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Technical Report

Orbital control for a class of planar impulsive hybrid systems with controllable resets

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Abstract

This paper investigates the stabilization of a novel subclass of impulsive hybrid systems (IHS) that feature controllable guards and resets maps. While the structure of the guards and the reset map is assumed to be completely determined from the physical set-up, both mappings are jointly adjustable through exogenous control inputs at run-time. As a core feature, all adjustments result in *simultaneous*, inseparable changes of the *reset time* and the *reset action*. In real world, such systems are encountered in mechanical impact systems. In particular, this paper addresses the task of enforcing global orbital asymptotic stability of given periodic executions by means of dedicated control actions.

Contents

1 Introduction

This publication addresses the stabilization of a novel class of periodically operated impulsive hybrid systems (IHS), where the control input *simultaneously* affects the event time and the reset action. In particular, it focuses on an illustrative real-world example of a planar IHS, namely the *controlled billiards problem*, which admits a systematic, model-based solutions to the problem in closed-form.

Impulsive systems [1, 2, 3], switched systems [4, 5, 6, 7, 8] and recently also impulsive hybrid systems [9, 10, 11] have been extensively study in the literature. Impulsive control systems exploit externally controlled discontinuous state resets, typically at fixed, predetermined time instants to control the output trajectory according to given specifications. Switched systems, on the other hand, only admit to switch the continuous dynamics among a finite set of operation modes. However, the instants, at which mode transitions occur, are usually at the decision of the operator. Therefore, the control task of impulsive systems predominantly concentrates on determining control impulses of appropriate amplitude, whereas for switched systems, the input sequence is typically fix, but the *event times* are the outcome of the controller design. Impulsive hybrid systems combine the key properties of both impulsive and switched systems, namely switching and state resets. At the same time, at which a mode transition occurs, the continuous state is also updated according to a reset mapping.

This paper obtains a novel perspective on the control task of IHS. It addresses the design of an *event-surface controller*, which jointly adapts the event time sequence and the reset map to regulate the control output. While the reset map's structure is completely defined by the physical set-up, its parameters can be adapted by means of a control input in order to adjust the reset action. Due to the inseparable coupling of the two control influences, the controller has to compromise between choosing the "best" reset action and the "best" event-time to meet the system specifications. Indeed, such a coupling is found in many real-world examples, predominantly in mechanical impacting systems [12, 13], which provide the primary motivation for this study.

While previous publications exclusively address the stabilization of equilibria, this paper concentrates on system exhibiting periodic or chaotic stationary executions, which are inherent to many real world applications. Orbital stability has already been intensely discussed for switched and for impulsive systems [14, 15, 16, 17, 18], but not yet for impulsive hybrid systems.

2 Impulsive hybrid systems with constrained resets (IHSCR)

2.1 Continuous-time hybrid model

Impulsive hybrid systems can be modeled as hybrid automata, with locations identifying discrete modes of operation, edges indicating possible transitions and labels specifying the associated events. The hybrid evolution is completely described by the following set of equations:

Impulsive hybrid system with controlled resets (IHSCR):

Autonomous continuous dynamics:

$$\dot{\zeta}(t) = \boldsymbol{f}(\zeta(t), q(t)), \quad \zeta(0) = \zeta_0 \tag{2.1}$$

Controlled event generator (guards):

$$\Phi(\zeta(t), \boldsymbol{u}(t), q(t), e) = \begin{cases} \Phi_{e_1}(\zeta(t), \boldsymbol{u}(t), q(t)) \\ \vdots \\ \Phi_{e_E}(\zeta(t), \boldsymbol{u}(t), q(t)) \end{cases} \tag{2.2}$$

Event generation:

$$e(t) = \arg\min_{e \in \mathcal{E}} |\Phi(\zeta(t), \boldsymbol{u}(t), q(t), e)| \tag{2.3}$$

$$\bar{t}(k) = \arg\min_{t > \bar{t}(k-1)} t \ : \ |\Phi(\zeta(t), \boldsymbol{u}(t), q(t), e(t))| = 0 \tag{2.4}$$

$$\bar{e}(k+1) = e(\bar{t}(k)^-) \tag{2.5}$$

Mode selector:

$$\bar{q}(k+1) = g(\bar{q}(k), \bar{e}(k+1)), \quad \bar{q}(0) = q_0$$

$$q(t) = \bar{q}(k), \quad \forall t \in (\bar{t}(k), \bar{t}(k+1)] \tag{2.6}$$

Controlled reset map:

$$\zeta(\bar{t}(k)^+) = \boldsymbol{R}(\zeta(\bar{t}(k)^-), \boldsymbol{u}(\bar{t}(k)^-), \bar{q}(k-1), \bar{e}(k),) \tag{2.7}$$

Here, $\zeta(t) \in \mathbb{R}^n$ and $q(k) \in \mathcal{Q} = \{1, \ldots, Q\}$ denote the continuous state and plant's current discrete mode at time t. The continuous flow $\zeta(t)$ evolves uncontrolled according to the differential equation (2.1), with a mode-dependent vector field $\boldsymbol{f}$.

The occurrence of an event is determined by the event generator (EG), which implements an event function (2.2) consisting of a finite set of exogenously controlled event conditions $\Phi_e,\ e \in \mathcal{E}$, which can be dynamically adjusted through the control input $\boldsymbol{u}(t)$. The event generator outputs the symbol $e(t) = e_i$ indicating the event potentially issued at the next *event time* $\overline{t}(k)$. An event only *occurs*, whenever the corresponding event condition is satisfied (2.4). Due to the exogenous input, it is possible to control the generated event sequence, even though the continuous evolution is autonomous. The k-th *event distance* is denoted by $\overline{\tau}(k) = \overline{t}(k+1) - \overline{t}(k)$. Event-based sampled signal values are denoted by $\boldsymbol{x}(k)\,, \overline{q}(k)$, etc.

At the occurrence of an event, the discrete subsystem (2.6) executes a discrete transition depending on the current mode $\overline{q}(k)$ and the previously issued event $\overline{e}(k+1)$. At the same time, the continuous state $\boldsymbol{\zeta}$ of the system is discontinuously updated according to the mode-dependent, controlled reset map (2.7). The input-dependent subsets

$$\mathcal{S}(q, e, \boldsymbol{u}) = \{\boldsymbol{\zeta}\ :\ \Phi(\boldsymbol{\zeta},\ \boldsymbol{u},\ q,\ e) = 0\} \tag{2.8}$$

constitute *controlled event surfaces* in state-space for each mode-event pair (q, e).

An *execution* of the IHS over N events is represented by the collection

$$\chi\left(\boldsymbol{\zeta}_0^h, \boldsymbol{u}(t), t_0, N\right) = (\boldsymbol{\zeta}(t)\,,\ q(t)\,,\ \mathcal{T}(N))\quad ,\qquad \boldsymbol{\zeta}_0^h = \begin{pmatrix} \boldsymbol{\zeta}_0 & \overline{q}_0^\star \end{pmatrix}^{\mathrm{T}},\ \ t \in [t_0,\ \overline{t}(N)]\quad ,$$

with $\mathcal{T}(N) = ((\,\overline{t}(0), \overline{t}(1)]\,, \ldots)$. Executions $\chi^\star(\boldsymbol{\zeta}_0^{h,\star}, \boldsymbol{u}^\star(t)\,, 0, N)$ that satisfy

$$\begin{aligned} \overline{q}^\star(k+p) &= \overline{q}^\star(k) \\ \boldsymbol{\zeta}^\star\left(\tau, \overline{\boldsymbol{\zeta}}^\star(k+p), \overline{q}^\star(k+p)\right) &= \boldsymbol{\zeta}^\star\left(\tau, \overline{\boldsymbol{\zeta}}^\star(k), \overline{q}^\star(k)\right),\quad \forall \tau \in (0,\ \overline{\tau}^\star(k)] \end{aligned}$$

for all $k \le N-p$ are called *periodic* of *order* p. The closed orbit Γ traced out by $\chi^\star(\boldsymbol{\zeta}_0^{h,\star}, \boldsymbol{u}^\star(t)\,, 0, p)$ is called the *limit cycle* corresponding to the stationary input $\boldsymbol{u}^\star(t)$. Let $\overline{\boldsymbol{x}}^\star\left(\overline{q}_k^\star\right) \in \Gamma$ denote the *impact point*, at which mode $\overline{q}^\star(k) = \overline{q}_k^\star$ is activated for the next $\overline{\tau}^\star\left(\overline{q}_k^\star\right)$ time units. The event, which is issued at the activation of $\overline{q}_k^\star$, is $\overline{e}_k^\star$. Furthermore, denote the extrapolated orbit sections by

$$\Gamma_{\overline{q}_k^\star} = \left\{\boldsymbol{\zeta}\ :\ \exists \tau \in \mathbb{R}\ \text{s.t.}\ \boldsymbol{\zeta} = \boldsymbol{\zeta}\left(\tau, \overline{\boldsymbol{\zeta}}^\star\left(\overline{q}_k^\star\right), \overline{q}_k^\star\right)\right\}\quad . \tag{2.9}$$

2.2 Example of a IHSCR: controllable billiards

A problem extensively studied in mathematics and physics is the *polygonal billiards problem* [19](Fig. 2.1). The problem set-up consists of a polygonal, convex table with E bounding walls $\mathcal{W}_e$ and a billiard ball modeled as a point mass, which flows along straight line sequences on the table. Contrary to the classical problem, let the walls be pivot-mounted at $\boldsymbol{x}_{\mathrm{p},e}$ and externally rotatable. For reasons of complexity, let the walls execute rotations instantaneously.

The billiard ball starts off at an initial position $\boldsymbol{x}(0)$ with an initial heading $\boldsymbol{v}(0)$. Collisions are assumed to be elastic, i.e. energy preserving, and solely change the ball's heading with the angle of reflection γ being equal to the angle of incidence (Fig 2.1). The system's behavior is exactly described by a single mode IHSCR. The continuous dynamics

$$\dot{\boldsymbol{\zeta}}(t) = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \boldsymbol{I} & \mathbf{0} \end{array}\right) \boldsymbol{\zeta}(t), \quad \boldsymbol{\zeta}(0) = \left(\begin{array}{cc|cc} v_{x,0} & v_{y,0} & x_{x,0} & x_{y,0} \end{array}\right)^{\mathrm{T}} \tag{2.10}$$

are mode-independent and linear, with x_i, v_i denoting the position and heading components. The event function

$$\Phi(\boldsymbol{\zeta}(t), \boldsymbol{u}(t), q(t), e) = \boldsymbol{D}_e \boldsymbol{u}(t) \left(\boldsymbol{x}_{\mathrm{p},e} - \boldsymbol{C}\boldsymbol{\zeta}(t)\right) = \bar{\boldsymbol{n}}^{\mathrm{T}}_{\bar{q}_k^{\star}}(t) \left(\boldsymbol{x}_{\mathrm{p},e} - \boldsymbol{C}\boldsymbol{\zeta}(t)\right) \tag{2.11}$$

is piecewise affine and defines event planes. The input $\boldsymbol{u}(t) = \left(\boldsymbol{n}_1^{\mathrm{T}}(t) \quad \ldots \quad \boldsymbol{n}_N^{\mathrm{T}}(t)\right)^{\mathrm{T}}$ comprises the normal vectors of all walls $\mathcal{W}_e$, $\boldsymbol{D}_e$ extracts the relevant input elements and $\boldsymbol{C} = \left(\mathbf{0} \;\middle|\; \boldsymbol{I}\right)$. Each event $e \in \{1, \ldots, N\}$ indicates the collision of the ball with the associated wall. At an impact the velocity component of $\boldsymbol{\zeta}$ changes discontinuously according to

$$\boldsymbol{\zeta}(\bar{t}(k)^{+}) = \left(\begin{array}{c|c} \boldsymbol{I} - 2\dfrac{\boldsymbol{n}_{\bar{e}(k)}(\bar{t}(k)^{-})\boldsymbol{n}^{\mathrm{T}}_{\bar{e}(k)}(\bar{t}(k)^{-})}{\boldsymbol{n}^{\mathrm{T}}_{\bar{e}(k)}(\bar{t}(k)^{-})\boldsymbol{n}_{\bar{e}(k)}(\bar{t}(k)^{-})} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{I} \end{array}\right) \boldsymbol{\zeta}(\bar{t}(k)^{-}) \tag{2.12}$$

Remark 1. *Without dynamically controlling the wall orientations, the initial conditions of the ball completely determine the impacting sequence $e(t)$. However, by applying rotations to the walls, $e(t)$ can obviously be altered. A changing impact sequence is known to result in drastic changes to the ball's future evolution, which is indeed the source for chaos in billiards [19].*

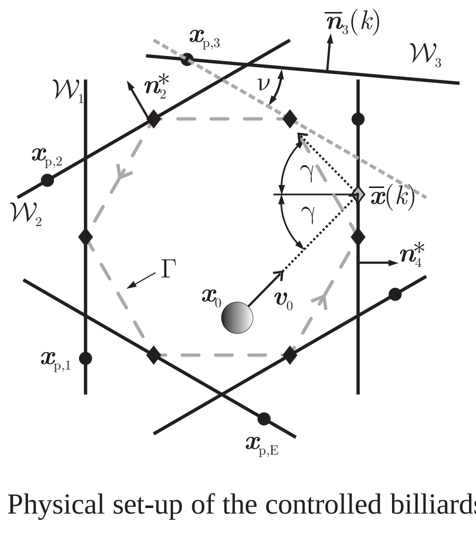

Figure 2.1: Physical set-up of the controlled billiards problem.

2.3 Essential properties of the example system

By looking at the example, several interesting properties are observed, which crucially influence the control of IHSCR in general.

1. Although the dynamics (2.10) of the example are LTI and admit a unique equilibrium, it is not the goal to stabilize this point. This statement applies to many IHSCR.

2. Even second order IHSCR typically exhibit chaotic behavior, which is caused by the state resets and changes in the event sequence $\{\overline{e}(k)\}_{k=0}^{\infty}$.

3. IHSCR always appear as feedback loops. The inner loop's task is to guarantee the perpetual generation of events to provide the means for control. Only the outer loop enables to regulate the output by adjusting the parameters of the inner loop.

4. The primary control influence, which is the state reset, cannot be decoupled from the event time, as the event surfaces are fixed at points in state-space. *This dependence is a unique feature of IHSCR and complicates the control task significantly.*

5. Although the control input may be varied in continuous-time, it only takes effect at event times. Therefore, a continuous-time model contains excessive, irrelevant information and is rather unsuited for the design.

3 Problem formulation for the billiards problem

For IHSCR it is clear, that the systematic design of the event-surface controller requires detailed knowledge about the structure of the event function (2.2) and the reset map (2.7). To devise a uniformly applicable control strategy for general IHSCR is very hard, if not impossible. Therefore, all subsequent sections concentrate on the controller design for the controlled billiards system, which represents a particular subclass of planar IHSCR. Here, the primary emphasis is to design a globally stabilizing control strategy for prespecified unstable periodic executions $\chi^\star$.

Definition 3.1 (Asymptotic orbital stability)**.** *[20] A periodic execution $\chi^\star$ is called asymptotically orbitally stable, iff for any $\epsilon > 0$ there exists a $\delta > 0$, such that any execution χ, with $\boldsymbol{\zeta}(t)$ starting at $\|\zeta(0) - \zeta^\star(0)\| < \delta$, converges towards $\chi^\star$, which implies that* $\mathrm{dist}(\boldsymbol{\zeta}(t), \Gamma) < \epsilon$, $\forall t > 0$ *and* $\lim_{t\to\infty} \mathrm{dist}(\boldsymbol{\zeta}(t), \Gamma) = 0$.

Definition 3.2 (Global asymptotic orbital stability)**.** *A periodic execution $\chi^\star$ is called globally asymptotically orbitally stable, iff it is asymptotically stable for arbitrary initial conditions, i.e. $\delta \to \infty$*

Problem 1. Given *the hybrid model* (2.10)-(2.12) *of the billiards problem, which represents a particular class of planar IHSCR and an admissible desired stationary orbit* Γ, *the objective is to* determine *an event surface control law*

$$\boldsymbol{u}(t) = \boldsymbol{f}_c(\boldsymbol{\zeta}(t), q(t)) \tag{3.1}$$

in state-feedback form, which ensures global asymptotic orbital stability.

Generally, the event surface controller (3.1) must be stated explicitly in feedback form, as many IHSCR are inherently chaotic. Thus, open-loop approaches will fail, as infinitely small perturbations will cause the trajectory to diverge from its planned path.

Remark 2. *To be consistent with the defintion of a periodic execution $\chi^\star(\boldsymbol{\zeta}_0^{h,\star}, \boldsymbol{u}^\star(t), 0, N)$ and to avoid presentation issues that arise, when modes are being activated multiple times over one period, the single-mode billiards problem will be subsequently modeled as a multi-modal system. Each event $\overline{e}_k^\star$ generated by the execution $\chi^\star$ is associated with a mode $\overline{q}_k^\star = \overline{e}_{k+1}^\star$. Hence, mode $\overline{q}_k^\star$ being active indicates that event $\overline{e}_{k+1}^\star$ is to occur next.*

4 Sampled-data abstraction for the controlled billiards system

A model-based design procedure requires a model, which explicitly establishes the connection between the design parameters and their effect on the control output. For IHSCR, one possible approach of achieving this is to utilize an event-based sampled description, denoted as the *controlled embedded map* [21]. This map describes the state evolution from one impact $\bar{\boldsymbol{x}}(k)$ to the next $\bar{\boldsymbol{x}}(k+1)$. For the controlled billiards system, this embedded map decomposes into two series-connected subsystems:

Embedded map of the billiard system:

$$\bar{\boldsymbol{v}}(k+1) = \left(\boldsymbol{I} - 2\frac{\bar{\boldsymbol{n}}_{\bar{q}(k)}(k)\,\bar{\boldsymbol{n}}^{\mathrm{T}}_{\bar{q}(k)}(k)}{\bar{\boldsymbol{n}}^{\mathrm{T}}_{\bar{q}(k)}(k)\,\bar{\boldsymbol{n}}_{\bar{q}(k)}(k)} \right) \bar{\boldsymbol{v}}(k)\,, \quad \bar{\boldsymbol{v}}(0) = \boldsymbol{v}_0 \tag{4.1}$$

$$\bar{\boldsymbol{x}}(k+1) = \left(\boldsymbol{I} - \frac{\bar{\boldsymbol{v}}(k)\,\bar{\boldsymbol{n}}^{\mathrm{T}}_{\bar{q}(k)}(k)}{\bar{\boldsymbol{n}}^{\mathrm{T}}_{\bar{q}(k)}(k)\,\bar{\boldsymbol{v}}(k)} \right) \bar{\boldsymbol{x}}(k) + \frac{\bar{\boldsymbol{v}}(k)\,\boldsymbol{x}^{\mathrm{T}}_{p,\bar{q}(k)}}{\bar{\boldsymbol{n}}^{\mathrm{T}}_{\bar{q}(k)}(k)\,\bar{\boldsymbol{v}}(k)} \bar{\boldsymbol{n}}_{\bar{q}(k)}(k)\,, \quad \bar{\boldsymbol{x}}(0) = \boldsymbol{x}_0 \tag{4.2}$$

In between two events, only the ball's position $\boldsymbol{x}(t)$ evolves according to the current heading, while resets solely update the heading $\boldsymbol{v}(t)$. This specific coupling between the heading, the position and the event-surface controller is illustrated by the block diagram of Fig. 4.1.

Given the desired stationary orbit Γ, its mode sequence Q_Γ and constraining all inputs to $\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\bar{\boldsymbol{n}}_{\bar{q}(k)}(k) = 1$, it is possible to restate equations (4.1), (4.2) in terms of the errors $\Delta\bar{\boldsymbol{x}}(k) = \bar{\boldsymbol{x}}(k) - \bar{\boldsymbol{x}}^{\star}\left(\bar{q}^{\star}_k\right)$, $\Delta\bar{\boldsymbol{v}}(k) = \bar{\boldsymbol{v}}(k) - \bar{\boldsymbol{v}}^{\star}\left(\bar{q}^{\star}_k\right)$:

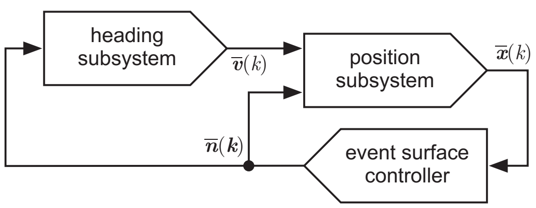

Figure 4.1: Block diagram illustrating the coupling of the billiards system.

Embedded map of the billiard system with respect to Γ:

$$\Delta\bar{\boldsymbol{v}}(k+1) = \Delta\bar{\boldsymbol{v}}(k) - \frac{2\bar{\boldsymbol{n}}_{\bar{q}(k)}(k)}{\bar{\boldsymbol{n}}^{\mathrm{T}}_{\bar{q}(k)}(k)\,\bar{\boldsymbol{n}}_{\bar{q}(k)}(k)} + \frac{2\bar{\boldsymbol{n}}^{\star}_{\bar{q}(k)}}{\bar{\boldsymbol{n}}^{\star\mathrm{T}}_{\bar{q}(k)}\bar{\boldsymbol{n}}^{\star}_{\bar{q}(k)}}, \quad \bar{\boldsymbol{v}}(0) = \boldsymbol{v}_0 \tag{4.3}$$

$$\Delta\bar{\boldsymbol{x}}(k+1) = \Delta\bar{\boldsymbol{x}}(k) + \bar{\tau}^{\star}(\bar{q}(k))\,\Delta\bar{\boldsymbol{v}}(k) + \bar{\boldsymbol{v}}(k)\underbrace{\left\{\Delta\boldsymbol{x}^{\mathrm{T}}_{p,k}\bar{\boldsymbol{n}}_{\bar{q}(k)}(k) - \bar{\tau}(\bar{q}(k))\right\}}_{\bar{\alpha}(k)} \tag{4.4}$$

Equation (4.3), (4.4) reveal the following information: 1. Only by controlling the wall orientations at operation, the heading error can be compensated, 2. only by deviating from the nominal heading, the position error can be reduced, and, 3. a vanishing position error $\Delta\bar{\boldsymbol{x}}(k) = 0,\ \forall k > K$ implies $\Delta\bar{\boldsymbol{v}}(k) = 0,\ \forall k > K$. The last observation might suggest to concentrate the control effort on monotonically reducing the position error at every impact. This, however, requires to find a proper distance measure for deviations from the desired orbit, for instance in terms of a periodic Lyapunov function, which is by no means trivial for time-varying systems. An easier way to solve the control problem is indeed to apply control actions that monotonically decrease the heading error from impact to impact, as shown in the following.

5 Design of globally stabilizing event surface controllers

5.1 Outline of the hybrid control strategy

The following sections present a globally stabilizing hybrid control strategy for the controlled billiards problem. As a consequence of the chaotic behavior, the control must be of *feedback form*. To simplify the presentation, the following assumptions and definitions are made:

1. the pivots $\boldsymbol{x}_{\bar{q}_k^\star}$ of the walls are design variables

2. pivots are placed, such that $\bar{\boldsymbol{v}}^{\star\mathrm{T}}\left(\bar{q}_k^\star\right)\left(\boldsymbol{x}_{\mathrm{p},\bar{q}_k^\star} - \bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right)\right) < 0$

3. the initial conditions of the ball admit an initial mode $q(0)$, such that by application of dedicated control actions, the stationary impacting sequence $\mathcal{Q}_\Gamma$ can be maintained at all times.

4. without loss of generality, the vector $\bar{\boldsymbol{v}}(0)$ is assumed to be of unit length.

Definition 5.1. *The orthogonal complement* $\bar{\boldsymbol{v}}_\perp^\star\left(\bar{q}_k^\star\right)$ *of the k-th stationary heading* $\bar{\boldsymbol{v}}^\star\left(\bar{q}_k^\star\right)$ *satisfies*

$$\bar{\boldsymbol{v}}_\perp^{\star\mathrm{T}}\left(\bar{q}_k^\star\right)\bar{\boldsymbol{v}}^\star\left(\bar{q}_k^\star\right) = 0 \tag{5.1}$$

$$\bar{\boldsymbol{v}}_\perp^{\star\mathrm{T}}\left(\bar{q}_k^\star\right)\left(\boldsymbol{x}_{\mathrm{p},\bar{q}_k^\star} - \bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right)\right) > 0 \ , \tag{5.2}$$

i.e. it points to the outside of the orbit Γ.

Definition 5.2. *The* $(k+1)$*-st impact of the ball with the subsequent wall* $\mathcal{W}_{\bar{q}_k^\star}$ *is said to occur on the* INSIDE *the orbit* Γ*, if* $\bar{\boldsymbol{x}}(k+1) \in \Gamma_{\bar{q}_{k+1}^\star}$ *and the conditions*

$$\left(\bar{\boldsymbol{x}}(k+1) - \bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right)\right)^{\mathrm{T}}\left(\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+2}^\star\right) - \bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right)\right) \geq 0 \tag{5.3}$$

$$\left(\bar{\boldsymbol{x}}(k+1) - \bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+2}^\star\right)\right)^{\mathrm{T}}\left(\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+2}^\star\right) - \bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right)\right) \leq 0 \tag{5.4}$$

are satisfied. An impact occurs on the OUTSIDE *of* Γ*, if* $\bar{\boldsymbol{x}}(k+1) \in \Gamma_{\bar{q}_{k+1}^\star}$ *and it does not occur on the inside.*

Definition 5.3. *Define the angles between the vectors* $\bar{\boldsymbol{v}}(k)$ *and* $\bar{\boldsymbol{v}}^{\star}\left(\bar{q}_k^{\star}\right)$ *and between* $\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}(k)$ *and* $\bar{\boldsymbol{n}}^{\star}_{\bar{q}_k^{\star}}$ *as*

$$\delta_k = \arccos\left(\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\bar{\boldsymbol{v}}^{\star}\left(\bar{q}_k^{\star}\right)\right)$$
$$\nu_k = \arccos\left(\frac{\bar{\boldsymbol{n}}^{\mathrm{T}}_{\bar{q}_k^{\star}}(k)\,\bar{\boldsymbol{n}}^{\star}_{\bar{q}_k^{\star}}}{\|\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}(k)\|\|\bar{\boldsymbol{n}}^{\star}_{\bar{q}_k^{\star}}\|}\right) .$$

The proposed control strategy, which ensures global orbital stability of Γ, involves two different control maneuvers:

1. point-to-point transfer in the $x - y$ plane
2. enforcing impacts on prespecified affine manifolds.

Given the desired orbit Γ, it possible to fix the pivots of $\mathcal{W}_{\bar{q}_k^{\star}}$, such that by appropriately concatenating these actions, the ball is driven to Γ from arbitrary initial positions and headings. The key ingredient of the control strategy is to reduce the heading error $\Delta\bar{\boldsymbol{v}}(k)$ asymptotically. To show this requires several intermediate steps, which are summarized in the following sections.

5.2 Feedback expressions for the control maneuvers

For both maneuvers, the control inputs, i.e. the normals $\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}$ for mode $\bar{q}(k) = \bar{q}_k^{\star}$, can be computed by explicit state-feedback expressions.

Lemma 5.1. *By setting* $\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}(k)$ *to*

$$\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}(k) = \begin{pmatrix} -\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\boldsymbol{Q}_{\bar{q}_k^{\star}}\bar{\boldsymbol{v}}(k)\left(\boldsymbol{x}_{p,\bar{q}_k^{\star}} - \bar{\boldsymbol{x}}(k)\right)^{\mathrm{T}} \\ \bar{\boldsymbol{v}}^{\mathrm{T}}(k) \end{pmatrix}^{-1} \begin{pmatrix} \bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\boldsymbol{Q}_{\bar{q}_k^{\star}}\left(\Delta\bar{\boldsymbol{x}}(k) - \bar{\tau}^{\star}\left(\bar{q}_k^{\star}\right)\bar{\boldsymbol{v}}^{\star}\left(\bar{q}_k^{\star}\right)\right) \\ 1 \end{pmatrix} \quad \textit{with} \tag{5.5}$$

$$\boldsymbol{Q}_q = \boldsymbol{I} - \frac{\boldsymbol{m}_q\boldsymbol{m}_q^{\mathrm{T}}}{\boldsymbol{m}_q^{\mathrm{T}}\boldsymbol{m}_q} \quad , \tag{5.6}$$

the impact between $\mathcal{W}_{\bar{q}_k^{\star}}$ *and the ball starting off at an arbitrary* $\bar{\boldsymbol{x}}(k)$ *with arbitrary heading* $\bar{\boldsymbol{v}}(k)$ *occurs on the affine mainfold*

$$\mathcal{P}\left(\bar{q}_k^{\star}\right) = \left\{\boldsymbol{x} \;:\; \boldsymbol{m}^{\mathrm{T}}_{\bar{q}_k^{\star},\perp}\left(\boldsymbol{x} - \bar{\boldsymbol{x}}^{\star}\left(\bar{q}_{k+1}^{\star}\right)\right) = 0\right\} \tag{5.7}$$

through the $(k+1)$*-st stationary impact point* $\bar{\boldsymbol{x}}^{\star}\left(\bar{q}^{\star}_{k+1}\right)$.

Proof. Parametrize $\Delta\bar{\boldsymbol{x}}(k)$ and $\bar{\boldsymbol{v}}(k)$ as

$$\Delta\bar{\boldsymbol{x}}(k) = \alpha_{p,k}\bar{\boldsymbol{v}}^{\star}\left(\bar{q}^{\star}_k\right) + \beta_{p,k}\boldsymbol{m}_{\bar{q}^{\star}_k} \tag{5.8}$$

$$\bar{\boldsymbol{v}}(k) = \alpha_{v,k}\boldsymbol{m}_{\bar{q}^{\star}_k} + \beta_{v,k}\boldsymbol{m}_{\bar{q}^{\star}_k,\perp} \quad , \tag{5.9}$$

with coefficients satisfying $-1 \leq \alpha_{v,k},\ \beta_{v,k} \leq 1, \beta_{v,k} \neq 0$. Without loss of generality, let $\|\boldsymbol{m}_{\bar{q}^{\star}_k}\| = \|\boldsymbol{m}_{\bar{q}^{\star}_k,\perp}\| = 1$. Plugging (5.8) into the (4.4), the propagated position error becomes

$$\Delta\bar{\boldsymbol{x}}(k+1) = \alpha_{p,k}\bar{\boldsymbol{v}}^{\star}\left(\bar{q}^{\star}_k\right) + \beta_{p,k}\boldsymbol{m}_{\bar{q}^{\star}_k} + \bar{\tau}^{\star}\left(\bar{q}^{\star}_k\right)\Delta\bar{\boldsymbol{v}}(k) + \bar{\alpha}(k)\,\bar{\boldsymbol{v}}(k) \ \text{ with} \tag{5.10}$$

$$\begin{aligned}\bar{\alpha}(k) &= -\alpha_{p,k}\frac{\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\boldsymbol{Q}_{\bar{q}^{\star}_k}\bar{\boldsymbol{v}}^{\star}\left(\bar{q}^{\star}_k\right)}{\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\boldsymbol{Q}_{\bar{q}^{\star}_k}\bar{\boldsymbol{v}}(k)} - \bar{\tau}^{\star}\left(\bar{q}^{\star}_k\right)\frac{\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\boldsymbol{Q}_{\bar{q}^{\star}_k}\Delta\bar{\boldsymbol{v}}(k)}{\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\boldsymbol{Q}_{\bar{q}^{\star}_k}\bar{\boldsymbol{v}}(k)} \\ &= -\alpha_{p,k} + \left(\alpha_{p,k} - \bar{\tau}^{\star}\left(\bar{q}^{\star}_k\right)\right)\frac{\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\boldsymbol{Q}_{\bar{q}^{\star}_k}\Delta\bar{\boldsymbol{v}}(k)}{\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\boldsymbol{Q}_{\bar{q}^{\star}_k}\bar{\boldsymbol{v}}(k)}\end{aligned} \tag{5.11}$$

$$\Rightarrow \quad \Delta\bar{\boldsymbol{x}}(k+1) = \beta_{p,k}\boldsymbol{m}_{\bar{q}^{\star}_k} - \left(\alpha_{p,k} - \bar{\tau}^{\star}\left(\bar{q}^{\star}_k\right)\right)\left(\boldsymbol{I} - \frac{\bar{\boldsymbol{v}}(k)\,\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\boldsymbol{Q}_{\bar{q}^{\star}_k}}{\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\boldsymbol{Q}_{\bar{q}^{\star}_k}\bar{\boldsymbol{v}}(k)}\right)\Delta\bar{\boldsymbol{v}}(k) \tag{5.12}$$

Now, applying the second parametrization (5.9) of the current heading and accounting for (5.6), the previous expression further simplifies to

$$\Delta\bar{\boldsymbol{x}}(k+1) = \beta_{p,k}\boldsymbol{m}_{\bar{q}^{\star}_k} - \left(\alpha_{p,k} - \bar{\tau}^{\star}\left(\bar{q}^{\star}_k\right)\right)\left(\boldsymbol{I} - \frac{\bar{\boldsymbol{v}}(k)\,\boldsymbol{m}^{\mathrm{T}}_{\bar{q}^{\star}_k,\perp}}{\beta_{v,k}}\right)\Delta\bar{\boldsymbol{v}}(k) \quad . \tag{5.13}$$

Finally, $\boldsymbol{m}^{\mathrm{T}}_{\bar{q}^{\star}_k,\perp}\Delta\bar{\boldsymbol{x}}(k+1) = 0$ is easily recovered, which proves, that the subsequent impact occurs on the manifold $\mathcal{P}\left(\bar{q}^{\star}_k\right)$. □

Lemma 5.2. *A point-to-point transfer from* $\bar{\boldsymbol{x}}(k)$ *to an arbitrary point* $\boldsymbol{x}_T$ *can be achieved with just one intermediate impact by setting the normal of the subsequently intersected wall* $\mathcal{W}_{\bar{q}^{\star}_k}$ *to*

$$\boldsymbol{n}_e(k) = \bar{\boldsymbol{v}}(k) + \gamma_k\bar{\boldsymbol{v}}_{\perp}(k) \quad , \tag{5.14}$$

if and only if

$$\gamma_k = -\frac{a_{12} - b_{12}}{a_{22}} + \sqrt{\frac{(a_{12} - b_{12})^2}{a_{22}^2} - \frac{c + a_{11} - 2b_{11}}{a_{22}}} \tag{5.15}$$

is positive and real-valued, where $\bar{\boldsymbol{v}}_\perp^{\mathrm{T}}(k)\,\bar{\boldsymbol{v}}_\perp^{\star}\big(\bar{q}_k^{\star}\big) > 0$ *and*

a_{11}	a_{12}	a_{22}	b_{11}	b_{12}
$\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\boldsymbol{M}\bar{\boldsymbol{v}}(k)$	$\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\boldsymbol{M}\bar{\boldsymbol{v}}_\perp(k)$	$\bar{\boldsymbol{v}}_\perp^{\mathrm{T}}(k)\,\boldsymbol{M}\bar{\boldsymbol{v}}_\perp(k)$	$\boldsymbol{t}^{\mathrm{T}}\bar{\boldsymbol{v}}(k)$	$\boldsymbol{t}^{\mathrm{T}}\bar{\boldsymbol{v}}_\perp(k)$

$\boldsymbol{M}$	$\boldsymbol{t}^{\mathrm{T}}$	c
$\left(\Delta\bar{\boldsymbol{x}}_{p,k}\Delta\bar{\boldsymbol{x}}_{p,k}^{\mathrm{T}} - \Delta\bar{\boldsymbol{x}}_{p,T}\Delta\bar{\boldsymbol{x}}_{p,T}^{\mathrm{T}}\right)$	$\Delta\bar{\boldsymbol{x}}_{T,k}^{\mathrm{T}}\bar{\boldsymbol{v}}(k)\,\Delta\bar{\boldsymbol{x}}_{p,k}^{\mathrm{T}}$	$\Delta\bar{\boldsymbol{x}}_{T,k}^{\mathrm{T}}\Delta\bar{\boldsymbol{x}}_{T,k}$
$\Delta\bar{\boldsymbol{x}}_{p,T}$	$\Delta\bar{\boldsymbol{x}}_{p,k}$	$\Delta\bar{\boldsymbol{x}}_{T,k}$
$\boldsymbol{x}_{\mathrm{p},\bar{q}_k^{\star}} - \boldsymbol{x}_T$	$\boldsymbol{x}_{\mathrm{p},\bar{q}_k^{\star}} - \bar{\boldsymbol{x}}(k)$	$\boldsymbol{x}_T - \bar{\boldsymbol{x}}(k)$

.

Proof. First, note that the parametrization (5.14) allows to choose γ_k arbitrarily, without violating the length constraint $\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}^{\mathrm{T}}(k)\,\bar{\boldsymbol{v}}(k) = 1$. Given $\bar{\boldsymbol{x}}(k)$, $\bar{\boldsymbol{v}}(k)$, the subsequent impact point satisfies

$$\bar{\boldsymbol{x}}(k{+}1) = \bar{\boldsymbol{x}}(k) + \bar{\boldsymbol{v}}(k)\,\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}^{\mathrm{T}}(k)\,\Delta\bar{\boldsymbol{x}}_{p,k} \tag{5.16}$$

Alternatively, it can also be expressed in terms of the end-point $\boldsymbol{x}_T$ as

$$\bar{\boldsymbol{x}}(k{+}1) = \boldsymbol{x}_T - \bar{\boldsymbol{v}}(k{+}1)\,\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}^{\mathrm{T}}(k)\,\Delta\bar{\boldsymbol{x}}_{p,T} \quad , \tag{5.17}$$

which utilizes that $\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}(k) = 1$ and (4.1) together imply $\bar{\boldsymbol{v}}^{\mathrm{T}}(k{+}1)\,\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}(k) = -1$. Therefore

$$\Delta\bar{\boldsymbol{x}}_{k+1,k} = \bar{\boldsymbol{v}}(k)\,\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}^{\mathrm{T}}(k)\,\Delta\bar{\boldsymbol{x}}_{p,k} \tag{5.18}$$

$$\Delta\bar{\boldsymbol{x}}_{T,k+1} = \bar{\boldsymbol{v}}(k{+}1)\,\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}^{\mathrm{T}}(k)\,\Delta\bar{\boldsymbol{x}}_{p,T} \tag{5.19}$$

Expanding the equality

$$(\Delta\bar{\boldsymbol{x}}_{T,k} - \Delta\bar{\boldsymbol{x}}_{k+1,k})^{\mathrm{T}}\,(\Delta\bar{\boldsymbol{x}}_{T,k} - \Delta\bar{\boldsymbol{x}}_{k+1,k}) = \Delta\bar{\boldsymbol{x}}_{T,k+1}^{\mathrm{T}}\Delta\bar{\boldsymbol{x}}_{T,k+1} \quad , \tag{5.20}$$

plugging in (5.18), (5.19), remembering $\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\bar{\boldsymbol{v}}(k) = \bar{\boldsymbol{v}}^{\mathrm{T}}(k{+}1)\,\bar{\boldsymbol{v}}(k{+}1) = 1$ and rearranging terms yields the expression

$$0 = \bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}^{\mathrm{T}}(k)\underbrace{\left(\Delta\bar{\boldsymbol{x}}_{p,k}\Delta\bar{\boldsymbol{x}}_{p,k}^{\mathrm{T}} - \Delta\bar{\boldsymbol{x}}_{p,T}\Delta\bar{\boldsymbol{x}}_{p,T}^{\mathrm{T}}\right)}_{\boldsymbol{M}}\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}(k) - 2\underbrace{\Delta\bar{\boldsymbol{x}}_{T,k}^{\mathrm{T}}\bar{\boldsymbol{v}}(k)\,\Delta\bar{\boldsymbol{x}}_{p,k}^{\mathrm{T}}}_{\boldsymbol{t}^{\mathrm{T}}}\bar{\boldsymbol{n}}_{\bar{q}_k^{\star}}(k) + \underbrace{\Delta\bar{\boldsymbol{x}}_{T,k}^{\mathrm{T}}\Delta\bar{\boldsymbol{x}}_{T,k}}_{c} \quad . \tag{5.21}$$

Substitution of (5.14) into (5.21) and solving for γ_k finally yields two solutions, where only (5.15) is valid, as the other solution always yields a negative event distance $\bar{\tau}(k)$. □

Remark 3. *Note, that the terminal point* $\boldsymbol{x}_T$ *appears in almost every coefficient* a_{ij}, b_{1i} *and* c. *Thus, the scaling factor* γ_k *of the normal* $\boldsymbol{n}_{\bar{q}(k)}(k)$ *is a highly nonlinear function of this point. In planar IHSCR, it is indeed possible to drive any trajectory to the orbit* Γ *with exactly two resets. However, the control input cannot be stated in closed form.*

5.3 Generating decaying error sequences $\Delta\bar{\boldsymbol{v}}(k)$, $\Delta\bar{\boldsymbol{x}}(k)$ by enforcing impacts on $\Gamma_{\bar{q}_k^\star}$.

After the derivation of explicit expressions for the control input, the next step is to show how to stitch the two maneuvers together, such that global orbital stability is achieved. Sect. 4 highlighted that finding a control input sequence, which makes the position error become a strictly decreasing sequence, is difficult. This is due to the cascaded coupling structure of the subsystems (Fig. 4.1). How to derive a control input sequence that results in a strictly decreasing heading error sequence, on the other hand, is intuitively clear.

Lemma 5.3. *The heading error $\Delta\bar{\boldsymbol{v}}(k)$ decays at every impact, iff the $|\nu_k| \leq |\delta_k|$ and* $\mathrm{sgn}(\nu_k) = \mathrm{sgn}(\delta_k)$.

Proof. A decaying heading error implies

$$\Delta\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\Delta\bar{\boldsymbol{v}}(k) - \Delta\bar{\boldsymbol{v}}^{\mathrm{T}}(k+1)\,\Delta\bar{\boldsymbol{v}}(k+1) \geq 0 \ . \tag{5.22}$$

Expanded this expression using $\|\bar{\boldsymbol{v}}(k)\| = 1$, $\bar{\boldsymbol{n}}^{\mathrm{T}}_{\bar{q}_k^\star}(k)\,\bar{\boldsymbol{n}}_{\bar{q}_k^\star}(k) = l$, $\bar{\boldsymbol{n}}^{\star\mathrm{T}}_{\bar{q}_k^\star}\bar{\boldsymbol{n}}^{\star}_{\bar{q}_k^\star} = l_0$, $\Delta\bar{\boldsymbol{v}}^{\star\mathrm{T}}_{k,k+1}\Delta\bar{\boldsymbol{v}}^{\star}_{k,k+1} = 4/l_0$, $\Delta\bar{\boldsymbol{v}}^{\star}_{k,k+1} = 2\bar{\boldsymbol{n}}^{\star}_{\bar{q}_k^\star}/l_0$ and noting that

$$\Delta\bar{\boldsymbol{v}}(k+1) = \left(\Delta\bar{\boldsymbol{v}}(k) + \Delta\bar{\boldsymbol{v}}^{\star}_{k,k+1}\right) - \frac{2\bar{\boldsymbol{n}}_{\bar{q}_k^\star}(k)}{\bar{\boldsymbol{n}}^{\mathrm{T}}_{\bar{q}_k^\star}(k)\,\bar{\boldsymbol{n}}_{\bar{q}_k^\star}(k)} \tag{5.23}$$

yields

$$\begin{aligned}
&-l\left(2\Delta\bar{\boldsymbol{v}}^{\star\mathrm{T}}_{k,k+1}\Delta\bar{\boldsymbol{v}}(k) + \frac{4}{l_0}\right) + 4\left(\left(\Delta\bar{\boldsymbol{v}}(k) + \Delta\bar{\boldsymbol{v}}^{\star}_{k,k+1}\right)^{\mathrm{T}}\bar{\boldsymbol{n}}_{\bar{q}_k^\star}(k) - 1\right) \geq 0\\
&\Leftrightarrow -l\left(4\underbrace{\frac{\bar{\boldsymbol{n}}^{\star\mathrm{T}}_{\bar{q}_k^\star}}{l_0}\bar{\boldsymbol{v}}(k)}_{\leq 1} + \underbrace{\frac{4}{l_0} - \frac{4}{l_0}}_{=0}\right) + 4\left(-\underbrace{\bar{\boldsymbol{v}}^{\star\mathrm{T}}\left(\bar{q}^\star_{k+1}\right)\bar{\boldsymbol{n}}_{\bar{q}_k^\star}(k)}_{\leq 0}\right) \geq 0\\
&\Leftrightarrow -4l\left(\underbrace{\frac{\bar{\boldsymbol{n}}^{\star\mathrm{T}}_{\bar{q}_k^\star}}{l_0}\bar{\boldsymbol{v}}(k)}_{\leq 1} + \underbrace{\frac{\bar{\boldsymbol{n}}^{\mathrm{T}}_{\bar{q}_k^\star}(k)}{l}\bar{\boldsymbol{v}}^{\star}\left(\bar{q}^\star_{k+1}\right)}_{\leq 0}\right) \geq 0
\end{aligned} \tag{5.24}$$

Furthermore parametetrize the normal $\bar{\boldsymbol{n}}_{\bar{q}_k^\star}(k)$ and the current heading $\bar{\boldsymbol{v}}(k)$ with respect to their stationary counterparts

$$\bar{\boldsymbol{n}}_{\bar{q}_k^\star}(k) = \boldsymbol{T}(\nu_k)\,\bar{\boldsymbol{n}}^{\star}_{\bar{q}_k^\star} \tag{5.25}$$

$$\bar{\boldsymbol{v}}(k) = \boldsymbol{T}(\delta_k)\,\bar{\boldsymbol{v}}^{\star}\left(\bar{q}_k^\star\right) \tag{5.26}$$

by using Def. 5.3. Then, the two terms of (5.24) evaluate to

$$\Rightarrow \left(\frac{\bar{\boldsymbol{n}}_{\bar{q}_k^\star}^{\star\mathrm{T}}}{l_0}\bar{\boldsymbol{v}}(k)\right) = (\bar{\boldsymbol{v}}^{\star\mathrm{T}}\left(\bar{q}_{k+1}^\star\right)\bar{\boldsymbol{v}}^\star\left(\bar{q}_{k+1}^\star\right))\cos(\theta_k + \delta_k)\cos(\theta) \tag{5.27}$$

$$\Rightarrow \left(\frac{\bar{\boldsymbol{n}}^{\mathrm{T}}(k)}{l}\bar{\boldsymbol{v}}^\star\left(\bar{q}_{k+1}^\star\right)\right) = -(\bar{\boldsymbol{v}}^{\star\mathrm{T}}\left(\bar{q}_{k+1}^\star\right)\bar{\boldsymbol{v}}^\star\left(\bar{q}_{k+1}^\star\right))\cos(\theta_k + \delta_k - \nu_k)\cos(\theta_k + \nu_k) \quad , \tag{5.28}$$

where θ_k is the angle between $\bar{\boldsymbol{v}}^\star\left(\bar{q}_k^\star\right)$ and $\bar{\boldsymbol{n}}_{\bar{q}_k^\star}^\star$. After substituting (5.27), (5.28) into (5.24), expanding the expression and rearranging terms the condition

$$-4(\bar{\boldsymbol{v}}^{\star\mathrm{T}}\left(\bar{q}_{k+1}^\star\right)\bar{\boldsymbol{v}}^\star\left(\bar{q}_{k+1}^\star\right))\left(\cos(\delta_k) - \cos(\delta_k)\cos(\nu_k)^2 - \frac{\sin(\delta_k)}{\sin(\nu_k)}\sin(\nu_k)^2\cos(\nu_k)\right) \geq 0 \tag{5.29}$$

is finally obtained. Accordingly, the above proves that, regardless of the value of θ_k, (5.24) always holds, as long as $|\nu_k| \leq |\delta_k|$ and $\mathrm{sgn}(\nu_k) = \mathrm{sgn}(\delta_k)$ is satisfied. □

Lemma 5.4. *If all impacts occur on the orbit sections $\Gamma_{\bar{q}_k^\star}$, the impacting sequence alternates between the inside and the outside of Γ, iff $2\,|\nu_k| \leq |\delta_k|$ and $\mathrm{sgn}(\nu_k) = \mathrm{sgn}(\delta_k)$. If $2\nu_k = \delta_k$, complete compensation of the heading error is accomplished.*

Proof. As the system matrix of (4.1) constitutes a freely designable Householder reflector and $\|\bar{\boldsymbol{v}}(k)\| = \|\bar{\boldsymbol{v}}(k+1)\|$, a complete heading error compensation at the next impact is always possible, if impacts are not restricted to lie on $\Gamma_{\bar{q}_{k+1}^\star}$. Hence, (5.22) possesses a maximum with the maximizer $\nu_{k,\max}$ to be computed by solving its derivative

$$\frac{\mathrm{d}(\Delta\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\Delta\bar{\boldsymbol{v}}(k) - \Delta\bar{\boldsymbol{v}}^{\mathrm{T}}(k+1)\,\Delta\bar{\boldsymbol{v}}(k+1))}{\mathrm{d}\nu_k} = \\ -4(\bar{\boldsymbol{v}}^{\star\mathrm{T}}\left(\bar{q}_{k+1}^\star\right)\bar{\boldsymbol{v}}^\star\left(\bar{q}_{k+1}^\star\right))(\sin(\delta_k) - \cos(\nu_{k,\max})\sin(\delta_k - \nu_{k,\max})) = 0 \quad , \tag{5.30}$$

for zero. The above expression only vanishes for $\nu_{k,\max} = \delta_k/2$ and its derivative is always negative, such that there exists no other maximizer.

Impacts, which alter between on the inside and on the outside of Γ require that the terms $\bar{\boldsymbol{n}}_{\bar{q}_k^\star}^{\star\mathrm{T}}\Delta\bar{\boldsymbol{v}}(k)$ and $\bar{\boldsymbol{n}}_{\bar{q}_k^\star}^{\star\mathrm{T}}\Delta\bar{\boldsymbol{v}}(k+1)$ possess opposite signs. If $\mathcal{W}_{\bar{q}_k^\star}$ is oriented nominally, i.e. $\bar{\boldsymbol{n}}_{\bar{q}_k^\star}(k) = \xi_k\bar{\boldsymbol{n}}_{\bar{q}_k^\star}^\star$, $\xi_k = 1/(\bar{\boldsymbol{n}}_{\bar{q}_k^\star}^{\star\mathrm{T}}\bar{\boldsymbol{v}}(k))$, then from (4.3) we get

$$\bar{\boldsymbol{n}}_{\bar{q}_k^\star}^{\star\mathrm{T}}\Delta\bar{\boldsymbol{v}}(k+1) = -\frac{1-\xi_k}{\xi_k} = -\bar{\boldsymbol{n}}_{\bar{q}_k^\star}^{\star\mathrm{T}}\Delta\bar{\boldsymbol{v}}(k) \tag{5.31}$$

Thus, the condition for alternating impacts holds for $\nu_k = 0$. By the previous result, this condition must indeed hold for the whole interval $\nu_k \in [0, \delta_k/2]$. □

As stated before, a one-step compensation of the heading error is rarely appropriate. If all impacts occur on $\Gamma_{\bar{q}_k^\star}$, the critical situation of $\nu_k = \delta_k = \delta_{\text{crit}}$ arises, when the condition

$$L_p^2 \frac{\sin^2(\delta_{\text{crit}})}{\cos^2(\theta_k + \delta_{2\text{crit}})} = L_\star^2 + L_k^2(\delta_{\text{crit}}) - 2L_\star L_k(\delta_{\text{crit}}) \cos(\delta_{\text{crit}}), \quad \delta_{\text{crit}} \geq 0 \tag{5.32}$$

holds true, where the following abbreviations have been used:

$$\boldsymbol{T}(\delta) = \begin{pmatrix} \cos(\delta) & \sin(\delta) \\ -\sin(\delta) & \cos(\delta) \end{pmatrix} \tag{5.33}$$

$$\Delta\bar{\boldsymbol{x}}(k,\delta) = \bar{\boldsymbol{x}}(k+1) - \bar{\boldsymbol{x}}(k) = \frac{\boldsymbol{T}(\delta)\,\bar{\boldsymbol{v}}^\star\left(\bar{q}_k^\star\right)\bar{\boldsymbol{v}}_\perp^{\star\mathrm{T}}\left(\bar{q}_{k+1}^\star\right)\left(\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right) - \bar{\boldsymbol{x}}(k)\right)}{\bar{\boldsymbol{v}}_\perp^{\star\mathrm{T}}\left(\bar{q}_{k+1}^\star\right)\boldsymbol{T}(\delta)\,\bar{\boldsymbol{v}}^\star\left(\bar{q}_k^\star\right)} \tag{5.34}$$

$$L_k(\delta) = \sqrt{\Delta\bar{\boldsymbol{x}}^\mathrm{T}(k,\delta)\,\Delta\bar{\boldsymbol{x}}(k,\delta)} \tag{5.35}$$

$$L_\star = \sqrt{\left(\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right) - \bar{\boldsymbol{x}}(k)\right)^\mathrm{T}\left(\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right) - \bar{\boldsymbol{x}}(k)\right)} \tag{5.36}$$

$$L_{p,k} = \sqrt{\left(\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right) - \boldsymbol{x}_{\mathrm{p},\bar{q}_k^\star}\right)^\mathrm{T}\left(\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right) - \boldsymbol{x}_{\mathrm{p},\bar{q}_k^\star}\right)}\ , \tag{5.37}$$

Conditions (5.32) is established from basic geometry and the cosine law. It shows that δ_{crit} provides a lower bound for δ_k, i.e. $||\nu_k|| > ||\delta_k||, \forall ||\delta_k|| < ||\delta_{\text{crit}}||$. Therefore, if a δ_{crit} satisfying (5.32) exist, asymptotic decay of the heading error from impact to impact is not necessarily guaranteed. Note, however, that the condition is violated for any δ_k, if $L_{p,k}, k$ is sufficiently large. Thus, by placing the pivots sufficiently far away from $\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right)$, global asymptotic convergence of the heading error is guaranteed for all trajectories.

Lemma 5.5. *For all nominal wall set-ups and stationary orbits Γ, there exist pivot points $\boldsymbol{x}_{\mathrm{p},\bar{q}_k^\star}$, which guarantee a monotonous decrease of the heading error sequence $\{\Delta\bar{\boldsymbol{v}}(k)\}_{k=0}^{\infty}$, provided that switching is executed on $\Gamma_{\bar{q}_k^\star}$ and the worst case position error $\Delta\bar{\boldsymbol{x}}(k) \leq \Delta_{x,wc}$ error can be bounded for all k.*

Proof. To prove Lem. 5.5, simply solve (5.32) for L_p when setting $\nu_k = \delta_k = \delta_{\text{crit}}$ and $\bar{\boldsymbol{x}}(k) = \boldsymbol{x}_{wc}$:

$$L_{p,k}^2 = \frac{\left(L_\star^2 + L_k^2(\delta_{\text{crit}}) - 2L_\star L_k(\delta_{\text{crit}})\cos(\delta_{\text{crit}})\right)\cos^2(\theta_k + \delta_{\text{crit}})}{\sin^2(\delta_{\text{crit}})}\ . \tag{5.38}$$

The minimal stabilizing pivot distance $L_p^\star$, which will guarantee uniform convergence of all trajectories starting on the limit cycle Γ, is the one obtained for limit

$$L_{p,k}^\star > \lim_{\delta_{\text{crit}} \to 0} \sqrt{\frac{\left(L_\star^2 + L_k^2(\delta_{\text{crit}}) - 2L_\star L_k(\delta_{\text{crit}})\cos(\delta_{\text{crit}})\right)\cos^2(\theta_k + \delta_{\text{crit}})}{\sin^2(\delta_{\text{crit}})}}\ . \tag{5.39}$$

For all pivots beyond this critical point, i.e. $||\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right) - \boldsymbol{x}_{\mathrm{p},\bar{q}_k^\star}|| > L_{p,k}^\star$, the inequality $\nu_k < \delta_k$

always holds. Both the nominator and the denominator approach zero, as δ_{crit} goes to zero. Therefore, the application d'Hopital's rule twice finally yields

$$L_{p,k}^\star = \left(\frac{\left(\Delta\bar{\boldsymbol{x}}^{\mathrm{T}}(k,0)\,\frac{\mathrm{d}\Delta\bar{\boldsymbol{x}}}{\mathrm{d}\delta_k}(k,0)\right)^2}{\Delta\bar{\boldsymbol{x}}^{\mathrm{T}}(k,0)\,\Delta\bar{\boldsymbol{x}}(k,0)} + \Delta\bar{\boldsymbol{x}}^{\mathrm{T}}(k,0)\,\Delta\bar{\boldsymbol{x}}(k,0) \right) \cos(\gamma) \quad , \tag{5.40}$$

$$\begin{aligned} \frac{\mathrm{d}\Delta\bar{\boldsymbol{x}}}{\mathrm{d}\delta_k}(k,0) = {} & \frac{\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}\delta}(0)\,\bar{\boldsymbol{v}}^\star\left(\bar{q}_k^\star\right)\bar{\boldsymbol{v}}_\perp^{\star\mathrm{T}}\left(\bar{q}_{k+1}^\star\right)\left(\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right) - \bar{\boldsymbol{x}}(k)\right)\bar{\boldsymbol{v}}_\perp^{\star\mathrm{T}}\left(\bar{q}_{k+1}^\star\right)\boldsymbol{T}(0)\,\bar{\boldsymbol{v}}^\star\left(\bar{q}_k^\star\right)}{\left(\bar{\boldsymbol{v}}_\perp^{\star\mathrm{T}}\left(\bar{q}_{k+1}^\star\right)\boldsymbol{T}(0)\,\bar{\boldsymbol{v}}^\star\left(\bar{q}_k^\star\right)\right)^2} \\ & - \frac{\boldsymbol{T}(0)\,\bar{\boldsymbol{v}}^\star\left(\bar{q}_k^\star\right)\bar{\boldsymbol{v}}_\perp^{\star\mathrm{T}}\left(\bar{q}_{k+1}^\star\right)\left(\bar{\boldsymbol{x}}^\star\left(\bar{q}_{k+1}^\star\right) - \bar{\boldsymbol{x}}(k)\right)\bar{\boldsymbol{v}}_\perp^{\star\mathrm{T}}\left(\bar{q}_{k+1}^\star\right)\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}\delta}(0)\,\bar{\boldsymbol{v}}^\star\left(\bar{q}_k^\star\right)}{\left(\bar{\boldsymbol{v}}_\perp^{\star\mathrm{T}}\left(\bar{q}_{k+1}^\star\right)\boldsymbol{T}(0)\,\bar{\boldsymbol{v}}^\star\left(\bar{q}_k^\star\right)\right)^2} \end{aligned} \tag{5.41}$$

which is a generic expression and only contains known or worst case quantities related to the limit cycle Γ and the nominal wall orientations. As a consequence, there exist critical pivot distances $L_{p,k}^\star$ for each limit cycle Γ, which can be computed according to (5.40). Knowing the nonimal wall orientations, the pivot points follows uniquely. □

Remark 4. *Note, that large pivot distances $L_{p,k}$ guarantee a monotonic decay of the heading error. However, at the same time, they result in smaller heading corrections at each impact. Therefore, robustness of the control strategy comes at the price of a reduced convergence rate.*

Proposition 5.1. *Placing the wall pivots at infinity destroys controllability of the billiards system.*

Proof. The proposition follows directly from Lem. 5.3 and the fact that increasing the pivot distance results in the smaller actuaction angles ν_k, if impacts are restricted to lie on $\Gamma_{\bar{q}_{k+1}^\star}$. In the extreme case of $L_{p,k} \to \infty$, $\nu_k = 0$ for all k, regardless of where the impacts occur. Thus, initial heading errors can never be corrected during operation □

The critical pivot distance $L_{p,k}^\star$ can indeed be computed for arbitrary angle ratios ν_k/δ_k, in particular, for $\nu_k = \delta_k/2$. This ratio is important, as it ensures that impacts alternate between the inside and the outside of Γ.

Knowledge about how to make the heading error decay monotonically, is the key to drive the position error to zero as well.

Lemma 5.6. *Due to the product of position and heading error in* (4.4)*, the position error $\Delta\bar{\boldsymbol{x}}(k)$ vanishes for $t \to \infty$, if all impacts occur on the orbit sections $\Gamma_{\bar{q}_k^\star}$, the heading error $\Delta\bar{\boldsymbol{v}}(k)$ converges to zero and the position error $\Delta\bar{\boldsymbol{x}}(k)$ stays bounded for all k. However, $\|\Delta\bar{\boldsymbol{x}}(k)\|$ does not necessarily decrease monotonically.*

Proof. Consider (5.13) for $\boldsymbol{m}_q = \bar{\boldsymbol{v}}^\star\left(\bar{q}^\star_{k+1}\right)$. If $\Delta\bar{\boldsymbol{v}}(k) = 0$ for $k \to \infty$ and $\left|\alpha_{p,k}\right| \leq \infty, \forall k$, which requires boundedness of $\Delta\bar{\boldsymbol{x}}(k)$, then the ball eventually moves along $\bar{\boldsymbol{v}}^\star\left(\bar{q}^\star_{k+1}\right)$. As long as $\left(\bar{\boldsymbol{x}}(k+1) - \bar{\boldsymbol{x}}^\star\left(\bar{q}^\star_{k+2}\right)\right)^{\mathrm{T}} \bar{\boldsymbol{v}}^\star\left(\bar{q}^\star_{k+1}\right) > 0$ holds, then the $(k+2)$-st impact point $\bar{\boldsymbol{x}}(k+2) = \bar{\boldsymbol{x}}^\star\left(\bar{q}^\star_{k+2}\right)$. □

5.4 Bounding the position error by execution of sporadic point-to-point transfers.

By Lem. 5.6, a vanishing heading error implies a vanishing position error, as long as the position error stays bounded. The latter can be guaranteed by execution of point-to-point transfers, whenever two subsequent ball impacts are about to occur on the outside of the orbit. In this situation, a transfer from $\bar{\boldsymbol{x}}(k)$ to a point $\bar{\boldsymbol{x}}(k+2)$ on the inside of Γ, reduces both the position *and* the heading error, as shown by the following results.

Lemma 5.7. *Suppose that* $\Delta\boldsymbol{x}^{\mathrm{T}}_{p,k+1}\bar{\boldsymbol{v}}(k) < 0$, $\Delta\boldsymbol{x}^{\mathrm{T}}_{p,k+1}\bar{\boldsymbol{v}}_\perp(k) > 0$ *hold true after the k-th impact. Perturbing the ball's subsequent nominal impact at* $\bar{\boldsymbol{x}}'(k+1) = \bar{\boldsymbol{x}}(k) + (\bar{\tau}(k) + \Delta\tau)\bar{\boldsymbol{v}}(k)$ *by an arbitrary* $\Delta\tau \geq 0$ *results in a subsequent ball motion* $\boldsymbol{x}'(\tau_2) = \bar{\boldsymbol{x}}'(k+1) + \tau_2\bar{\boldsymbol{v}}'(k+1)\,,\ \tau_2 \geq 0$, *which will not intersect with the successor trajectory* $\boldsymbol{x}(\tau_1) = \bar{\boldsymbol{x}}(k+1) + \tau_1\bar{\boldsymbol{v}}(k+1)\,,\ \tau_1 \geq 0$ *starting at the unperturbed impact point* $\bar{\boldsymbol{x}}(k+1)$.

Proof. The first step of this proof is to express the perturbed quantities $\bar{\boldsymbol{x}}'(k+1)$ and $\bar{\boldsymbol{v}}'(k+1)$ in terms of $\Delta\tau$, $\bar{\boldsymbol{x}}(k+1)$ and $\bar{\boldsymbol{v}}(k)$. Subsequently, it needs to be shown that $\nexists \tau_1, \tau_2 \geq 0$ s.t. $\boldsymbol{x}'\left(\tau_2, \bar{\boldsymbol{x}}'(k+1)\,, \bar{q}^\star_{k+1}\right) = \boldsymbol{x}\left(\tau_1, \bar{\boldsymbol{x}}(k+1)\,, \bar{q}^\star_{k+1}\right)$.

For the first step, parametrize the perturbed impact point

$$\bar{\boldsymbol{x}}'(k+1) = \bar{\boldsymbol{x}}(k+1) + \Delta\tau\bar{\boldsymbol{v}}(k) \tag{5.42}$$

in terms of the unperturbed point and the current heading. Taking $\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\bar{\boldsymbol{n}}(k) = 1$ into consideration, the nominal and the perturbed successor heading can be expressed as

$$\bar{\boldsymbol{n}}(k) = \bar{\boldsymbol{v}}(k) + \beta\bar{\boldsymbol{v}}_\perp(k) \ \text{ with } \beta = -\frac{\Delta\bar{\boldsymbol{x}}^{\mathrm{T}}_{p,k+1}\bar{\boldsymbol{v}}(k)}{\Delta\bar{\boldsymbol{x}}^{\mathrm{T}}_{p,k+1}\bar{\boldsymbol{v}}_\perp(k)} \geq 0 \tag{5.43}$$

$$\bar{\boldsymbol{n}}'(k) = \bar{\boldsymbol{n}}(k) + \gamma\bar{\boldsymbol{v}}_\perp(k) = \bar{\boldsymbol{v}}(k) + (\beta + \gamma)\,\bar{\boldsymbol{v}}_\perp(k) \quad . \tag{5.44}$$

where the parametrization of $\bar{\boldsymbol{n}}'(k)$ ensures satisfaction of $\bar{\boldsymbol{v}}^{\mathrm{T}}(k)\,\bar{\boldsymbol{n}}'(k) = 1$. The impact condition

$\Delta\bar{\boldsymbol{x}}^{\mathrm{T}}_{p,k+1}\bar{\boldsymbol{n}}(k) = 0$ together with (5.42), (5.44) implies

$$\left(\Delta\bar{\boldsymbol{x}}_{p,k+1} - \Delta\tau\bar{\boldsymbol{v}}(k)\right)^{\mathrm{T}}(\bar{\boldsymbol{n}}(k) + \gamma\bar{\boldsymbol{v}}_{\perp}(k)) = 0$$
$$\Rightarrow \quad \gamma = \frac{\Delta\tau}{\Delta\bar{\boldsymbol{x}}^{\mathrm{T}}_{p,k+1}\bar{\boldsymbol{v}}_{\perp}(k)}$$

Note, that γ linearly depends on $\Delta\tau$ and by construction of the lemma has the same sign. According to (4.1), the successor heading after impacting at $\bar{\boldsymbol{x}}(k+1)$ and $\bar{\boldsymbol{x}}'(k+1)$ becomes

$$\bar{\boldsymbol{v}}(k+1) = \bar{\boldsymbol{v}}(k) - 2\frac{\bar{\boldsymbol{n}}(k)}{1+\beta^2} \tag{5.45}$$
$$\bar{\boldsymbol{v}}'(k+1) = \bar{\boldsymbol{v}}(k) - 2\frac{\bar{\boldsymbol{n}}(k) + \gamma\bar{\boldsymbol{v}}_{\perp}(k)}{1+(\beta+\gamma)^2} \tag{5.46}$$

As $\bar{\boldsymbol{v}}'(k+1) \neq \bar{\boldsymbol{v}}(k+1)$, the two rays $\boldsymbol{x}\left(\tau_1, \bar{\boldsymbol{x}}(k+1), \bar{q}^{\star}_{k+1}\right)$, $\boldsymbol{x}\left(\tau_2, \bar{\boldsymbol{x}}'(k+1), \bar{q}^{\star}_{k+1}\right)$ must intersect, i.e. $\exists\tau_1, \tau_2 \in \mathbb{R}$

$$\bar{\boldsymbol{x}}(k+1) + \tau_1\bar{\boldsymbol{v}}(k+1) - \bar{\boldsymbol{x}}'(k+1) - \tau_2\bar{\boldsymbol{v}}'(k+1) = 0$$
$$\Leftrightarrow \quad \left((\Delta\tau + \tau_2 - \tau_1) - 2\left(\frac{\tau_2 - \tau_1}{1+(\beta+\gamma)^2} - \frac{\tau_1(2\beta\gamma+\gamma^2)}{(1+(\beta+\gamma)^2)(1+\beta^2)}\right)\right)\bar{\boldsymbol{v}}(k)$$
$$- 2\left(\frac{\tau_2(\beta+\gamma) - \tau_1\beta}{1+(\beta+\gamma)^2} - \frac{\tau_1(2\beta^2\gamma+\beta\gamma^2)}{(1+(\beta+\gamma)^2)(1+\beta^2)}\right)\bar{\boldsymbol{v}}_{\perp}(k) = 0 \tag{5.47}$$

Because $\bar{\boldsymbol{v}}(k)$ and $\bar{\boldsymbol{v}}_{\perp}(k)$ are orthogonal, both terms of (5.47) have to vanish independently. This implies

$$\frac{\tau_2 - \tau_1}{1+(\beta+\gamma)^2} - \frac{\tau_1(2\beta\gamma+\gamma^2)}{(1+(\beta+\gamma)^2)(1+\beta^2)} = -\frac{\tau_2\gamma}{\beta(1+(\beta+\gamma)^2)} \tag{5.48}$$

Replacing (5.48) in (5.47) and solving for τ_1 yields

$$\tau_1 = \Delta\tau + \left(1 + \frac{2\gamma}{\beta(1+(\beta+\gamma)^2)}\right)\tau_2 \tag{5.49}$$

Finally, plugging this result back into (5.48) and collecting terms results in

$$2(\beta^2\gamma + \gamma^2\beta + \gamma)\tau_2 + 2\left(2\beta^2\gamma + \beta\gamma^2 + \beta + \beta^3\right)\Delta\tau = 0 \tag{5.50}$$

As $\Delta\tau, \gamma, \beta > 0$ yields $\tau_2 < 0$, which implies $\tau_1 < 0$. Thus, if perturbing the subsequent impact point $\bar{\boldsymbol{x}}(k+1)$, the perturbed successor trajectory does not intersect with the nominal one, which completes the proof. □

Lemma 5.8. *Given a position-heading pair* $(\bar{\boldsymbol{x}}(k), \bar{\boldsymbol{v}}(k))$, *with* $\bar{\boldsymbol{v}}^{\star\mathrm{T}}_{\perp}\left(\bar{q}^{\star}_{k}\right)\Delta\bar{\boldsymbol{x}}(k) \geq 0$. *If* $\left(\bar{\boldsymbol{x}}^{\star}\left(\bar{q}^{\star}_{k+2}\right) - \bar{\boldsymbol{x}}^{\star}\left(\bar{q}^{\star}_{k+3}\right)\right)^{\mathrm{T}}\left(\bar{\boldsymbol{x}}(k+1) - \bar{\boldsymbol{x}}^{\star}\left(\bar{q}^{\star}_{k+3}\right)\right) < 0$, *then by executing a point-to-point transfer* *from* $\bar{\boldsymbol{x}}(k)$ *to* $(\bar{\boldsymbol{x}}^\star(\bar{q}^\star_{k+2}) + \bar{\boldsymbol{x}}^\star(\bar{q}^\star_{k+3}))/2$ *according to Lem. 5.2, instead of enforcing an impact on the orbit section* $\Gamma_{\bar{q}^\star_{k+1}}$, *heading error convergence* $\|\Delta\bar{\boldsymbol{v}}(k+1)\| < \|\Delta\bar{\boldsymbol{v}}(k)\|$ *is still enforced.*

Proof. As a result of Lem. 5.7, the execution of a point transfer from $\bar{\boldsymbol{x}}(k)$ to $\bar{\boldsymbol{x}}(k+2) = (\bar{\boldsymbol{x}}^\star(\bar{q}^\star_{k+2}) + \bar{\boldsymbol{x}}^\star(\bar{q}^\star_{k+3}))/2$ requires to postpone $(k+1)$-st impact $\bar{\boldsymbol{x}}'(k+1) = \bar{\boldsymbol{x}}(k+1) + \Delta\tau_1\bar{\boldsymbol{v}}(k)$, $\Delta\tau_1 > 0$ with respect to impacting on $\Gamma_{\bar{q}^\star_{k+1}}$ in the considered situation. To compensate the heading error completely the $(k+1)$-st impact needs to be delayed even further. Because in the latter case $\bar{\boldsymbol{v}}'(k+1) = \bar{\boldsymbol{v}}^\star(\bar{q}^\star_{k+1})$, the subsequent trajectory $\boldsymbol{x}(\tau, \bar{\boldsymbol{x}}'(k+1), \bar{\boldsymbol{v}}'(k+1))$ intersects with $\Gamma_{\bar{q}^\star_{k+2}}$ at some point

$$\bar{\boldsymbol{x}}'(k+2) = \bar{\boldsymbol{x}}^\star(\bar{q}^\star_{k+2}) + \Delta\tau_2\bar{\boldsymbol{v}}^\star(\bar{q}^\star_{k+2}), \Delta\tau_2 < 0 \ . \tag{5.51}$$

Thus, by executing a point transfer to any point $\bar{\boldsymbol{x}}(k+2) = \bar{\boldsymbol{x}}^\star(\bar{q}^\star_{k+2}) + \gamma\bar{\boldsymbol{v}}^\star(\bar{q}^\star_{k+2}), 0 < \gamma < 1$ implies

$$\|\Delta\bar{\boldsymbol{v}}(k+1)\| < \|\Delta\bar{\boldsymbol{v}}(k)\| \tag{5.52}$$

$$\bar{\boldsymbol{v}}^{\star\mathrm{T}}_\perp(\bar{q}^\star_{k+1})\,\Delta\bar{\boldsymbol{v}}(k+1) < 0 \ . \tag{5.53}$$

□

Besides guaranteeing boundedness of the position error for all k, the result of Lem. 5.8 also ensures that the stationary impacting sequence can be preserved during the transient motion of the ball.

Theorem 5.1. *By alternating the impacts on the inside and on the outside of the orbit, the position error* $\Delta\bar{\boldsymbol{x}}(k)$ *stays bounded for all k and asymptotically approaches zero as* $\Delta\bar{\boldsymbol{v}}(k) \to 0$.

Proof. By Lem. 5.4, an alternating impact sequence is attained, iff $\nu_k < \delta_k/2$, which simultaneously guarantees a strictly decreasing heading error sequence. Moreover, Lem. 5.8 states, that $\Delta\bar{\boldsymbol{x}}(2k)$ is bounded at every second impact for sure. Since the event distance $\bar{\tau}^\star(\bar{q}^\star_k)$ is bounded for all modes as well and knowing that $\Delta\bar{\boldsymbol{v}}(k)$ is bounded and approaches zero as k increases, every other $\Delta\bar{\boldsymbol{x}}(2k+1)$ must be bounded as well. Hence, $\bar{\boldsymbol{x}}(k)$ is bounded for all k and by (5.13), it must ultimately approach zero, since almost all impacts occur on the orbit. □

5.5 Summary of the control strategy

With all results of the previous subsections, the hybrid control strategy can be summarized in the following Algorithm. Note that, although this control strategy was derived for the specific example of the billiards problem, it applies to all IHSCR, which possess the same system structure, i.e. integrator dynamics, event-planes and reflecting resets.

Algorithm 5.1. *Globally stabilizing control strategy.*

Given: *The billiards system* (2.10)-(2.12) *with E walls, an admissible orbit* Γ *to be globally stabilized and an associated nominal wall set-up, i.e.* $\left\{\bar{\boldsymbol{n}}^{\star}_{\bar{q}^{\star}_1}, \ldots \bar{\boldsymbol{n}}^{\star}_{\bar{q}^{\star}_E}\right\}$.

Initialization:

1. *Determine the worst case scenario for impacting on the outside of the orbit* $\Gamma \rightarrow \boldsymbol{x}_{wc}$.
2. *For this worst case scenario, compute the critical pivot distances* $L_{p,\bar{q}^{\star}_k}$ *for each wall and* Γ *according to* (5.32) *with* $2\nu_k = \delta_k$.
3. *Determine the best initial mode* $q(0) = \bar{q}^{\star}_0$.
4. *Adjust the subsequently impacted wall* $\mathcal{W}_{\bar{q}^{\star}_0}$ *according to Lem. 5.2, such that a transfer from* $\bar{\boldsymbol{x}}(0)$ *to* $\bar{\boldsymbol{x}}^{\star}\left(\bar{q}^{\star}_2\right)$ *is executed.*

During operation (**$k \geq 2$**):

1. *If the previous impact occurred on the outside of* Γ, *adjust the subsequently impacted wall* $\mathcal{W}_{\bar{q}^{\star}_k}$ *according to Lem. 5.1, such that the next impact occurs on the orbit section* $\Gamma_{\bar{q}^{\star}_{k+1}}$.
2. *Else, compute both actuations according to Lem. 5.1* $\left(\bar{\boldsymbol{n}}^{(1)}_{\bar{q}^{\star}_k}(k)\right)$ *and Lem. 5.2* $\left(\bar{\boldsymbol{n}}^{(2)}_{\bar{q}^{\star}_k}(k)\right)$ *for a transfer from* $\bar{\boldsymbol{x}}(k)$ *to* $\left(\bar{\boldsymbol{x}}^{\star}\left(\bar{q}^{\star}_{k+2}\right) + \bar{\boldsymbol{x}}^{\star}\left(\bar{q}^{\star}_{k+3}\right)\right)/2$. *Afterwards apply the control value* $\bar{\boldsymbol{n}}_{\bar{q}^{\star}_k}(k) = \arg\max_i \left(\Delta\bar{\boldsymbol{x}}^{\mathrm{T}}_{p,k}\bar{\boldsymbol{n}}^{(i)}_{\bar{q}^{\star}_k}(k)\right)$.

Result: *Global orbital stability of* Γ *for arbitrary* $(\boldsymbol{x}(0), \boldsymbol{v}(0))$.

6 Simulation results of the controlled billiards system

Consider the nominal rectangular billiard table set-up centered at the origin with four walls and a ball starting at $\boldsymbol{\zeta}(0) = (-1\ 0.4\ 0.5\ \ -0.3)^{\mathrm{T}}$ (Fig. 6.1(a)). The four stationary impact points and the heading of the desired orbit Γ (dashed bold line) are:

$\overline{\boldsymbol{x}}^{\star}\left(\overline{q}_0^{\star}\right)$	$\overline{\boldsymbol{x}}^{\star}\left(\overline{q}_1^{\star}\right)$	$\overline{\boldsymbol{x}}^{\star}\left(\overline{q}_2^{\star}\right)$	$\overline{\boldsymbol{x}}^{\star}\left(\overline{q}_3^{\star}\right)$	$\overline{\boldsymbol{v}}^{\star}\left(\overline{q}_0^{\star}\right)$	$\overline{\boldsymbol{v}}^{\star}\left(\overline{q}_1^{\star}\right)$	$\overline{\boldsymbol{v}}^{\star}\left(\overline{q}_2^{\star}\right)$	$\overline{\boldsymbol{v}}^{\star}\left(\overline{q}_3^{\star}\right)$
$(-0.5\ \ -1)^{\mathrm{T}}$	$(-1\ \ -0.5)^{\mathrm{T}}$	$(0.5\ 1)^{\mathrm{T}}$	$(1\ 0.5)^{\mathrm{T}}$	$(-1\ \ -1)^{\mathrm{T}}$	$(-1\ 1)^{\mathrm{T}}$	$(1\ 1)^{\mathrm{T}}$	$(1\ \ -1)^{\mathrm{T}}$

For this table and the chosen Γ, the worst case impacts on the outside of the orbit occur, when the ball previously rolled along one of the nominal walls. Therefore, $\boldsymbol{x}_{wc} = (-2.5\ \ -1)$. Given Γ and $\boldsymbol{x}_{wc}$, the pivot positions, which guarantee global orbital stability are located far out of the table, too far to be depicted. Their distance to the stationary impact points $\overline{\boldsymbol{x}}^{\star}\left(\overline{q}_k^{\star}\right)$ is $L_{p,\overline{q}_k^{\star}} = 4$.

Having fixed the pivots and applying the control law of Algo. 5.1 the trajectory depicted in Fig. 6.1(a) is obtained. At the beginning, an initial transition onto the orbit is executed. The third impact $\overline{\boldsymbol{x}}(3)$ does also not occur on the orbit, as $\overline{\boldsymbol{x}}(4)$ would otherwise be located outside of the orbit, again. Instead, another point transfer to the inside of Γ is performed. Subsequently, all impacts alternate between the inside and the outside of the orbit, which guarantees the orbital stability of Γ. This is clearly observable from the figure.

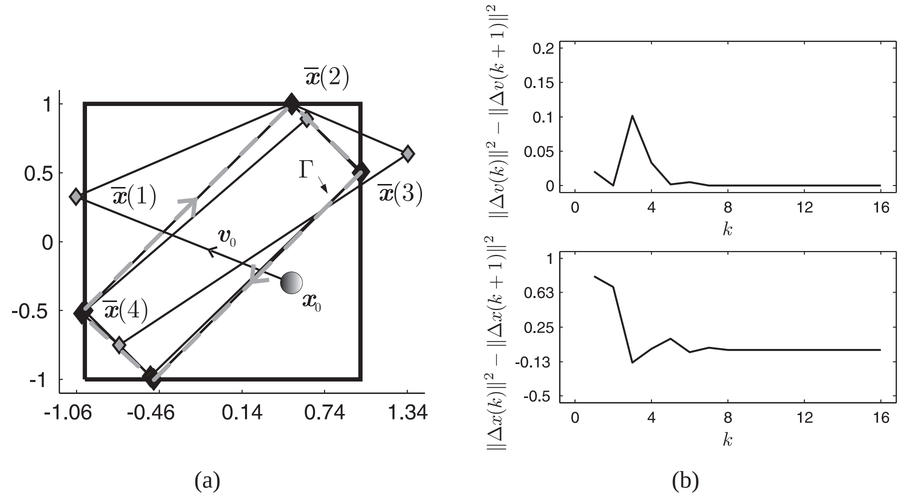

Figure 6.1: Simulation of the controlled billiards systems: (a) Ball motion on the table, (b) Sampled heading and position error trajectories.

The ball converges towards Γ, until it finally evolves on the unstable orbit after the 8-th impact. Even though the pivots are located far out of the table, the decay rate of the heading error is still decent and monotonous (Fig 6.1(b)). Furthermore, the position error reduces from period to period, but it *does not* decrease at every impact. This fact, which complicates the control of the position error directly as previously discussed, is observed in the lower part of Fig 6.1(b). Repeating the simulation for different initial conditions results in the same converging behavior, as the orbital stability is guaranteed globally.

7 Conclusion

This paper focussed on the orbital control of a novel subclass of impulsive hybrid systems with exogenously controllable resets (IHSCR). It highlighted the complications that arise from the inseperable coupling of the reset time and the reset action. The latter is a unique system property. Because of the complexity of the control problem in general, the paper concentrated its attention to a particular simple subclass of planar IHSCR with specific structure. For these systems, a hybrid control strategy was proposed, that achieves global orbital stability of a prespecified periodic execution. As a key element to the solution, the embedded map of the underlying hybrid model, which explicitly establishes the connection between the control input and the output, was stated in closed form. The derivation and the performance evaluation of the control law was executed explicitly for the chaotic billiards system, which is intensely studied in mathematics and physics. However, the result applies to all systems exhibiting the same structure.

Despite the simplicity of the system, the ideas and experiences gained in deriving the hybrid control law provides a valuable starting point for generalizing the design to more complicated IHSCR. Moreover, the integration of performance criteria into the problem formulation, like minimizing control effort or maximizing the convergence rate, and the investigation of controllability for this class of systems are other interesting, open questions.

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