Non-Pathological Sampling of Switched Linear Systems

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***Abstract*— We study the conservation of observability after sampling in the class of switched linear systems with no inputs and switching in the measurement equation only. Using van der Waerden's Theorem, a classic result from Ramsey Theory, we give a sufficient condition on the components of the resulting discretized switched linear system for it to be observable under arbitrary switching sequences. This preliminary result then enables us to extend the Kalman-Bertram criterion for non-pathological sampling to the class of switched linear systems under consideration.**

***Index Terms*— observability, sampling, switched linear systems**

I. Introduction

Consider the following class of switched linear systems

$$\begin{aligned} \dot{x}_t &= Ax_t \\ y_t &= C(\theta_t)x_t, \end{aligned} \tag{1}$$

where $x_t \in \mathbb{R}^n$ and $y_t \in \mathbb{R}^p$, where A and $C(\cdot)$ are real matrices of compatible dimensions, and where θ_t assumes values in the set $\{1, \ldots, s\}$, so that $C(\theta_t)$ switches among s different measurement matrices $C(1), \ldots, C(s)$. We assume the switching signal θ to be piecewise constant, i.e., to exhibit a finite number of switches in any finite time interval, thus ruling out Zeno behaviors. An important consequence of this assumption is that no minimum dwell time is imposed. Furthermore, θ is assumed to be independent of the state trajectory of the system, i.e. an exogenous input to the system.

The discretized counterpart of (1), obtained by sampling the states and outputs at a constant period T, i.e. (for any continuous-time quantity z_t, we let $\bar{z}_k \triangleq z_{kT}$):

$$\begin{aligned} \bar{x}_{k+1} &= e^{AT}\bar{x}_k \\ \bar{y}_k &= C(\bar{\theta}_k)\bar{x}_k, \end{aligned} \tag{2}$$

is still a switched linear system, which would not be the case if the dynamics (i.e. the A matrix) was subject to switching as well (unless, e.g., θ switches only at the sampling times). Note that, under our assumptions on θ, the sequences $\{\bar{\theta}_k\}_{k=1}^{\infty}$ are arbitrary.

These models have been used extensively in the literature to capture the behavior of linear systems undergoing intermittent sensor failures or packet losses in the measurements path [12], [22], and a great deal of research activity has lately concerned the design of observers or estimators for such systems [4], [24].

It is usually desirable for a discretized system to conserve some properties of its continuous-time originator, and in this paper we study the conservation of observability after sampling. For unimodal systems (i.e., with $s = 1$), for which (1) reduces to the classical autonomous linear time-invariant system

$$\begin{aligned} \dot{x}_t &= Ax_t \\ y_t &= Cx_t, \end{aligned} \tag{3}$$

and (2) becomes

$$\begin{aligned} \bar{x}_{k+1} &= e^{AT}\bar{x}_k \\ \bar{y}_k &= C\bar{x}_k, \end{aligned} \tag{4}$$

the following result first appeared in [19]:

Theorem 1 (Kalman-Bertram Criterion): Let $\sigma(A)$ denote the spectrum of A. If (A, C) is an observable pair, then whenever the sampling period T satisfies, for all $\{\lambda, \lambda'\} \in \sigma(A) \times \sigma(A)$,

$$\lambda \neq \lambda' + \frac{(2\pi j)k}{T} \quad \forall\, k \in \mathbb{Z}\backslash\{0\}, \tag{5}$$

the discrete-time pair (e^{AT}, C) is observable. ◊

A proof can be found in [19], but the result can easily be established using the Popov-Belevitch-Hautus rank test (see, e.g., [8]). Further research on this subject has focused mainly on generalized hold functions [18], [21] (for controllability) and on robust sampling techniques [20]. However, to the best of our knowledge, no generalization to switched linear systems has ever been established. Perhaps one of the reasons this has been the case is that, until very recently, little was understood about observability in switched linear systems, partly because of the addition of the discrete variable θ. Therefore, we first need to specify exactly what we mean by "observability" in the switched setting, which is the objective of the next section, in which we recall some results on observability in switched linear systems. We will then, in Section III, establish some preliminary results. Finally, in Section IV, we present the main results of this paper, consisting in carrying over the Kalman-Bertram criterion to the switched linear system (1).

II. Observability in Switched Linear Systems

By *observability*, we mean the ability to infer the initial state from the measurements and switching signals under arbitrary switching signals. In other words, we assume the switching signals to be known, and we are concerned with inferring the initial states for all possible switching signals. Before proceeding to define our problem precisely, it is important that we point out that observability analysis for switched and hybrid systems has lately experienced a surge of interest. An important problem that has received a fair amount of attention is the unobserved switching case, where one is concerned with inferring the initial state without access to the switching signal (see, e.g., [2], [5], [28], [29] where more general classes of systems were studied). Note also the difference between our framework and that of *hybrid* linear systems, where the switching signal actually depends on the state trajectory, thus rendering observability more complex to establish (see, e.g., [7], [14], [25], [26] for discrete-time systems, and [6], [9], [11] for continuous-time systems). For instance, by contrast to the hybrid *piecewise affine* case [10], [25], where the mode in effect is a piecewise constant function of the state, observability under unknown modes has recently been shown to be decidable in autonomous switched linear systems [2].

A. Continuous-Time Systems

Let us recall the model

$$\begin{aligned} \dot{x}_t &= Ax_t \\ y_t &= C(\theta_t)x_t. \end{aligned} \tag{6}$$

The problem we consider is whether, for all switching signals θ, one can infer x_0 uniquely by observing the output y. In other words,

$$\forall\theta\forall x_0\forall x_0'\ x_0 \neq x_0' \Rightarrow y \neq y', \tag{7}$$

where y and y' are the outputs in (6) when the initial states are x_0 and x_0', respectively, and when the mode signals are both equal to θ. We then say y determines x_0 under known and arbitrary mode signals, and we have:

Proposition 1 ([13], [29]): The output signal y determines the initial state x_0 under known and arbitrary mode signals θ if and only if every pair $(A, C(i))$ is observable. ◊

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B. Discrete-Time Systems

Once again, we consider the general version of (2),

$$\begin{aligned} x_{k+1} &= Ax_k \\ y_k &= C(\theta_k)x_k, \end{aligned} \tag{8}$$

where the A matrix is not necessarily a matrix exponential, and we ask whether there exists an observation horizon N such that $y_1, \ldots, y_N$ determine the initial state x_1 under known and arbitrary mode sequences, or

$$\forall\theta\forall x_1\forall x_1' \;\; x_1 \neq x_1' \Rightarrow Y_N \neq Y_N', \tag{9}$$

where $Y_N \triangleq (y_1^T \cdots y_N^T)^T$. We have:

Definition 1 (Pathwise Observability [1]): The switched linear system (8) is pathwise observable if there exists an integer N such that the successive measurements $y_1, \ldots, y_N$ determine the initial state x_1 under known and arbitrary mode sequences $\{\theta_k\}_{k=1}^{\infty}$. $\diamondsuit$

Let us define a path θ of length N as a string of length N whose elements take values in $\{1, \ldots, s\}$, and lets us denote its length by $|\theta| = N$. We can now define the observability matrix $\mathcal{O}(\theta)$ of a path θ as

$$\mathcal{O}(\theta) \triangleq \begin{pmatrix} C(\theta_1) \\ \vdots \\ C(\theta_N)A^{N-1} \end{pmatrix}. \tag{10}$$

A straightforward characterization of pathwise observability is then obtained as:

Proposition 2: The switched linear system (8) is pathwise observable if and only if there exists an integer N such that all paths θ of length N are observable, i.e. satisfy $\text{rank}(\mathcal{O}(\theta)) = n$. We refer to the smallest such integer as the index of pathwise observability. $\diamondsuit$

In the remainder of the paper, we will often abuse our notation and refer to a pathwise observable set of pairs $\{(A, C(1)), \ldots, (A, C(s))\}$ rather than to a pathwise observable switched linear system. Pathwise observability has recently been shown to be decidable in [1], [17], where the indexes of pathwise observability were shown to be bounded by numbers depending only on n and s.

III. Preliminary Results

In this section, we focus on the discrete-time system (8), and establish a sufficient condition on the individual pairs $(A, C(i))$ for the set $\{(A, C(1)), \ldots, (A, C(s))\}$ to be pathwise observable. More precisely, we show the existence of an integer L such that if the pairs $(A^l, C(i))$ are observable for all $l \leq L$, then the set of pairs $\{(A, C(1)), \ldots, (A, C(s))\}$ is pathwise observable. The main result is an upper bound on such L that is independent of the system. The idea is that if a pair $(A^l, C(i))$ is observable, $l \in \mathbb{N}$, then whenever $\theta_{a+lk} = i$ for $k = 0, \ldots, n-1$ and for some integer a, i.e., whenever some mode i occurs n times in θ at constant interval l, $\mathcal{O}(\theta)$ will contain the following matrix as a submatrix:

$$\begin{pmatrix} C(i) \\ C(i)A^l \\ \vdots \\ C(i)(A^l)^{n-1} \end{pmatrix} A^{a-1}, \tag{11}$$

which has rank n if A is invertible, and therefore ensures that $\text{rank}(\mathcal{O}(\theta)) = n$. Note that this would not be the case if there were switching among different A-matrices as well. In that case, the matrix in (11) would, in general, still exhibit coupling with modes other than i. What we thus want to show is that whenever a pair $(A^l, C(i))$ is observable for all modes i and all l smaller than a certain number, then the system is pathwise observable. This implies the possibility to assert that, in every path of at least a certain length $\mathcal{W}$, some mode i has to occur n equally separated times. It turns out that a solution to this problem is provided in a branch of combinatorial analysis referred to as *Ramsey Theory* [16]. Indeed, we wish to capitalize on the fact that any mode sequence has to exhibit certain regularity properties as long as it is long enough, which is a type of statement that falls precisely under the domain of Ramsey theory, whose main assertion is that complete disorder is an impossibility and that the appearance of disorder is really a matter of scale. As it turns out, our question finds its answer in van der Waerden's Theorem [27] (in its finite version), which is one of the central results of Ramsey theory:

Theorem 2 (van der Waerden [27]): For every pair of positive integers n and s, there exists a positive integer N such that whenever the set $\{1, \ldots, N\}$ is partitioned into s subsets $S_1, \ldots, S_s$, at least one of the subsets contains an arithmetic progression of length n.[1] $\diamondsuit$

The smallest such integers, denoted by $\mathcal{W}(n, s)$, are usually referred to as the van der Waerden numbers. It is indeed easy to see how the solution to our problem follows from Theorem 2 by simply taking every S_i to be the set of times at which mode i occurs in θ. In other words, ignoring the trivial case $n = 1$ and assuming $n \geq 2$ throughout the remainder of this paper, we get:

Corollary 1: Let θ be a path assuming values in $\{1, \ldots, s\}$. If $|\theta| \geq \mathcal{W}(n, s)$, then there exist an integer $i \in \{1, \ldots, s\}$ and two positive integers $a \leq |\theta| - n + 1$ and $l < |\theta|/(n-1)$ such that $\theta_{a+lk} = i$ for every $k = 0, \ldots, n-1$. $\diamondsuit$

Proof: Let $S_i = \{k \in \{1, \ldots, |\theta|\} \mid \theta_k = i\}$ for all $i \in \{1, \ldots, s\}$. Clearly, $S_1, \ldots, S_s$ is a partition of $\{1, \ldots, |\theta|\}$. By Theorem 2, since $|\theta| \geq \mathcal{W}(n, s)$, some S_i contains an arithmetic progression of length n. In other words, there exist two positive integers a and l such that $a + lk \in S_i$, and therefore $\theta_{a+lk} = i$, for $k = 0, \ldots, n-1$. Finally, $l < |\theta|/(n-1)$ because $l(n-1) < a + l(n-1) \leq |\theta|$. $\square$

Before establishing the main result of this section, which is a direct consequence of Corollary 1, we define, for $n \geq 2$,

$$\mathcal{W}'(n, s) \triangleq \left\lceil \frac{\mathcal{W}(n, s)}{n-1} \right\rceil - 1, \tag{12}$$

where $\lceil \cdot \rceil$ denotes the ceiling function (i.e. $\lceil \alpha \rceil = \min\{i \in \mathbb{N} \mid \alpha \leq i\}$).

Theorem 3: If A is invertible, and if $(A^l, C(i))$ is an observable pair for all $i \in \{1, \ldots, s\}$ and all positive integers $l \leq \mathcal{W}'(n, s)$, then $\{(A, C(1)), \ldots, (A, C(s))\}$ is pathwise observable with an index no larger than $\mathcal{W}(n, s)$. $\diamondsuit$

Proof: Let θ be any path of length $\mathcal{W}(n, s)$. By Corollary 1, there exist an integer $i \in \{1, \ldots, s\}$ and two integers $a \leq |\theta| - n + 1$ and $l < \mathcal{W}(n, s)/(n-1)$ such that $\theta_{a+lk} = i$ for $k = 0, \ldots, n-1$. Therefore, the submatrix of $\mathcal{O}(\theta)$ consisting of the blocks $a + lk$ of $\mathcal{O}(\theta)$, $k = 0, \ldots, n-1$, which can be expressed as

$$\begin{pmatrix} C(i) \\ C(i)A^l \\ \vdots \\ C(i)(A^l)^{n-1} \end{pmatrix} A^{a-1},$$

has rank n since A (and therefore A^{a-1}) is invertible, and since the pair $(A^l, C(i))$ is observable, because $l \leq \mathcal{W}'(n, s)$. Therefore $\mathcal{O}(\theta)$ has rank n, which completes the proof. $\square$

[1] An arithmetic progression is a sequence of positive integers such that the difference between successive terms is constant.

Remarks:

- These conditions are *not* necessary. For instance, the set of pairs $\{(A, C(1)), (A, C(2))\}$, where:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{cases} C(1) = (1\ 0) \\ C(2) = (2\ 0) \end{cases} \tag{13}$$

  is pathwise observable with index 2, but while $\mathcal{W}'(2, 2) = 2$, neither $(A^2, C(1))$ nor $(A^2, C(2))$ is an observable pair.
- The index of pathwise observability in Theorem 3 is not necessarily equal to $\mathcal{W}(n, s)$. For instance, the set of pairs $\{(A, C(1)), (A, C(2))\}$, where:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{cases} C(1) = (1\ 1) \\ C(2) = (1\ 2) \end{cases} \tag{14}$$

  satisfies the assumptions of Theorem 3, but is pathwise observable with index 2, while $\mathcal{W}(2, 2) = 3$. $\Diamond$

The numbers $\mathcal{W}(n, s)$ are referred to as the van der Waerden numbers. Unfortunately, the only van der Waerden numbers known exactly fit in Table I. Only upper bounds are known for the rest. Those bounds grow at an enormous rate, which limits the computational applicability of Theorem 3, despite sporadic improvements on some sequences of bounds (see, e.g., [15], [23]). However, Theorem 3 is fortunately all we need in order to show the more practical results of the next section concerning sampled systems.

s \ n	2	3	4	5	$\cdots$	n
1	2	3	4	5	$\cdots$	n
2	3	9	35	178		
3	4	27				
4	5	76				
$\vdots$	$\vdots$					
s	$s+1$					

TABLE I
KNOWN VALUES OF $\mathcal{W}(n, s)$

IV. Non-Pathological Sampling

What we wish to establish here is whether observability of every pair $(A, C(i))$ implies pathwise observability of the set of pairs $\{(e^{AT}, C(1)), \ldots, (e^{AT}, C(s))\}$. The following theorem follows almost directly from Theorems 1 and 3.

Theorem 4: Let $\sigma(A)$ denote the spectrum of A. If $(A, C(i))$ is an observable pair for all $i \in \{1, \ldots, s\}$, then whenever the sampling period T satisfies, for all $\{\lambda, \lambda'\} \in \sigma(A) \times \sigma(A)$,

$$\lambda \neq \lambda' + \frac{(2\pi j)k}{lT} \quad \forall\, k \in \mathbb{Z}\backslash\{0\},\ \forall\, l \leq \mathcal{W}'(n, s), \tag{15}$$

the set of pairs $\{(e^{AT}, C(1)), \ldots, (e^{AT}, C(s))\}$ is pathwise observable with an index no larger than $\mathcal{W}(n, s)$. $\Diamond$

Proof: First, since AT commutes with itself and l is an integer, $e^{AlT} = (e^{AT})^l$. Therefore, by Theorem 1, (15) implies that the pair $((e^{AT})^l, C(i))$ is observable for all $i \in \{1, \ldots, s\}$ and all $l \leq \mathcal{W}'(n, s)$. Moreover, e^{AT} being a matrix exponential, it is an invertible matrix. The result then follows from Theorem 3. $\square$

Now, even though some numbers $\mathcal{W}(n, s)$ may be unknown, they are finite, as discussed earlier. The following corollary follows:

Corollary 2: If $(A, C(i))$ is an observable pair for all $i \in \{1, \ldots, s\}$, then the set of pairs $\{(e^{AT}, C(1)), \ldots, (e^{AT}, C(s))\}$ of the discretized system is pathwise observable for all but a countable number of sampling periods T. $\Diamond$

Proof: If every eigenvalue of A is real, then (15) always holds and the set is pathwise observable for all $T > 0$. Otherwise, defining the set F of frequencies as

$$F \triangleq \left\{ \frac{|\mathrm{Im}(\lambda_i) - \mathrm{Im}(\lambda_j)|}{2\pi}\ |\lambda_i \neq \lambda_j \in \sigma(A),\ \mathrm{Re}(\lambda_i) = \mathrm{Re}(\lambda_j) \right\}, \tag{16}$$

we get that the set of pathological sampling periods is, by (15), a subset of

$$\left\{ k/(fl),\ k \in \mathbb{N}^*,\ f \in F,\ l \leq \mathcal{W}'(n, s) \right\}, \tag{17}$$

which is countable. Hence the result. $\square$

Finally, note that what needs to be avoided in Theorem 1 is the interaction between the natural frequencies of the linear system and the sampling frequency. It is therefore easily established that, under the same conditions, conservation of observability is guaranteed when the sampling period T is small enough. The importance of this observation is actually further motivated by robust control problems, as pointed out in [20]. The following theorem extends this result to switched linear systems (1-2):

Theorem 5: If $(A, C(i))$ is an observable pair for all $i \in \{1, \ldots, s\}$, then there exists a positive real number T such that whenever $0 < \tau < T$, the set of pairs $\{(e^{A\tau}, C(1)), \ldots, (e^{A\tau}, C(s))\}$ of the discretized system is pathwise observable with an index smaller than or equal to $\mathcal{W}(n, s)$. $\Diamond$

Proof: Clearly,

$$T = \frac{1}{\max(F)\mathcal{W}'(n, s)}, \tag{18}$$

which is the smallest element of the set in (17), works. $\square$

What Theorem 5 really says is that if all continuous-time components are observable, then the sampled-data system is always observable provided it is sampled fast enough, the key result being that the upper bounds on the sampling period and on the number of observations are independent of the system and of the switching signal.

V. Conclusion

In this paper, we have introduced an application of Ramsey Theory to the study of pathwise observability of a class of switched linear systems. The result presented has enabled, for the first time, the study of the conservation of observability properties after the introduction of sampling in switched systems, which has resulted in a criterion very similar to the well-known Kalman-Bertram criterion. It is noteworthy that all of these results can be dualized to the study of the effect of sampling on controllability for systems with *actuator* failures, although with less significance due to the inexistence, in practice, of a perfect impulsive hold, the dual of sampling. For details, we refer the reader to [3].

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