are 2×2 matrices. Let $A:=\{1,2\}$, and let $g_1(x)=A_1x$ and $g_2(x)=A_2x$. The time interval is $[0,T]$ with $T=10$, and the initial condition is $x_0=x(0)=(1,1)^T$. The cost criterion (functional) is $J=\frac{1}{2}\int_0^{10}||x(t)||^2dt$, and the variable parameter consists of the switching sequence (including the number of switches) between the modes. The matrices A_1 and A_2 are the following,

$$A_1 \;=\; \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}, \qquad \text{and} \qquad A_2 \;=\; \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}.$$

We observe that each one of the matrices has one positive eigenvalue and one negative eigenvalue, and the respective eigenvectors of the negative eigenvalues do not coincide. Consequently, the switching is used to manage the unstable parts of the state trajectories. In fact, given the ”energy” cost functional, we expect the optimal switching schedule (if an optimum exists) to frequently switch between the two modes.

The algorithm for minimizing J alternates between two phases, corresponding to its Step 1 and Step 2, below.

Algorithm 5.1.

Given: A grid $\Theta:=\{\theta_\ell,\ \ell=0,1,\ldots,L\}$ of equally-spaced points covering the interval $[0,T]$, and an $\epsilon > 0$.

Step 0. Let $\sigma := \{1\}$, namely $\dot{x} = A_1 x$ for all $t \in [0, T]$ and there are no switching times. Set $N = 0$.

Step 1. For every $j = 0, \ldots, N$, do (i) and (ii) as follows:

(i) Compute a point $\theta_j \in \Theta$ such that

$$\theta_j \ \in \ \text{argmin}\{\frac{dJ}{d\lambda^+}(0;\tau) \mid \tau \in \Theta \cap [\tau_j, \tau_{j+1})\}. \tag{27}$$

(ii) If $\frac{dJ}{d\lambda^+}(0;\theta_j) < 0$ then modify the switching schedule σ by adding two switching points at θ_j and the switching mode $A_{j+1(\text{mod } 2)}$ between them, and set $N = N + 2$. On the other hand, if $\frac{dJ}{d\lambda^+}(0;\theta_j) = 0$, then stop and exit.

Step 2. Use Algorithm 3.1 to solve the problem P_σ to the extent of computing a point $\bar{\tau}$ such that $||\bar{h}(\bar{\tau})|| < \epsilon$.

Step 3. Go to Step 1. ■

We chose ϵ to be $\epsilon = 0.3$, and the grid Θ consisted of 200 equally-spaced points in the interval $[0, 10]$. Also, whenever we added the two switching times in Step 1 we separated them by 0.02, i.e., one was at a point θ_j and the other was at 0.02 to the right of θ_j.

Iteration (k)	$\bar{\tau}(k)$	$J(\bar{\tau}(k))$	$\|\|\nabla(J(\bar{\tau}))\|\|$
(*)	-	113.74	-
0	(4.25, 4.27)	112.32	99.94
1	(3.70, 4.82)	59.91	82.67
9	(2.01, 7.11)	24.53	0.18
(**)	(0.45, 0.47, 2.01, 4.40, 4.42, 7.11, 8.60, 8.62)	24.45	3.58
17	(0.22, 0.90, 2.05, 3.82, 5.05, 7.16, 8.05, 9.16)	22.40	0.26

Table 1: Numerical Results

Numerical results are shown in Table 1. In the first row, indicated by $(*)$ there were no switching times and the only mode was $A_1 x$. Step 1 then computed the switching points at 4.25 and 4.27. Thence, the algorithm considers the modal sequence $\sigma = 1, 2, 1$ having two switching times, and in Step 2 involving 9 iterations of Algorithm 3.1, the cost is reduced from 112.32 to 24.53 while the gradient's magnitude declines from 99.94 to 0.18. At this point the algorithm reverted to step 1 and inserted the complementary mode at each one of the three modal domains, with the resulting switching schedule shown in the row marked by (**). The algorithm then continued in Step 2 until $||\nabla(J(\bar{\tau}))|| = 0.26$, for a cost reduction from 24.45 to 22.40. At this point (row 17) Step 1 attempted to insert a complementary mode at each one of the 9 modal domains. The minimizers of the right-hand side of (25) ranged between -0.42 and -0.11, deemed too small to proceed. It is for this reason that the algorithm's run was stopped after iteration 17.

VI. Conclusions

This paper concerns an optimal control problem defined on switched-mode dynamical systems, whose variational parameter consists of the switching schedule. It first derives a formula for the gradient of the cost functional with respect to a given sequence of switching times. It then extends the formula, based on its special structure, to the directional derivative of the cost functional with respect to the length of a time interval at which additional switching modes can be inserted. The approach described in this paper holds out promise of computing suboptimal solutions to optimal scheduling problems, including the determination of modal sequencing, in hybrid dynamical systems. Numerical examples testify to the potential viability of the proposed approach.

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