Optimal Motion Primitives for Multi-UAV Convoy Protection

A. Rahmani, X. C. Ding and M. Egerstedt

Abstract— **In this paper we study the problem of controlling a number of Unmanned Aerial Vehicles (UAVs) to provide convoy protection to a group of ground vehicles. The UAVs are modeled as Dubins vehicles flying at a constant altitude with bounded turning radius. This paper first presents time-optimal paths for providing convoy protection to static ground vehicles. Then this paper addresses paths and control strategies to provide convoy protection to ground vehicles moving on a straight line. Minimum numbers of UAVs required to provide perpetual convoy protection for both cases are derived.**

I. Introduction

Coordination of heterogeneous unmanned vehicles is one of the canonical problems and the key to success of a number of proposed unmanned missions. We explore this coordination in the framework of providing ground convoy protection for a group of UGVs (Unmanned Ground Vehicles), using a group of dynamically more capable UAVs (Unmanned Aerial Vehicles). From the early days airplanes have been used to provide close air support or simply large-scale area surveillance to the ground convoys moving in unknown and potentially dangerous environment. Wide spread use of the unmanned vehicles to conduct tasks in inherently dangerous environments arises the need for efficient UAV and UGV coordination algorithms ([9], [12]). Our goal is to provide an optimal control strategy for a single UAV, as well as optimal path planning for multiple UAVs in order to provide successful convoy protection.

In this paper the UAVs are modeled as Dubins vehicles [4] flying at a constant altitude. Due to kinematic constraints of the UAVs and limited ranges of sensors on-board the UAVs, it may be impossible to provide coverage to the ground vehicles with a single UAV. In this case, the problem of interest becomes that of providing an optimal path for a single UAV so that it can monitor the ground vehicle for the longest time, and coordinating multiple UAVs so that the ground vehicles are visible to at least one UAV at any given time. Figure 1 visualizes this concept.

A Dubins vehicle is a planar vehicle with bounded turning radius and constant forward speed. L.E. Dubins was the first to give a characterization of time-optimal trajectories for such a vehicle using geometric methods [4]. Shortest-path problems for Dubins vehicles have been since studied extensively (see [5], [10] for example). Walsh *et al.* [13] found optimal paths for an airplane on $SE(2)$. Dubins vehicle has been used as a simplified model to describe planar motion of UAVs in [10], [2]. Chitsaz *et al.* [2] extend

Authors are with the School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332; Emails: {*amir.rahmani,ding,magnus*}*@ece.gatech.edu*. This work was sponsored by a grant from Rockwell Collins Advanced Technology Center.

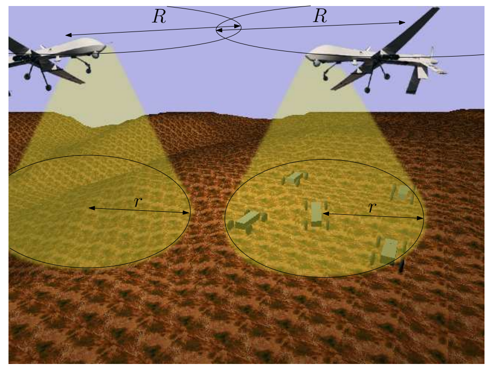

Fig. 1. UAVs providing convoy protection to UGVs. The UAV is assumed to be kinematically restricted by its minimum turning radius R. The sensors on-board the UAV also have limited range and assumed to be able to observe a disk of radius r on the ground.

the Dubins' model from $SE(2)$ to $SE(2) \times \mathbb{R}$ to account for altitude changes and gave a characterization of the time-optimal trajectories for this model based on the final altitude.

Motion primitives are often used to produce optimal trajectories for Dubins vehicle (see [1], [6] for example). In point to point minimum time transfer, it has been shown that the optimal solutions are curves consisting of only three motion primitives: line-segment and circular arcs turning maximumly to the left and to the right (see [4], [8], [11]). The optimal paths for the point-to-point minimum-time transfer problem are characterized by sequences of these three motion primitives. This paper shows that in the case of stationary convoy protection, optimal paths are characterized by sequence of only two motion primitives (maximumly turning left or right) and do not include a line-segment.

In this paper we address the problem of coordinated convoy protection for both stationary UGVs and UGVs moving on a straight line. We first consider the convoy as stationary and find the optimal path for a single UAV to maximize the coverage time, we then show how to coordinate a group of UAVs to provide continuous coverage of the convoy. Next, we focus on moving convoy and introduce a control strategy that guarantees periodical meet-up with the convoy. We introduce a bound on the speed of the convoy that enables one UAV to provide continuous convoy protection. In the case when speed of the convoy is outside of this bound we find the minimum number of UAVs to provide continuous protection to the convoy.

The rest of the paper is organized as follows. Section II formulates the problem. Section III address the problem in the case of stationary convoy. Section IV address the problem when the convoy is moving on a straight line. Section V concludes the paper.

II. PROBLEM FORMULATION

In this paper the UAVs are modeled as Dubins vehicles flying at constant altitude with unit speed and minimum turn radius of R. [1] Therefore, we can write the kinematics of the UAV as

$$\begin{cases} \dot{x} = \cos(\theta) \\ \dot{y} = \sin(\theta) \\ \dot{\theta} = \omega \end{cases} \tag{1}$$

where x and y are the position of the UAV in the x-y plane at the altitude the UAV is flying, and ω is the angular velocity of the vehicle. The angular velocity is bounded by the inverse of the minimum turn radius R of the vehicle, i.e., $\omega \in [-\frac{1}{R}, \frac{1}{R}]$. Let the state of the system be defined as $q(t) = [x(t), y(t), \theta(t)]^T$.

Definition 2.1: *Successful convoy protection* is achieved when the centroid of the UGVs is visible to at least one of the UAVs at any time, assuming UGVs travel relatively close to each other.

We assume the cameras on-board the UAVs can monitor a disk of radius r on the ground (see Figure 1 for an illustration of this problem). The disk of observation certainly depends on the altitude of the UAV, but to ensure quality of observation and successful protection, cameras or sensors on-board the UAV has narrower field of view than the UAV's turning radius in many cases, especially for cameras and sensors that carry out a specific objective such as executing computer vision algorithms (higher resolution in the expense of narrower field of view). We further assume the camera is gimbaled and the area viewed by the UAV remains a disk of radius r centered at projection of the position of the UAV on the ground regardless of the UAV's bank angle.

If $R > r$ and the UGVs are stationary, then a single UAV is not capable of providing convoy protection to the ground vehicles indefinitely and a control strategy is needed to optimize the time in which convoy protection is achieved. Note that in case of static convoy if $R \leq r$, then the convoy protection problem is solved by using a single UAV flying on a circular path of radius R with the center being the ground vehicles.

In this paper we consider the problem of controlling and coordinating the UAVs to provide convoy protection for both stationary UGVs and UGVs moving on a straight line. In both cases we assume that $R > r$. We denote the *convoy circle* as a disk of radius r around the centroid of the UGVs. Convoy protection is achieved if at least one UAV is present inside the convoy circle at any time. Because of the kinematic constraint (turning radius of the UAVs), the UAVs are required to be coordinated so that they collectively provide constant convoy protection despite flying in and out of the convoy circle.

[1]The unit speed assumption is justified by defining the unit length as the speed of the UAVs.

III. OPTIMAL PROTECTION OF STATIONARY CONVOYS

In this section we find optimal paths for a single UAV to provide maximum protection for some stationary convoy. We determine both the time-optimal path for a single UAV starting at a fixed initial condition and optimal paths if the UAV is allowed to pick the position and heading when entering the convoy circle. We also provide optimal paths for multiple UAVs to provide convoy protection for all time. The minimum number of UAVs required to achieve this task is also obtained.

A. Optimal path for a single UAV

First consider the problem of using one UAV to provide convoy protection to some stationary UGVs for maximum amount of time. Fix the origin of the x-y plane at the centroid of the UGVs. This problem can be considered as a maximum-time optimal control problem with state constraint $x^2 + y^2 - r^2 \leq 0$ and input constraint $|w| \leq \frac{1}{R}$. Furthermore, it can be assumed that the UAV starts at a point on the state constraint boundary (convoy circle). This assumption do not limit generality of the result since if the UAV starts inside the convoy circle, we can use Bellman's principle to obtain the remaining optimal path for the UAV by integrating backwards in time. Finally, it is useful to impose a terminal manifold constraint since the optimal solution always involves the terminal state being on the boundary of the state constraint set (exiting the circle).

The optimal control problem can be defined as:

Problem 3.1:

$$\min_{\omega(t)} J = \int_0^T -1 dt, \tag{2}$$

subject to the dynamics of (1), and the input constraint

$$-\frac{1}{R} \leq \omega(t) \leq \frac{1}{R}, \tag{3}$$

the state constraint

$$\begin{aligned} x(t)^2 + y(t)^2 - r^2 &\leq 0 \\ x(0)^2 + y(0)^2 - r^2 &= 0 \end{aligned} \tag{4}$$

and the terminal manifold constraint

$$M(q(T)) = x(T)^2 + y(T)^2 - r^2 = 0. \tag{5}$$

For simplicity of notation, we assume that all angles are taken modulus 2π.

To address the state constraint we use an auxiliary state τ

$$\dot{\tau}(t) = (x^2 + y^2 - r^2)^2 \, \xi(x^2 + y^2 - r^2), \tag{6}$$

where $\xi(.)$ is a Heaviside function

$$\xi(x^2 + y^2 - r^2) = \begin{cases} 0 : x^2 + y^2 - r^2 \leq 0 \\ 1 : \text{otherwise.} \end{cases} \tag{7}$$

The state can be augmented as $\bar{q}(t) = [q(t), \tau(t)]^T$. Let us require that $\tau(0) = 0$ and $\tau(T) = 0$. This enforces the constraint since being outside of the constraint produces a positive derivative of $\tau(t)$ and thus the terminal condition is violated. When there is no ambiguity, we assume that the state constraint is satisfied and we still call $q(t)$ the state trajectory.

The Hamiltonian for this optimal control problem can be written as:

$$\begin{aligned}\mathcal{H} \;&=\; -1 + \lambda_1 \cos\theta + \lambda_2 \sin\theta + \lambda_3 \omega + \\ &\quad \lambda_4 (x^2 + y^2 - r^2)^2 \xi(x^2 + y^2 - r^2), \end{aligned} \tag{8}$$

where $\lambda = [\lambda_1, \cdots, \lambda_4]^T$ are the trajectories of the costates. The necessary optimality condition from the Pontryagin's minimum principle states that

$$\begin{aligned}\mathcal{H}(\bar{q}^*(t), \lambda^*(t), \omega^*(t), t) \leq \mathcal{H}(\bar{q}^*(t), \lambda^*(t), \omega(t), t), \\ \forall \omega \in [-\frac{1}{R}, \frac{1}{R}], t \in [0, T]. \end{aligned} \tag{9}$$

Using the necessary optimality condition, and substituting the Hamiltonian from (8), one can see that the optimal controller is a function of the costate $\lambda_3(t)$ as:

$$\omega^*(t) = \begin{cases} -\frac{1}{R} & : \lambda_3^*(t) > 0 \\ \frac{1}{R} & : \lambda_3^*(t) < 0 \\ \text{undetermined} & : \lambda_3^*(t) = 0 \end{cases} \tag{10}$$

Thus it can be seen that when $\lambda_3^*(t) > 0$, the optimal control is maximum turning right, and when $\lambda_3^*(t) < 0$, the optimal control is maximum turning left. Hence, the optimal control trajectory is in the form of bang-bang control. It should be noted that when $\lambda_3^*(t) = 0$ for a finite time interval, then any control $\omega(t) \in [-\frac{1}{R}, \frac{1}{R}]$ satisfies (9) and this case is referred to as a singular condition (see [7]). For a singular condition to occur, it is necessary that there exist a time t such that $\lambda_3(t) = 0$ and $\dot{\lambda}_3(t) = 0$. For Dubins vehicles with dynamics specified in equation (1), singular intervals result in line segments as part of the optimal path. Line segments are usually part of the optimal paths for shortest-path (or minimum-time) Dubins vehicle problems. However, later in this section, we will show that line segments can not be part of the optimal path for Problem 3.1, and as a result the optimal control always switches between $\omega^*(t) = -\frac{1}{R}$ and $\omega^*(t) = \frac{1}{R}$.

Definition 3.1: For a state trajectory $q(t), t \in [0, T]$ satisfying the state constraint (4), if the costate trajectory and corresponding input satisfies the control strategy (10), then $q(t)$ is referred to as a *Candidate Optimal Trajectory* (COT).

Pontryagin's minimum principle states that being a COT is a necessary condition for being the optimal solution.

Assuming a trajectory $q(t)$ is a COT, it can be shown that the the costate $\lambda(t)$ satisfying the necessary optimality condition (9) can be uniquely determined. Given a terminal state $q(T)$, denote $\theta_T = \theta(T)$ and define the angle $\psi_T = \operatorname{atan2}(y(T), x(T))$. Using the terminal manifold and transversality condition we can solve the costate equations backwards, and obtain the following lemma (for detailed analysis see our technical report [3]):

Lemma 3.1: For any terminal state $q(T)$, a unique COT $q(t)$ and its corresponding input and costate history can be reconstructed. Furthermore, if $\theta_T \neq \psi_T$, then $q(t)$ is composed of maximumly turning right or left curves, or combination of both at some switching times. If $\theta_T = \psi_T$, then $q(t)$ is a line that goes through the origin.

A direct consequence of the Lemma 3.1 is that, an optimal trajectory can not contain both a circular arc and a line segment. Hence, the optimal control law can not exhibit switching from turning to going straight or vice versa. We denote the state when the controller switches from maximum turning left to right or from maximum turning right to left as a switching point.

A COT may contain uncountably many switching points, and obtaining their location can be a difficult task. However, the structure of this problem (state and input constraint as well as the terminal manifold) allows a very powerful theorem that characterizes the optimal trajectory (the proof of this theorem is contained in [3]).

Theorem 3.2: For the optimal control problem (3.1), the optimal trajectory of the UAV can not contain more than one switching point.

Since the optimal trajectory can only switch at most once, the number of COT that can be optimal is drastically reduced. It is then possible to construct optimal curves for any initial condition. Similar to many other Dubins car path planning approaches (see [4], [11], [8], [2] for example), we can define 2 motion primitives $\{\mathbf{L}, \mathbf{R}\}$, where $\mathbf{L}$ and $\mathbf{R}$ motion primitives turn the car maximumly to the left and right, respectively. For this problem, there is only one case where a straight line is a COT (initial condition $q(0) = [-r, 0, 0]^T$ and rotation of this point by any angle). However, in this case, there are 2 other COTs that are both longer in length and involve one-switching. Therefore unlike the Dubins vehicle shortest-path problem ([8]), there is no straight line motion primitive since it can not be optimal. Furthermore, since the optimal trajectory only switch once, there are only 4 possible sequences of the $\{\mathbf{L}, \mathbf{R}\}$ motion primitives, namely $\{\mathbf{L}, \mathbf{R}, \mathbf{LR}, \mathbf{RL}\}$, where $\mathbf{LR}$ stands for turning left then right and $\mathbf{RL}$ for turning right then left.

The problem of finding the optimal path is then reduced to one of finding whether or not the optimal trajectory contains a switching point and location of the optimal switching point. By studying the state and costate trajectory corresponding to a COT, one can obtain the following property for the optimal switching point (for detailed proof see [3]).

Theorem 3.3: For any initial condition $q(0)$, the optimal switching point of the optimal trajectory lies on the line passing through the origin and the exit point.

The above theorem is useful because it completely determines the optimal control law. A set of optimal paths for initial conditions with heading $\theta(0) = \frac{\pi}{2}$ are shown in Figure 2. The optimal control law is determined as follows. Assuming $\theta(0) = \frac{\pi}{2}$, the optimal motion sequence depends on the initial position. If $x(0) \in (-r, -\frac{r^2}{R}]$, then **R** is optimal. If $x(0) \in (-\frac{r^2}{R}, 0]$, then **LR** is optimal. If $x(0) \in [0, \frac{r^2}{R})$, then **RL** is optimal. If $x(0) \in [\frac{r^2}{R}, r)$, then **L** is optimal. Figure 2 also shows the optimal switching surface on which switchings are optimal. The switching point is determined by theorem 3.3 since the exit point (obtained by projecting the state forward), the center of the convoy circle and the switching point must be on the same line. If the initial heading of the UAV is not $\frac{\pi}{2}$, then one can always rotate the state of the system around the convoy circle until this is true.

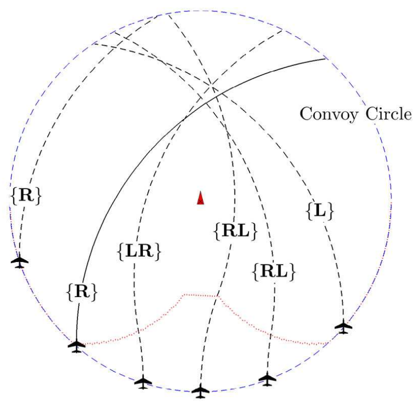

Fig. 2. A number of optimal state trajectories with initial heading $\frac{\pi}{2}$. The optimal switching points are plotted together to form the optimal switching surface. In this case, $R = 1.5r$. The dashed paths corresponds to optimal paths. The solid path corresponds to one initial position and this path is longer than all other optimal paths with the same initial heading.

It can be shown that the optimal entry states $q(0) = [-r, 0, \arcsin(\frac{r}{R})]^T$ and $q(0) = [-r, 0, -\arcsin(\frac{r}{R})]^T$ (and their rotations around the origin) produce the longest optimal paths (length of all these paths are equal and maximal). These points are then the optimal entry points and the optimal paths with these initial conditions are referred to as the *globally optimal paths*. A globally optimal path characterizes the maximum possible time a UAV can stay inside the convoy circle. The set of all optimal entry points is denoted as $\mathbb{E}^*$ and it can be described as:

$$\mathbb{E}^* = \{q = [-r\cos(\theta), -r\sin(\theta), \pm\arcsin(\frac{r}{R}) + \theta]^T, \quad \theta \in [-\pi, \pi]\}. \tag{11}$$

An easy way to recognize a globally optimal path is to observe the fact that the entry point of a globally optimal path is always on the same line as the origin and the exit point (as result of theorem 3.2).

B. Multi-UAV convoy protection

Now we consider the problem of coordinating multiple UAVs to achieve convoy protection for a set of UGVs. Due to kinematic constraint of the UAVs ($r < R$), it is impossible for one UAV to provide complete convoy protection for all time. In this situation, multi-UAV coordination is required in order to successful carry out convoy protection. It should be noted that a globally optimal path not only specifies an optimal path inside the convoy circle, but also a path for a single UAV to come back to the convoy circle without changing direction. As shown in Figure 3, the path constitutes a circle of radius R and part of the path is the globally optimal path inside the convoy circle. There are many similar optimal paths, and they are referred to as optimal convoy protection paths. These paths maximize the ratio of time inside the convoy circle over outside of the convoy circle, since it is the quickest path to come back to the circle, always reenter optimally and repeat as a limit-cycle. All of the globally

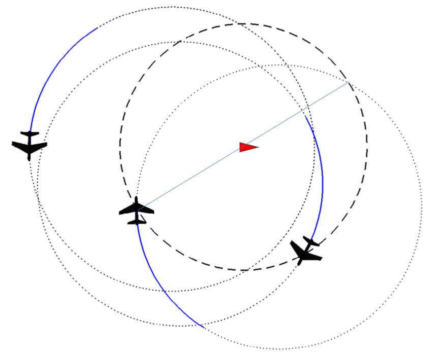

Fig. 3. Three optimal convoy protection paths are shown. They maximize the time spent inside the convoy circle over the time outside of the convoy circle. The smaller dashed circle is the convoy circle, and the larger dotted circles are optimal convoy protection paths. The solid circle is the past trajectories of the UAV.

optimal paths have the same length as function of r and R. Given this length, we can compute the minimum number of UAVs required to sustain convoy protection for ground convoy for all time. This results in the following corollary:

Corollary 3.4: Given the convoy circle of radius r for the UGVs and maximum turning radius R for the UAVs, the minimum number of UAVs needed to provide convoy protection for all time is:

$$N = \left\lceil \frac{\pi}{\arcsin(\frac{r}{R})} \right\rceil, \tag{12}$$

where $\lceil \cdot \rceil$ denotes the ceiling function.

Assume that there is N UAVs and they can start at an optimal initial condition $q^*(0) \in \mathbb{E}^*$, the UAVs need to space themselves evenly in terms of the time entering the convoy circle. This can be achieved by slowing down and speeding up with respect to the other UAVs so that the i-th UAV enters the convoy circle at time $\frac{2\pi R}{N} i$. This strategy is possible since the optimal paths derived for this problem remain the same for UAVs of any speed (instead of unit speed).

IV. Moving Convoy protection strategies

In this section we focus on convoy protection strategy for moving UGVs. Again, we assume that the location of the UGVs are represented by their centroid as a point. Instead of being static, here we consider that this point is moving in a constant direction with a constant and bounded speed. Denote the speed of the UGVs as V_G. The UGVs are assumed to be moving in a constant heading of angle ϕ.

For the UAVs, we again normalize their speed to 1. Hence the UAVs follow the dynamics in equation (1) and their states are denoted by $[x, y, \theta]^T$. The UAVs are assumed to be capable of flying with faster speed than the UGVs (this agrees with current state of technologies in terms of speed of ground robots versus UAVs). Hence, we assume that $V_G \leq 1$.

Now, we propose a control strategy with a corresponding lower bound $V_G^\star$ so that if the speed of the UGVs is in this bound ($V_G \in [V_G^\star, 1]$), then one UAV is guaranteed to provide convoy protection for all time.

Inspired by the motion primitives defined in the static convoy protection problem, we fix the motion of the UAV to a sequence of maximally left and right turns, i.e., $\mathcal{M} = \{\mathbf{L}, \mathbf{R}, \mathbf{L}, \mathbf{R}, \cdots\}$ or $\mathcal{M} = \{\mathbf{R}, \mathbf{L}, \mathbf{R}, \mathbf{L}, \cdots\}$. We assume that the UAV and UGVs are initially on top of each other; i.e., the initial x-y coordinates of the UGVs is $[x(0), y(0)]^T$. Now, we define the angle between the heading of the UGVs and initial heading of the UAV as β. Hence, $\beta = \phi - \theta(0)$. Again, to simplify notations, we assume that all angles are taken modulus 2π.

We switch the motion primitive between $\mathbf{L}$ and $\mathbf{R}$ every time the paths of UAV and UGVs intersect. With this control strategy, the path of the UAV and the UGVs intersect every time the UAV flies for a circular arc of angle 2β. An example of the trajectory of the UAV and the UGVs are shown in Figure 4. The initial motion primitive of the motion sequence $\mathcal{M}$ depends on β. If $\beta \in [0, \pi)$, then the path of the UGVs is to the left of the initial heading of the UAV and the first motion primitive is $\mathbf{L}$, otherwise, the first motion primitive is $\mathbf{R}$.

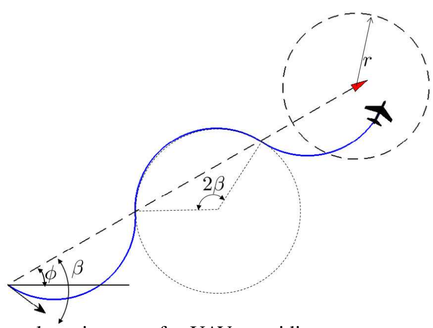

Fig. 4. Example trajectory of a UAV providing convoy protection for the UGVs with the proposed control strategy. The solid curve is the path of the UAV. The dashed line is the path of the UGVs. The dashed circle is the convoy circle. The angle of the circular arc for each motion primitive is 2β. In this case, $\beta \in [0, \frac{\pi}{2}]$.

In the following discussion, we focus on the case that $\beta \in [0, \pi)$ and the motion sequence is $\mathcal{M} = \{\mathbf{L}, \mathbf{R}, \mathbf{L}, \mathbf{R}, \cdots\}$, because if $\beta \in (-\pi, 0]$, then the path of the UAV is symmetric to the path corresponding to the angle of $-\beta$.

It is desirable to control the UAV to meet the UGVs periodically. This goal can be achieved by carefully choosing the initial heading of the UAV based on the speed of the UGVs. The following lemma relates the speed of the UGVs with the desired initial heading of the UAV.

Lemma 4.1: Assume that the UGVs move with constant speed V_G and heading ϕ, and the UAV starts at the same position as the UGVs with the initial heading $\theta(0)$. $\beta = \phi - \theta(0)$. Assume that $\beta \in [0, \pi)$ and hence $\mathcal{M} = \{\mathbf{L}, \mathbf{R}, \mathbf{L}, \mathbf{R}, \cdots\}$. Then if the UAV executes the proposed control strategy, and $V_G = \frac{\sin(\beta)}{\beta}$, then the UAV and the UGVs meet at the end of each motion primitive.

Proof: If the UAV executes the proposed control strategy, then it flies for a circular arc of angle 2β for each motion primitive in $\mathcal{M}$. Assume that the UAV meet with the UGVs at the end of each motion primitive. For each motion primitive, the UAV travels for a distance of $2R\beta$ and the UGVs travel for a distance of $2R\sin(\beta)$. Since the UAV is unit speed, we have $V_G 2R\beta = 2R\sin(\beta)$, and therefore $V_G = \frac{\sin(\beta)}{\beta}$. ■

Note that, if the UGVs travel with the same speed as the UAV, i.e. $V_G = 1$, then from Lemma 4.1, we have $\beta = 0$. In this case, the UAV will fly exactly on top of the UGVs.

Using Lemma 4.1, we can obtain the lower bound for the speed of UGVs to achieve perpetual convoy protection.

Theorem 4.2: Using the proposed control strategy, one UAV is sufficient to provide continuous convoy protection for all time, if V_G is bounded below by $V_G^\star$, where

$$V_G^\star = \frac{\sqrt{2rR - r^2}}{R \arccos(1 - \frac{r}{R})}. \quad (13)$$

Proof: Without loss of generality, we assume that the UAV and UGVs start at the origin and the heading of the UGVs is $\phi = 0$. If $\phi \neq 0$, We can always rotate the path of the UGVs so that $\phi = 0$. We first focus on the first motion primitive. Let us look at the positions of the UGVs and UAV after flying a circular arc of angle 2γ where $\gamma \in [0, \beta]$. Denote the x-y coordinates of the UGVs and the UAV as p_c and p_a, respectively. Note that p_c and p_a are both functions of γ, and they can be obtained after some algebra and trigonometry as

$$p_a(\gamma) = \begin{bmatrix} 2R\sin(\gamma)\cos(\beta - \gamma) \\ -2R\sin(\gamma)\sin(\beta - \gamma) \end{bmatrix}, \quad (14)$$

and

$$p_c(\gamma) = \begin{bmatrix} 2R\frac{\gamma}{\beta}\sin(\beta) \\ 0 \end{bmatrix}. \quad (15)$$

We denote the distance between the UAV and the UGVs as $d(\gamma)$, hence

$$d(\gamma) = \|p_a(\gamma) - p_c(\gamma)\|_2. \quad (16)$$

Note that $d(0) = d(\beta) = 0$, and $d(\gamma)$ is strictly concave in the interval $[0, \beta]$. Furthermore, $d(\gamma)$ is at the maximum exactly when $\gamma = \frac{\beta}{2}$. Thus, the distance between the UAV and the UGVs is at the maximum at the midpoint of the motion primitive.

The maximum distance between UAV and the UGVs can be computed as $d(\frac{\beta}{2}) = R(1 - \cos(\beta))$. If the UAV is sufficient to provide continuous convoy protection for the entire motion primitive, then we require that $d(\gamma) \leq r, \forall \gamma \in [0, \beta]$. This is true if $R(1 - \cos(\beta)) \leq r$. Since $\beta \in [0, \pi)$, we can obtain a bound on β as $\beta \leq \arccos(1 - \frac{r}{R})$. Note that $V_G = \frac{\sin(\beta)}{\beta}$ is strictly decreasing in $[0, \pi)$. Therefore, we have that $V_G \geq V_G^\star$, where

$$V_G^\star = \frac{\sin(\arccos(1 - \frac{r}{R}))}{\arccos(1 - \frac{r}{R})} = \frac{\sqrt{2rR - r^2}}{R \arccos(1 - \frac{r}{R})}. \quad (17)$$

This analysis can be applied to every motion primitive in the motion sequence $\mathcal{M}$. Thus, if $V_G \geq V_G^\star$, then one UAV is sufficient to provide continuous convoy protection for all time. ■

When $V_G < V_G^\star$, convoy protection cannot be provided with a single UAV and we need to coordinate multiple UAVs to provide perpetual convoy protection. We use a similar approach as the static convoys case to determine the minimum number of UAVs required. In this case, on the path of each execution of one motion primitive, there are two segments of the path when the distance between the UAV and the UGVs is less than or equal to r. Hence, convoy protection is provided by one UAV for two circular arcs of angle $\gamma^\star$ for each execution of one motion primitive, where $d(\frac{\gamma^\star}{2}) = r$ and d is defined in equation (16). Refer to Figure 5 for an example.

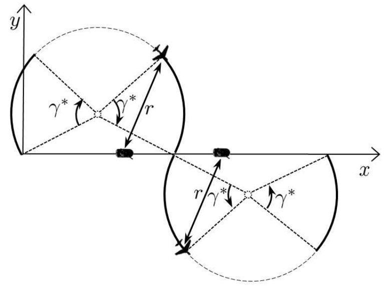

Fig. 5. For each execution of one motion primitive, there are two segments of the path corresponding to two circular arcs of angle $\gamma^\star$, so that the distance between the UAV and the UGVs is less or equal to r when the UAV is on these segments. In this figure, the dashed curve is the path of the UAV, the solid curves are the segments of the path in which convoy protection is provided. The UAV and the convoys are drawn at the times when the UAV enters and exits these segments.

Similar to the multi-UAV coordination approach in the previous section, we can use a timing strategy to schedule the UAVs such that, at any time, one of the UAVs is inside the convoy circle. First, note that the minimum number of UAVs required to provide continuous convoy protection can be obtained by the following corollary:

Corollary 4.3: Using the proposed control strategy, if $V_G < V_G^\star$, then the minimum number of UAVs needed to provide continuous convoy protection for all time is $N = \lceil \frac{\beta}{\gamma^\star} \rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function. $\gamma^\star$ can be obtained by solving a non-linear equation $d(\frac{\gamma^\star}{2}) = r$, using β obtained from V_G ($V_G = \frac{\sin(\beta)}{\beta}$).

Proof: Directly follows from the fact that, for each motion primitive, the length of the path in which one UAV stays inside the convoy circle is $2R\gamma^\star$, while the length of the entire path for the motion primitive is $2R\beta$. ■

Figure 6 shows how one can schedule the UAVs to provide continuous convoy protection for all time. The key is to synchronize the position of the UGVs with individual UAVs at different times, so that when one UAV exits the convoy circle, there is at least one UAV inside the convoy circle and it is on the segment of its path in which the distance to the UGVs is less or equal to r.

V. Concluding Remarks

This paper studies the problem of providing convoy protection to a group of UGVs using UAVs. The UAVs are kinematically restricted by their minimum turning radii and the limited field of view of the sensors onboard. We identify optimal paths for one UAV to provide convoy protection for maximum amount of time when UGVs are stationary. We also propose a coordination strategy as well as optimal paths for multiple UAVs to provide continuous convoy protection for all time. The minimum number of UAVs required to achieve this task is derived. For the case of UGVs moving on straight lines with constant speed, we provide a control strategy that guarantees periodical meet-up with the UGVs, as well as a corresponding bound on the speed of the UGVs, so that if this bound is satisfied, then one UAV is capable of providing convoy protection for all time. If the speed of the UGVs is outside this bound, we propose a coordination strategy and obtain the minimum number of UAVs required to achieve continuous convoy protection.

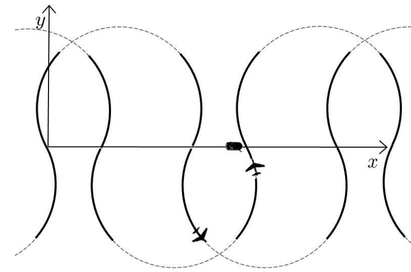

Fig. 6. Example of using two UAVs to provide continuous convoy protection using the proposed strategy. In this figure, the dashed curves are the paths of the UAVs, the solid curves are the segments of the paths in which convoy protection is provided. In this case, every time one UAV exits the convoy circle, the other UAV is inside the convoy circle. This is always true if $N = \lceil \frac{\beta}{\gamma^\star} \rceil = 2$ and the times when the UAVs synchronize with the UGVs are spaced out.

References

[1] C. Belta, A. Bicchi, M. Egerstedt, E. Frazzoli, E. Klavins, and G. J. Pappas. Symbolic planning and control of robot motion: State of the art and grand challenges. *IEEE Robotics and Automation Magazine*, Vol. 14, No. 1, pp. 61-70, 2007.

[2] H. Chitsaz, S. M. LaValle. Time-optimal paths for a Dubins airplane, *IEEE Conference on Decision and Control*, pp. 2379-2384, Dec. 2007.

[3] X. C. Ding, A. Rahmani and M. Egerstedt. Optimal Multi-UAV Convoy Protection. *Technical report*, GRITS, Georgia Institute of Technology, http://www.ece.gatech.edu/∼ding/technicalreport.pdf (2008)

[4] L. Dubins. On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents, *American J. of Mathematics*, Vol. 79, pp. 497-516, 1957.

[5] L. E. Dubins. On plane curves with curvature. *Pacific J. Math.*, Vol. 11, No. 2, pp. 471-481, 1961.

[6] E. Frazzoli, M. A. Dahleh, and E. Feron. Maneuver-based motion planning for nonlinear systems with symmetries. *IEEE Trans. on Robotics*, Vol. 21, No. 6, pp 1077-1091, 2005.

[7] D. E. Kirk. *Optimal Control Theory*, Dover Publications, 2004.

[8] S.M. Lavalle. *Planning Algorithms*, Cambridge Univ. Press, NY, 2006.

[9] J. Lee, R. Huang, A. Vaughn, X. Xiao, J. K. Hedrick, M. Zennaro, and R. Sengupta, Strategies of path-planning for a UAV to track a ground vehicle, *The Second Annual Symposium on Autonomous Intelligent Networks and Systems*, 2003.

[10] K. Savla, E. Frazzoli, and F. Bullo. Traveling Salesperson Problems for the Dubins vehicle. *IEEE Trans. on Automatic Control*, Vol. 53, No. 6, pp.1378-1391, 2008.

[11] P. Souèeres and J. Boissonnat. Optimal trajectories for nonholonomic mobile robots. *Robot Motion Planning and Control*, pp. 93-170. Springer, 1998.

[12] S. C. Spry, A. R. Girard and J. K. Hedrick. Convoy protection using multiple Unmanned Air Vehicles: Organization and Coordination. *American Control Conference*, Portland, Oregon, Jun. 2005.

[13] G. Walsh, A. Sarti, and S. Shankar Sastry. Algorithms for steering on the group of rotations. *Technical Report UCB/ERL M93/44*, EECS Department, University of California, Berkeley, 1993.