Robust Finite-Time Connectivity Preserving Consensus Tracking and Formation Control for Multi-Agent Systems

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***Abstract*— In this paper, we consider finite-time connectivity preserving coordination problems for second-order multi-agent systems with disturbances and limited sensing range. Based on integral sliding mode control and artificial potential field, a distributed controller is developed to achieve robust finite-time consensus and meanwhile maintain the connectivity of the communication network. An integral sliding mode based control strategy is proposed such that the disturbance is rejected by a discontinuous control term and the connectivity of the network is preserved by the nominal control. The method is further extended to address a finite-time formation tracking control problem.**

I. INTRODUCTION

Multi-agent systems have attracted much attention in the past decade, as it has a variety of civilian and military applications, such as power systems, sensor networks, and multi-vehicle formation [1], [2]. In practical applications, the agents usually have limited sensing and communication capabilities and the connection graph for the multi-agent system is dependent on the sensing range of the agent. Thus, the connectivity of the initial network topology cannot be guaranteed for all future time, which motivates the connectivity preservation problem [3], [4], [5], [6], [7], [8], [9], [10], [11]. In [3], a distributed gradient method was developed to maintain the initial network topology for a first-order multi-robot system. In [5], a new hysteresis rule was adopted to the network connectivity and potential functions with bounded values were constructed to solve the rendezvous problem with connectivity preservation for multi-robot systems with double-integrator dynamics.

However, most of the existing works didn't specify the convergence rate of the rendezvous algorithm. When considering the convergence rate and robustness, finite-time control laws usually have better performance [12]. Finite-time consensus problems were investigated in [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26]. [13] presented nonsmooth analysis methods for finite-time stability of continuous-time dynamical systems, and then the methods were applied to study finite-time consensus problems. [14] developed a terminal sliding mode based distributed control law to achieve finite-time consensus of second-order nonlinear multi-agent systems. In [16], finite-time observers were developed to achieve state consensus in the presence of disturbances. Based on homogeneity theory, nonlinear consensus protocols were developed in [17] and [18]. The robust finite-time consensus problem was investigated in [19] using integral sliding mode control. The authors in [20] and [21] studied the finite-time rendezvous problems where the agents' states come to an agreement in a finite time within certain communication range. The integrator-type dynamics were considered in [20], while [21] investigated the nonlinear agent dynamics based on the Lipschitz assumption.

In this paper, we consider the finite-time connectivity preservation problem for second-order multi-agent systems with disturbances. We propose an integral sliding mode based framework to achieve robust finite-time consensus and formation tracking, and meanwhile maintain the connectivity of the communication network. The main contributions of this paper are: 1) Disturbance rejection, finite-time coordination and connectivity preservation are simultaneously achieved using a distributed integral sliding mode controller. The disturbances may change the states of the agents and thus violate the network connectivity. By using integral sliding mode control, the system trajectory begins from the sliding manifold. Then, the reaching phase is removed and the robustness to disturbances is guaranteed. Moreover, network connectivity is preserved on the sliding manifold via the proposed potential function. Thus, the disturbances can be rejected without affecting the connectivity of the network topology. 2) We study the finite-time connectivity preserving consensus tracking problem for second-order systems, which are more complicated than first-order leaderless systems addressed in the literature. 3) Inspired by the derived consensus tracking controller, we propose a finite-time connectivity preserving formation control approach based on a new connectivity rule and potential function. Compared with the existing studies on the finite-time formation tracking problem (e.g., [22], [23], [27]), the proposed method is robust to bounded disturbances and can preserve the network pattern.

The structure of this paper is described as follows. In Section II, we introduce the background of graph theory and concepts on finite-time stability. In Section III, a robust finite-time connectivity preserving consensus tracking problem is formulated and solved. In Section IV, the proposed approach is extended to address a formation tracking control problem. Section V concludes the paper.

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II. Notation and Preliminaries

Notations: In this paper, we use $\mathbb{R}$ and $\mathbb{R}^n$ to denote the set of real numbers and n-dimensional real column vectors, respectively. $A \otimes B$ denotes the Kronecker product of matrices A, B. Let $\|\cdot\|$ be the 2-norm and $\|\cdot\|_1$ be the 1-norm. For a vector $e = [e_1, \cdots, e_n]^T \in \mathbb{R}^n$, $\text{sgn}(e) = [\text{sgn}(e_1), \cdots, \text{sgn}(e_n)]^T$. Let $\text{sig}^\alpha(e) = \|e\|^\alpha (e/\|e\|)$, if $e \neq \mathbf{0}$, and $\text{sig}^\alpha(e) = \mathbf{0}$ if $e = \mathbf{0}$, which is continuous if $\alpha > 0$. $\mathbf{1}$ and $\mathbf{0}$ are column vectors with appropriate dimensions.

Graph theory: Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ represent an undirected graph, where $\mathcal{V} = \{1, ..., N\}$ denotes the vertex set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denotes the edge set. Graph $\mathcal{G}$ is connected if there is an undirected path between every pair of distinct agents. A matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ denotes the adjacency matrix of $\mathcal{G}$, where $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$ else $a_{ij} = 0$. In this paper, we suppose $a_{ii} = 0$. A matrix $L \triangleq D - A \in \mathbb{R}^{N \times N}$ is called the Laplacian matrix of $\mathcal{G}$, where $D = [d_{ii}] \in \mathbb{R}^{N \times N}$ is a diagonal matrix with $d_{ii} = \sum_{j=1}^{N} a_{ij}$.

Definition 1: [28]**(Homogeneity)** For the system $\dot{x} = f(x)$, $f(\mathbf{0}) = \mathbf{0}$, $x = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$, where $f(x) = [f_1(x), \cdots, f_n(x)]^T$ is a continuous vector field. $f(x)$ is said to be homogeneous of degree $q \in \mathbb{R}$ with dilation $(r_1, \cdots, r_n)$, $r_i > 0$, $i = 1, 2, \cdots, n$, if for any $\varepsilon > 0$, $f_i(\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n) = \varepsilon^{r_i + q} f_i(x)$, $i = 1, 2, \cdots, n$. The system $\dot{x} = f(x)$ is said to be homogeneous if f is homogeneous.

The following lemma plays an important role in the subsequent stability analysis.

Lemma 1: [26]For the system $\dot{x} = f(x) + \tilde{f}(t, x)$, $f(\mathbf{0}) = \mathbf{0}$, $x \in \mathbb{R}^n$, where f is a continuous and homogeneous vector function of degree $q < 0$ with dilation $(r_1, \cdots, r_n)$, and $\tilde{f}(t, x)$ satisfies $\tilde{f}(t, \mathbf{0}) = \mathbf{0}$ for all t. Assume that $x = \mathbf{0}$ is an asymptotically stable equilibrium of the system $\dot{x} = f(x)$. Then $x = \mathbf{0}$ is a locally finite-time stable equilibrium if for all t and all $x \in \{x \in \mathbb{R}^n : \|x\| = 1\}$,

$$\lim_{\varepsilon \to 0^+} \frac{\tilde{f}(t, \varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n)}{\varepsilon^{r_i + q}} = \mathbf{0}, i = 1, 2, \cdots, n. \quad (1)$$

In addition, if the system is globally asymptotically stable and locally finite-time stable, then it is globally finite-time stable.

III. Robust Finite-Time Connectivity Preserving Consensus Tracking

A. Problem Formulation

Consider a second-order multi-agent system with N followers. The dynamics of the follower i ($i \in \{1, ..., N\}$) are described by

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t) + d_i(t), \end{aligned} \quad (2)$$

where $x_i(t) \in \mathbb{R}^n$ represents the position state of agent i, $v_i(t) \in \mathbb{R}^n$ represents the velocity state of agent i, $u_i(t) \in \mathbb{R}^n$ represents the control input, and $d_i(t) \in \mathbb{R}^n$ is the disturbance. Let $x(t) = [x_1^T(t), \cdots, x_N^T(t)]^T \in \mathbb{R}^{Nn}$ be the position vector and $v(t) = [v_1^T(t), \cdots, v_N^T(t)]^T \in \mathbb{R}^{Nn}$ be the velocity vector.

The leader for system (2) has the following dynamics

$$\begin{aligned} \dot{x}_0(t) &= v_0(t), \\ \dot{v}_0(t) &= a_0(t), \end{aligned} \quad (3)$$

where $x_0(t) \in \mathbb{R}^n$ represents the position state of the leader, $v_0(t) \in \mathbb{R}^n$ represents its velocity, and $a_0(t) \in \mathbb{R}^n$ is the acceleration of the leader.

The finite-time consensus problem for multi-agent systems with disturbances has been studied in the literature. For example, in [14], discontinuous terminal sliding mode controllers were designed to achieve finite-time consensus tracking control of multi-robot systems with input disturbances. In [29], the authors proposed a distributed control algorithm to guarantee fast finite-time consensus of uncertain first-order multi-agent systems. In reality, however, the communication ability of the agent is usually limited and each agent can only communicate with agents within its information range. If the relative distance of two neighboring agents are larger than this range, the communication link may be lost. Motivated by this observation, we study the finite-time connectivity preserving consensus tracking and disturbance attenuation problem, defined as follows

Definition 2: **(Robust Finite-time Connectivity Preserving Consensus Tracking)** Consider a multi-agent system with N followers and 1 leader, with (2) being the dynamics of the follower and (3) the leader. Each agent can sense only up to a distance R from it. Suppose that the initial topology of the followers is connected and at least one follower has access to the leader's information. The robust finite-time connectivity preserving consensus tracking problem is solved if the system has the following properties: 1) the connectivity of the initial graph $\mathcal{G}(0)$ is preserved for all $t \geq 0$; 2) $x_i(t) \to x_0(t)$ and $v_i(t) \to v_0(t)$ in a finite time, $i \in \{1, ..., N\}$.

The following assumptions will be used in the stability analysis.

Assumption 1: The disturbance $d_i(t)$ is bounded by a known constant, i.e., $||d_i|| \leq c_1$, $i \in \{1, ..., N\}$.

Assumption 2: The leader's acceleration $a_0(t)$ is bounded by a known constant, i.e., $||a_0|| \leq c_2$.

B. Network Connectivity

In this paper, we adopt the revised network connectivity setting as given in [10] where the edge set $\mathcal{E}(t)$ is defined as follows:

A1: The initial edges are generated by $\mathcal{E}(0) = \{(i, j) : \|x_i(0) - x_j(0)\| < R, i, j \in \{1, ..., N\}\} \cup \{(0, j) : \|x_0(0) - x_j(0)\| < R, j \in \{1, ..., N\}\}$;

A2: $(i, 0) \notin \mathcal{E}(t)$ for all $t \geq 0$, $i \in \{0, 1, ..., N\}$;

A3: If $\|x_i(t) - x_j(t)\| \geq R$, then $(i, j) \notin \mathcal{E}(t)$.

The information exchange among the $N+1$ agents is represented by graph $\mathcal{G}(t) = \{\mathcal{V}, \mathcal{E}(t)\}$ with $\mathcal{V} = \{0, 1, ..., N\}$. Let $\mathcal{G}_F(t) = \{\mathcal{V}_F, \mathcal{E}_F(t)\}$ be the subgraph of the overall graph $\mathcal{G}(t)$ where $\mathcal{V}_F = \{1, ..., N\}$ and $\mathcal{E}_F(t) \subset \mathcal{E}(t)$. We make the following assumption on $\mathcal{G}_F(0)$.

Assumption 3: $\mathcal{G}_F(0)$ is connected and at least one agent has access to the leader's information.

The Laplacian matrix $L(t) \in \mathbb{R}^{N\times N}$ associated with $\mathcal{G}_F(t)$ is written as

$$L(t) \triangleq D(t) - A(t), \tag{4}$$

where $D(t) = [d_{ii}(t)] \in \mathbb{R}^{N\times N}$ is a diagonal matrix with $d_{ii}(t) = \sum_{j=1}^{N} a_{ij}(t)$ and $A(t) = [a_{ij}(t)] \in \mathbb{R}^{N\times N}$ is the adjacency matrix defined as

$$a_{ij}(t) \triangleq \begin{cases} 1, & (i,j) \in \mathcal{E}_F(t), \\ 0, & (i,j) \notin \mathcal{E}_F(t). \end{cases} \tag{5}$$

Define a diagonal matrix $B(t) = [b_i(t)] \in \mathbb{R}^{N\times N}$, which represents the access of agents to the leader's trajectory, where $b_i(t) = 1$ if the i-th agent has access to the leader's trajectory signal at time t and 0 otherwise. Let $H(t) \triangleq L(t) + B(t) \in \mathbb{R}^{N\times N}$ be the information-exchange matrix. Then we have the following lemma.

Lemma 2: [30]If Assumption 3 holds, then matrix $H(0)$ is positive definite and symmetric.

Remark 1: In the subsequent development, it can be shown that the edges generated by Rule **A1** remains unchanged as time goes on. Thus, the communication graph is a static graph rather than a dynamic one. Before proving this, we still use the above-mentioned notations in order to avoid confusion.

C. Control Design

Consider a distributed integral sliding mode control law with the following form

$$u_i(t) = u_{nomi}(t) + u_{disconi}(t), \tag{6}$$

where the nominal control input $u_{nomi}(t)$ determines the performance of the system without disturbance and an additional discontinuous control input $u_{disconi}(t)$ deals with the disturbance [31]-[32]. The nominal control input $u_{nomi}(t)$ is designed as

$$\begin{aligned} u_{nomi} =& -\sum_{j=1}^{N} k_1 a_{ij}(t)\varphi(\|x_i - x_j\|)\text{sig}^{\alpha}(x_i - x_j) \\ & - k_1 b_i(t)\varphi(\|x_i - x_0\|)\text{sig}^{\alpha}(x_i - x_0) \\ & - \sum_{j=1}^{N} k_2 a_{ij}(t)\text{sig}^{\beta}(v_i - v_j) \\ & - k_2 b_i(t)\text{sig}^{\beta}(v_i - v_0), \end{aligned} \tag{7}$$

where $\varphi(\cdot)$ is a nonnegative potential function to be designed later, k_1, k_2, α and β are positive constants, and the symbol $\text{sig}^{\alpha}(\cdot)$ was previously defined in Section II.

Let $\vartheta_i(t) = v_i(t) - v_i(0)$ with $v_i(0)$ being the initial velocity of agent i, $i \in \{0,\ 1,\ ...,\ N\}$. The discontinuous control input $u_{disconi}(t)$ is designed as

$$u_{disconi} = -k_3 \text{sgn}(s_i), \tag{8}$$

where k_3 is a positive constant and the sliding manifold $s_i(t) \in \mathbb{R}^n$ is designed as

$$\begin{aligned} s_i =& \sum_{j=1}^{N} a_{ij}(t)(\vartheta_i - \vartheta_j) + b_i(t)(\vartheta_i - \vartheta_0) \\ & - \int_0^t \{\sum_{j=1}^{N} a_{ij}(\tau)(u_{nomi}(\tau) - u_{nomj}(\tau)) \\ & + b_i(\tau)u_{nomi}(\tau)\}d\tau. \end{aligned} \tag{9}$$

The nonnegative potential function $\varphi(\|x_i - x_j\|)$ is designed as a function of the distance between agent i and agent j. In order to guarantee finite-time convergence as well as connectivity preservation, we consider a class of functions $\varphi(\|x_i - x_j\|)$, $\|x_i - x_j\| \in [0, R)$ satisfying the following conditions:

B1: $\varphi(\|x_i - x_j\|)$ is continuous for $\|x_i - x_j\| \in [0, R)$;

B2: $\varphi(\|x_i - x_j\|) = c$ for $\|x_i - x_j\| \in [0, \varpi]$, where c is a positive constant and $\varpi \in (0, R)$;

B3: $\int_0^{\|x_i - x_j\|} \varphi(\xi)\xi^{\alpha} d\xi \to +\infty$ as $\|x_i - x_j\| \to R$.

Remark 2: The condition **B1** guarantees $\varphi(\|x_i - x_j\|)$ is integratable, **B2** guarantees the consensus is achieved in a finite time after all the edge-distances are less than ϖ, and **B3** is for the preservation of the initial network connection. For the problem in Definition 2, a potential function satisfying **B1**, **B2**, and **B3** can be selected as

$$\begin{aligned} &\varphi(\|x_i - x_j\|) \\ &= \begin{cases} 1, & \|x_i - x_j\| \in [0, \frac{R}{2}], \\ \frac{R^2}{4(R - \|x_i - x_j\|)^2}, & \|x_i - x_j\| \in (\frac{R}{2}, R). \end{cases} \end{aligned} \tag{10}$$

D. Stability Analysis

The robust finite-time connectivity preservation problem can be split into three subproblems: 1) to guarantee that the states of agents are on the sliding manifold; 2) to achieve connectivity preservation on this sliding manifold; 3) to enable finite-time convergence for all the agents. The derived results on the three subproblems are given in the subsequent lemmas.

Lemma 3: Consider a group of N mobile agents with dynamics (2) and control input (6)-(9). Suppose that Assumptions 1-3 hold and $\mathcal{G}(t) = \mathcal{G}(0)$ for $t \in [0, t_1)$. Then, each agent will maintain its trajectory on the sliding manifold, i.e., $s_i(t) = 0$, $i \in \{1, ..., N\}$, $t \in [0, t_1)$, provided that the control gain k_3 satisfies $k_3 > c_1 + c_2$.

Proof: Denote $s(t) = [s_1^T(t),\ \cdots,\ s_N^T(t)]^T$, $\vartheta(t) = [\vartheta_1^T(t),\ \cdots,\ \vartheta_N^T(t)]^T$, $u_{nom}(t) = [u_{nom1}^T(t),\ \cdots,\ u_{nomN}^T(t)]^T$, and $d(t) = [d_1^T(t),\ \cdots,\ d_N^T(t)]^T \in \mathbb{R}^{Nn}$. Let $H = H(t)$ and $\tilde{H} = H \otimes I_n$, $t \in [0,\ t_1)$. Then, (9) can be rewritten in a compact form as

$$s = \tilde{H}(\vartheta - \mathbf{1}\otimes\vartheta_0) - \int_0^t \tilde{H} u_{nom}(\tau)d\tau. \tag{11}$$

By Lemma 2, $\tilde{H}$ is of full rank. Taking the derivative of (11) and pre-multiplying both sides by $\tilde{H}^{-1}$, one gets

$$\begin{aligned} \tilde{H}^{-1}\dot{s} =& \dot{v} - \mathbf{1}\otimes\dot{v}_0 - u_{nom} \\ =& - k_3\text{sgn}(s) + d - \mathbf{1}\otimes\dot{v}_0. \end{aligned} \tag{12}$$

Let $V_1(t) = \frac{1}{2}s^T\tilde{H}^{-1}s$. Taking the time derivative of V_1 along (12) gives

$$\begin{aligned}\dot{V}_1 =& s^T(-k_3\text{sgn}(s) + d - \mathbf{1}\otimes\dot{v}_0)\\ \le& -(k_3 - c_1 - c_2)\left\|s\right\|_1\\ \le& -\sqrt{2}(k_3 - c_1 - c_2)(\frac{1}{2}s^Ts)^{\frac{1}{2}}\\ \le& -\sqrt{\frac{2}{\lambda_{\max}(\tilde{H}^{-1})}}(k_3 - c_1 - c_2)V_1^{\frac{1}{2}},\end{aligned} \quad (13)$$

where $\lambda_{\max}(\tilde{H}^{-1})$ denotes the maximum eigenvalue of $\tilde{H}^{-1}$.

From (11), $s(0) = \mathbf{0}$. Then, it can be obtained that $s(t) = \mathbf{0}$, $t \in [0, t_1)$, which implies that the system trajectory resides on the sliding manifold in $[0,\, t_1)$. □

Remark 3: For integral sliding mode control, sliding mode starts from the initial time [31]-[32]. Using regularization approach, [31] has shown that, when $s = \mathbf{0}$, the equivalent value of control is given by $\dot{s} = \mathbf{0}$. From (11), the nominal dynamics can be written as $\dot{v}_i = \dot{v}_0 + u_{nomi}$, which implies that $-k_3sgn(0) + d_i - \dot{v}_0 = 0$, i.e., the bounded terms d_i and $\dot{v}_0$ are compensated by the equivalent value $[-k_3sgn(0)]_{eq}$ of the discontinuous control (8), which has infinite switching frequency theoretically [33],[34]. In practice, chattering behaviour exists in sliding mode control. In the past decades, the researchers in this area developed a lot of methods to reduce the chattering effect, e.g., the super twisting algorithm [35]. Taking into consideration of connectivity preservation, chattering reduction becomes more complicated. In this paper, we do not consider it, which will be our future work.

Lemma 4: Consider a network of N mobile agents with dynamics (2) and control input (6)-(9). Assume that Assumptions 1-3 hold. Then $\mathcal{G}(t)$ remains invariant for all $t \ge 0$, provided that the control parameters k_1, k_2, k_3, α, β satisfy

$$\begin{aligned}&k_1 > 0,\ k_2 > 0,\ k_3 > c_1 + c_2,\\ &0 < \alpha < 1,\ \beta = \frac{2\alpha}{1+\alpha},\end{aligned} \quad (14)$$

where c_1 and c_2 are the upper bounds of the disturbances and leader's acceleration defined in Assumptions 1 and 2, respectively.

Proof: Assume that there exists a time t_2 such that $\mathcal{G}(t_2^-) = \mathcal{G}(0)$ and $\mathcal{G}(t_2) \ne \mathcal{G}(0)$. According to Lemma 3, $s_i(t) = s_i(0)$ for $t \in [0, t_2)$, $i \in \{1, ..., N\}$. Let $\zeta_i(t) = x_i(t) - x_0(t)$ and $\eta_i(t) = v_i(t) - v_0(t)$. Then, based on (11), on the sliding manifold, we have

$$\begin{aligned}\dot{\zeta}_i =& \eta_i,\\ \dot{\eta}_i =& -\sum_{j=1}^{N} k_1 a_{ij}\varphi(\|\zeta_i - \zeta_j\|)\text{sig}^{\alpha}(\zeta_i - \zeta_j)\\ &- k_1 b_i\varphi(\|\zeta_i\|)\text{sig}^{\alpha}(\zeta_i)\\ &- \sum_{j=1}^{N} k_2 a_{ij}\text{sig}^{\beta}(\eta_i - \eta_j) - k_2 b_i\text{sig}^{\beta}(\eta_i).\end{aligned} \quad (15)$$

Denote $\zeta(t) = [\zeta_1^T(t),\ \cdots,\ \zeta_N^T(t)]^T$ and $\eta(t) = [\eta_1^T(t),\ \cdots,\ \eta_N^T(t)]^T \in \mathbb{R}^{Nn}$. Define a non-negative energy function $W(\zeta, \eta)$ in $[0, t_2)$ as

$$\begin{aligned}W(\zeta, \eta) =& \sum_{i=1}^{N}\sum_{j=1}^{N} k_1 \int_0^{\|\zeta_i - \zeta_j\|} a_{ij}\varphi(s)s^{\alpha}ds\\ &+ 2\sum_{i=1}^{N} k_1 \int_0^{\|\zeta_i\|} b_i\varphi(s)s^{\alpha}ds + \sum_{i=1}^{N}\eta_i^T\eta_i,\end{aligned} \quad (16)$$

with the initial value $W_0 = W(\zeta(0), \eta(0))$, where a_{ij} and b_i are the edge weights in graph $\mathcal{G}(0)$.

Note that W is continuous and differentiable in $[0, t_2)$. The time derivative of W satisfies

$$\begin{aligned}\dot{W} =& 2\sum_{i=1}^{N}\sum_{j=1}^{N} k_1 a_{ij}\varphi(\|\zeta_i - \zeta_j\|)\dot{\zeta}_i^T\text{sig}^{\alpha}(\zeta_i - \zeta_j)\\ &+ 2\sum_{i=1}^{N}\eta_i^T\dot{\eta}_i + 2\sum_{i=1}^{N} k_1 b_i\varphi(\|\zeta_i\|)\dot{\zeta}_i^T\text{sig}^{\alpha}(\zeta_i).\end{aligned} \quad (17)$$

Substituting (15) into (17), we have

$$\begin{aligned}\dot{W} =& -\sum_{i=1}^{N}\sum_{j=1}^{N} k_2 a_{ij}(\eta_i - \eta_j)^T\text{sig}^{\beta}(\eta_i - \eta_j)\\ &- 2\sum_{i=1}^{N} k_2 b_i\eta_i^T\text{sig}^{\beta}(\eta_i)\\ \le& 0,\end{aligned} \quad (18)$$

which implies that $W(t) \le W_0 < \infty$ for $t \in [0, t_2)$. From **B3**, $\int_0^R \varphi(\xi)\xi^{\alpha}d\xi = \infty > W_0$. Therefore, no edge-distance will tend to R for $t \in [0, t_2)$ (i.e., no edge will be lost at t_2). □

Lemma 5: Consider a group of N mobile agents with dynamics (2) and control input (6)-(9). If $\mathcal{G}(t)$ is invariant for all $t \ge 0$, then $x_i(t) \to x_0(t)$ and $v_i(t) \to v_0(t)$ in a finite time ($i \in \{1, ..., N\}$), provided that the control parameters k_1, k_2, k_3, α, β satisfy (14).

Proof: From (18) $\dot{W} = 0$ if and only if $\eta_1 = \cdots = \eta_N = 0$. Thus, (15) becomes

$$\begin{aligned}\mathbf{0} =& -\sum_{j=1}^{N} a_{ij}\varphi(\|\zeta_i - \zeta_j\|)\text{sig}^{\beta}(\zeta_i - \zeta_j)\\ &- b_i\varphi(\|\zeta_i\|)\text{sig}^{\beta}(\zeta_i),\end{aligned}$$

and thus,

$$\begin{aligned}\mathbf{0} =& -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\zeta_i - \zeta_j)^T\varphi(\|\zeta_i - \zeta_j\|)\text{sig}^{\beta}(\zeta_i\\ &- \zeta_j) - 2\sum_{i=1}^{N} b_i\zeta_i^T\varphi(\|\zeta_i\|)\text{sig}^{\beta}(\zeta_i),\end{aligned} \quad (19)$$

which implies that $\zeta_1 = \cdots = \zeta_N = 0$. Based on Lasalle Invariance Principle, $x_i - x_0 \to 0$ and $v_i - v_0 \to 0$ as $t \to \infty$. Thus, there exists a time t_3 such that $\|x_i - x_j\| < \varpi$, for all $i \in \{1, ..., N\}$, $j \in \{0, 1, ..., N\}$, where ϖ is defined in

B2. Then after time $t > t_3$, the dynamics of the closed-loop system in $(\zeta_i,\, \eta_i)$ space can be rewritten as

$$\begin{aligned}\dot{\zeta}_i =&\eta_i,\\ \dot{\eta}_i =&-\sum_{j=1}^{N} k_1 a_{ij} c\text{sig}^{\alpha}(\zeta_i-\zeta_j) - k_1 b_i c\text{sig}^{\alpha}(\zeta_i)\\ &-\sum_{j=1}^{N} k_2 a_{ij}\text{sig}^{\beta}(\eta_i-\eta_j) - k_2 b_i \text{sig}^{\beta}(\eta_i). \qquad (20)\end{aligned}$$

Furthermore, with $0 < \alpha < 1$, $\beta = \frac{2\alpha}{1+\alpha}$, similar to the analysis in [18], the system (20) is homogeneous of degree $q = \alpha - 1 < 0$ with dilation $(r_1, \cdots, r_1, r_2, \cdots, r_2)$, where $r_1 = 2$, $r_2 = 1+\alpha$. Then, $\lim_{\varepsilon\to 0^+} \frac{\tilde{f}(t,\varepsilon^{r_1}\zeta,\varepsilon^{r_2}\eta)}{\varepsilon^{r_2+q}} = 0$ holds for any $\|[\zeta,\eta]\| = 1$. According to Lemma 1, the system (20) is locally finite-time stable which implies that the considered system (2) with control input (6)-(9) is globally finite-time stable. □

The main results of this section can be stated in the following theorem.

Theorem 1: Consider a group of N mobile agents with dynamics (2), each being controlled by (6)-(9). If Assumptions 1-3 hold, and the control parameters k_1, k_2, k_3, α and β are selected according to (8), then the robust finite-time connectivity preserving consensus tracking problem described in Definition 2 is solved, in the sense that 1) $\mathcal{G}(t)$ is invariant for all $t \geq 0$; 2) $x_i(t) \to x_0(t)$ and $v_i(t) \to v_0(t)$ in a finite time, $i \in \{1, ..., N\}$.

Proof: Based on Lemmas 3 and 4, all the agents maintain their trajectories on the sliding manifold and $\mathcal{G}(t)$ is invariant for all $t \geq 0$. Then, based on Lemma 5, the agents states achieve consensus and track the desired trajectory in a finite time. □

IV. Robust Finite-Time Connectivity Preserving Formation Tracking

In this section, we apply the result of the previous section to address a formation tracking problem. Consider a multi-agent system composed of N followers governed by (2) and a leader governed by (3). The objective of each follower is to track the leader and meanwhile maintain a desired formation. Denote $h = [h_1^T, \cdots, h_N^T]^T$ where $h_i \in \mathbb{R}^n, i \in \{1, ..., N\}$ represents the configuration of agent i in the desired formation with respect to the leader. Let $h_{ij} = h_i - h_j$ be the desired position of agent i with respect to agent j. Then the problem can be described as follows

Definition 3: **(Robust Finite-Time Connectivity Preserving Formation Tracking)** Consider a multi-agent system with N followers and 1 leader, with (2) being the dynamics of the follower and and (3) the leader. Each follower i is assigned with a desired formation vector h_i, $i \in \{1, ..., N\}$. Suppose that $\mathcal{G}_F(0)$ is connected and at least one follower has access to the leader's information, the robust finite-time formation tracking is achieved if the system has the following properties: 1) the connectivity of the sensing graph $\mathcal{G}(0)$ is preserved for all $t \geq 0$; 2) $x_i(t) - h_i \to x_0$ and $v_i(t) \to v_0(t)$ in a finite time, $i \in \{1, ..., N\}$.

Let $\theta_i = x_i - h_i$. Then the system (2) in $(\theta_i,\, v_i)$ space can be written as

$$\begin{aligned}\dot{\theta}_i(t) =&v_i(t),\\ \dot{v}_i(t) =&u_i(t) + d_i(t). \qquad (21)\end{aligned}$$

Design a distributed control law in the similar way as (6) where the nominal control input $u_{nomi}(t)$ is designed as

$$\begin{aligned}u_{nomi} =& -\sum_{j=1}^{N} k_1 a_{ij}(t)\varphi(\|x_i - x_j - h_{ij}\|)\text{sig}^{\alpha}(x_i\\ &- x_j - h_{ij}) - k_1 b_i(t)\varphi(\|x_i - x_0 - h_i\|)\\ &\times \text{sig}^{\alpha}(x_i - x_0 - h_i) - \sum_{j=1}^{N} k_2 a_{ij}(t)\\ &\times \text{sig}^{\beta}(v_i - v_j) - k_2 b_i(t)\text{sig}^{\beta}(v_i - v_0), \qquad (22)\end{aligned}$$

and the discontinuous control input $u_{disconi}(t)$ remains the same as that in (8). In order to achieve the formation tracking control objective, the potential function $\varphi(\|x_i - x_j - h_{ij}\|)$ is defined as follows

$$\begin{aligned}&\varphi(\|x_i - x_j - h_{ij}\|)\\ &= \begin{cases} 1, \quad \text{if } \|x_i - x_j - h_{ij}\| \in [0, \varpi], \\ \frac{(R-||h_{ij}||)^2}{4(R-||h_{ij}||-\|x_i-x_j-h_{ij}\|)^2}, \text{ if} \\ \|x_i - x_j - h_{ij}\| \in (\varpi, R - ||h_{ij}||), \end{cases} \qquad (23)\end{aligned}$$

where $\varpi \in (0, R - ||h_{ij}||)$ is a positive constant, $h_{i0} = h_i$, $i \in \{1, ..., N\}$, $j \in \{0, 1, ..., N\}$. In addition, the connectivity rule in **A1** is revised as follows

A1': The initial edges are generated by $\mathcal{E}(0) = \{(i,j) : \|x_i(0) - x_j(0)\| < R - 2\|h_{ij}\|,\, i,j \in \{1, ..., N\}\} \cup \{(0,j) : \|x_0(0) - x_j(0)\| < R - 2\|h_j\|,\, j \in \{1, ..., N\}\}$.

According to Rule **A1'**, the agents that are initially located within a distance less than $R - 2\|h_{ij}\|$ are neighboring agents in the initial topology. It can be easily verified that the conclusion of Lemma 3 still holds for formation tracking control. In the following, we show that the revised potential function and connectivity rule ensure connectivity preservation of the initial network.

Lemma 6: Under the control input designed in (6), (8), (9) and (22), if Assumptions 1-2 hold, $\mathcal{G}_F(0)$ with edges generated by Rule **A1'** is initially connected and at least one follower has access to the leader's information, then $\mathcal{G}(t)$ is invariant for all $t \geq 0$, provided that the control parameters k_1, k_2, k_3, α, β satisfy (14).

Proof: Based on Rule **A1'**, it can be derived that $\|x_i(0) - x_j(0) - h_{ij}\| < R - ||h_{ij}||$. Let $\gamma = \int_0^{\|x_i - x_j - h_{ij}\|} \varphi(\xi)\xi^{\alpha} d\xi$. Then $\gamma \to +\infty$ as $\|x_i - x_j - h_{ij}\| \to R - ||h_{ij}||$. Following similar analysis as in Lemma 4, one can obtain that the edge set $\mathcal{E}(t) = \{(i,j) \mid ||x_i(t) - x_j(t) - h_{ij}|| < R - ||h_{ij}||\}$ is invariant for the trajectory of the closed-loop system. Furthermore, the connectivity of the network is maintained since $\|x_i(t) - x_j(t) - h_{ij}\| < R - ||h_{ij}||$ implies $|\|x_i(t) - x_j(t)\| - ||h_{ij}||| < R - ||h_{ij}||$ and $\|x_i(t) - x_j(t)\| < R$. □

Following a similar analysis as in Lemma 5, we conclude that the agents converge to the desired formation in a finite time.

Theorem 2: Suppose that Assumptions 1-2 hold, the initial topology $\mathcal{G}_F(0)$ with edges generated by Rule **A1'** is connected and at least one follower has access to the leader's information. Using control inputs determined by (6), (8), (9) and (22) where the control parameters k_1, k_2, k_3, α, β are selected satisfying (14), the robust finite-time connectivity preserving formation tracking problem described in Definition 2 can be solved.

V. Conclusion

In this study, we investigated the finite-time consensus and formation tracking problems for second-order multi-agent systems with disturbance rejection and connectivity preservation. Distributed control laws were designed to achieve finite-time consensus tracking or formation tracking and meanwhile maintain the connectivity of the communication network. Sliding mode control together with artificial potential field were utilized to achieve finite-time convergence in the presence of disturbances. Chattering elimination and estimation of the convergence time will be our future work.

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