A Graph-Theoretic Characterization of Controllability for Multi-agent Systems*

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***Abstract*— In this paper, we continue our pursuit of conditions that render a multi-agent networked system controllable. In particular, such conditions are sought for networks in which a collection of the agents take on leader roles while the remaining agents execute local, consensus-like control laws. Equitable partitions are introduced in order to improve on previous controllability results and the main contribution of this paper is a new necessary condition for controllability.**

***Index Terms*— Multi-Agent systems, Networked system, Controllability, Graph theory, Equitable partitions.**

I. Introduction

This paper explores the issue of providing a graph-theoretic characterization of the controllability of certain leader-based, multi-agent systems. As such, it constitutes a continuation of our previous work [1], by considerably improving the necessary condition for controllability. The controllability issue in leader-follower multi-agent systems was first introduced by Tanner in [2], where a necessary and sufficient condition for controllability was given based on the eigenvectors of the graph Laplacian. Although elegant, this condition was not graph-theoretic in that controllability could not be directly decided from the graph topology itself. A more topological result was given by Mesbahi and Rahmani [3], in which a sufficient condition for the network to be uncontrollable in the case of one anchored (leader) agent was given. Their result was related to the symmetry and automorphism group of the underlying graph. In this paper, we further extend this notion and present a more general condition based on so-called equitable partitions of the underlying graph. Our result thus addresses a scenario where multiple leaders are possible, and from an equitable partition point-of-view, captures a larger set of graphs.

II. Notation and Preliminaries

In this section we start with some basic notions in graph theory and recall some known results about controllability of multi-agent networks. (We refer the readers to [4] for more details about algebraic graph theory.)

Graphs are broadly adopted in the multi-agent literature to encode the interactions in networked systems. An undirected graph $\mathcal{G}$ is defined by a set $\mathcal{V}_{\mathcal{G}} = \{1, \dots n\}$ of *nodes* and a set $\mathcal{E}_{\mathcal{G}} \subset \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{G}}$ of *edges*. A *cell* $C \subset \mathcal{V}_{\mathcal{G}}$ is a subset of the node set. Two nodes i and j are *neighbors* if $(i, j) \in \mathcal{E}_{\mathcal{G}}$, and the neighboring relation is indicated with $i \sim j$, while $\mathcal{P}(i) = \{j \in \mathcal{V}_{\mathcal{G}} : j \sim i\}$ collects all neighbors to the node i. The degree of a node is given by the number of its neighbors, and we say that a graph is *regular* if all nodes have the same degree.

A *path* $i_0 i_1 \dots i_L$ is a finite sequence of nodes such that $i_{k-1} \sim i_k$, $k = 1, \dots, L$, and a graph $\mathcal{G}$ is *connected* if there is a path between any pair of distinct nodes. A subgraph $\mathcal{G}'$ is said to be *induced* from the original graph $\mathcal{G}$ if it is obtained by deleting a subset of nodes and the edges connected to those nodes.

Let $\mathcal{A} \in \mathbb{R}^{n \times n}$ be the $(0, 1)$ *adjacency* matrix and $\mathcal{B} \in \mathbb{R}^{n \times m}$ be the $(-1, 0, 1)$ vertex-edge *incidence* matrix, whose orientation is arbitrarily imposed on the graph. The graph *Laplacian* matrix is defined by $\mathcal{L} = \mathcal{B}\mathcal{B}^T$ and it is easy to verify that $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D}$ is the diagonal degree matrix.

In the context of multi-agent systems, the nodes represent agents and the edges are communication links. In particular, an agent i has access to the information pertaining to its neighbors and can use this piece of information to compute its control law. Let $x_i \in \mathbb{R}^d$ denote the state of node i, whose dynamics is described by the single integer $\dot{x}_i = u_i$, where u_i is the control input.

Rendezvous problem is one of the multi-agent control problems that has attracted considerable attentions from many researchers [5], [6], [7]. Some other problems, e.g. formation control [8], [9], [10], consensus [11] or agreement [12], [13], flocking [14], [15] and etc., share the same distributive flavor with the rendezvous problem and thus adopt a similar methodology. In fact, a widely adopted control method for multi-agent rendezvous is the Laplacian-based feedback on the form

$$\dot{x} = -\mathbb{L}x,$$

where $x = [x_1^T,\ x_2^T, \dots,\ x_n^T]^T$ denotes the aggregated state vector of the multi-agent system, and $\mathbb{L} = \mathcal{L} \otimes I_d$ is the augmented Laplacian matrix.

Under some circumstances, some agents are required to take leader roles, while the others follow them. In this paper, we use subscript l to denote the affiliation with leaders while f for the followers. For example, a *follower graph* $\mathcal{G}_f$ is the subgraph induced by the follower set $\mathcal{V}_f$. As the leader roles are designated, the incidence matrix $\mathcal{B}$ can be partitioned as

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_f \\ \mathcal{B}_l \end{bmatrix},$$

where $\mathcal{B}_f \in \mathbb{R}^{n_f \times m}$, and $\mathcal{B}_l \in \mathbb{R}^{n_l \times m}$. Here n_f, n_l and m are the cardinalities of the follower group, the leader group,

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and the edge set respectively. As a result, the graph Laplacian $\mathcal{L}$ is given by

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{bmatrix},$$

where

$$\mathcal{L}_f = \mathcal{B}_f \mathcal{B}_f^T, \ \mathcal{L}_l = \mathcal{B}_l \mathcal{B}_l^T \ \text{ and } l_{fl} = \mathcal{B}_f \mathcal{B}_l^T.$$

As an example, Figure 1 shows a leader-follower network with $\mathcal{V}_l = \{5, 6\}$ and $\mathcal{V}_f = \{1, 2, 3, 4\}$. This gives

$$\mathcal{B}_f = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{B}_l = \left[\begin{array}{c|cccc} \mathbf{0}_{2\times 4} & 1 & -1 & 0 & 0 \\ & 0 & 0 & -1 & 1 \end{array} \right],$$

and

$$\mathcal{L}_f = \begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix}, \ l_{fl} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

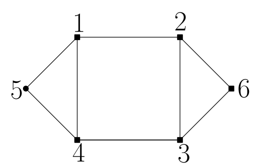


Fig. 1. A leader-follower network with: $\mathcal{V}_f = \{1, 2, 3, 4\}$ and $\mathcal{V}_l = \{5, 6\}$.

The system we are interested in is the leader-follower system, where the followers are governed by the Laplacian-based feedback law given by

$$\dot{x}_f \quad = \quad \mathcal{L}_f x_f + l_{fl} x_l, \tag{1}$$

while the leaders's movements are dictated by some exogenous control signals. Under this setup, a natural question is as follows: Can the leader(s) move the followers to any desired configuration? From a control theoretic point of view, we ask following questions:

1) Is this system controllable?
2) If not, what is the controllability (observability) decomposition?
3) What is the controllable subspace?
4) What is the structural significance of the controllable or uncontrollable subspace?

Before we answer these questions, let us review some basic controllability results:

Proposition 2.1: Given the system $(\mathcal{L}_f, l_{fl})$, the following statements are equivalent:

1) The system is completely controllable;
2) None of the eigenvectors[1] of $\mathcal{L}_f$ is in the nullspace of l_{fl}^T, i.e. $\forall\ v_i$, such that $\mathcal{L}_f v_i = \lambda v_i$ for some $\lambda \in \mathbb{R}$, $v_i \notin \mathcal{N}(l_{fl}^T)$, where $\mathcal{N}(\cdot)$ denotes the nullspace;
3) The controllability matrix $\mathcal{C}_{\mathcal{L}_f, l_{fl}}$ has full rank;
4) The matrix $[\lambda I - \mathcal{L}_f \mid l_{fl}]$ has full rank for all $\lambda \in \mathbb{R}$.

Based on these standard controllability results, together with some basic properties of the graph Laplacian, we can derive the following lemma.

Lemma 2.2: Given a connected graph, the system $(\mathcal{L}_f, l_{fl})$ is controllable if and only if $\mathcal{L}$ and $\mathcal{L}_f$ do not share any common eigenvalues.

Proof of Necessity: [2]
We can reformulate the lemma as stating that the system is uncontrollable if and only if there exists at least one common eigenvalue between $\mathcal{L}$ and $\mathcal{L}_f$. First, we show the necessity. Suppose the system is uncontrollable. Then from Theorem 2.1(2), there exists a vector $v_i \in \mathbb{R}^{n_f}$ such that $\mathcal{L}_f v_i = \lambda v_i$ for some $\lambda \in \mathbb{R}$, with

$$l_{fl}^T v_i = \mathbf{0}.$$

Now, since

$$\begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{bmatrix} \begin{bmatrix} v_i \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_f v_i \\ l_{fl}^T v_i \end{bmatrix} = \lambda \begin{bmatrix} v_i \\ \mathbf{0} \end{bmatrix},$$

λ is also an eigenvalue of $\mathcal{L}$, with eigenvector $[v_i^T, \mathbf{0}]^T$. ∎

III. Interlacing and Equitable Partitions

Equitable partitions and interlacing theory play an important role for our main results. In this section, we introduce some definitions and lemmas needed to support our main results.

Definition 3.1: A r-partition π of $\mathcal{V}(\mathcal{G})$, with cells $C_1, \dots, C_r$, is said to be *equitable* if each node in C_j has the same number of neighbors in C_i, for all i, j. We denote the cardinality of the partition π with $r = |\pi|$.

Let b_{ij} be the number of neighbors in C_j of a node in C_i. The directed graph with the r cells of π as its nodes and b_{ij} edges from the ith to the jth cells of π is called the *quotient* of $\mathcal{G}$ over π, and is denoted by $\mathcal{G}/\pi$. An obvious trivial partition is the n-partition, $\pi = \{\{1\}, \{2\}, \dots, \{n\}\}$. If a partition contains at least one cell with more than one node, we call it a nontrivial equitable partition (NEP), and the adjacency matrix of a quotient is given by

$$\mathcal{A}(\mathcal{G}/\pi)_{ij} = b_{ij}.$$

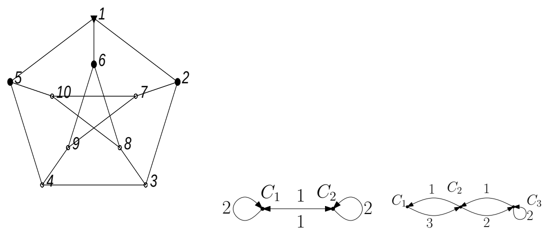
Now, the equitable partition can be derived from graph automorphisms. For example, in the so-called Peterson graph, shown in Figure 2(a), one equitable partition π_1 (Figure 2(b)) is given by the two orbit of the automorphism groups, namely the 5 inner vertices and the 5 outer vertices. The adjacency matrix of the quotient is given by

$$\mathcal{A}(\mathcal{G}/\pi_1) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

[1]Since $\mathcal{L}_f$ is symmetric, its left eigenvectors are equal to the right ones.
[2]The sufficiency proof will be given after Lemma 3.8.

The equitable partition can also be introduced by the equal distance partition. Let $C_1 \subset V(\mathcal{G})$ be a given cell, and let $C_i \subset V(\mathcal{G})$ be the set of vertices at distance $i-1$ from C_1. C_1 is said to be *completely regular* if its distance partition is equitable. For instance, every vertex in the Peterson graph is completely regular and introduces the partition π_2 as shown in Figure 2(c). The adjacency matrix of this quotient is given by

$$\mathcal{A}(\mathcal{G}/\pi_2) = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$



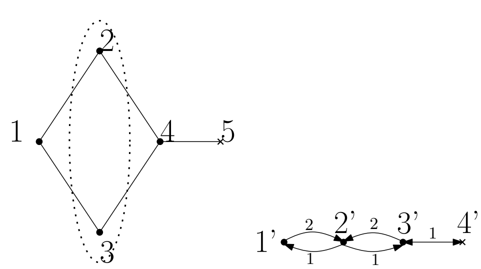
(a) (b) (c)

Fig. 2. Example of equitable partitions on (a) the Peterson graph $\mathcal{G} = J(5,2,0)$ and the quotients: (b) the NEP introduced by the automorphism is $\pi_1 = \{C_1^1, C_2^1\}, C_1^1 = \{1,2,3,4,5\}, C_2^1 = \{6,7,8,9,10\}$, and (c) the NEP introduced by equal-distance partition is $\pi_2 = \{C_1^2, C_2^2, C_3^2\}, C_1^2 = \{1\}, C_2^2 = \{2,5,6\}, C_3^2 = \{3,4,7,8,9,10\}$.

The adjacency matrix of the original graph and the quotient are closely related through the so-called interlacing theorem. But first, we introduce the following lemma.

Definition 3.2: A *characteristic vector* $p_i \in \mathbb{R}^n$ of a nontrivial cell C_i has 1's in the positions associated with C_i and 0's elsewhere. A *characteristic matrix* $P \in \mathbb{R}^{n\times r}$of a partition π of $V(\mathcal{G})$ is a matrix with the characteristic vectors of the cells as its columns. For example, the characteristic matrix of the equitable partition of the graph in Figure 3(a) is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2}$$



(a) (b)

Fig. 3. The (a) equitable partition and (b) the quotient of a graph.

Lemma 3.3: ([4] Lemma 9.3.1) Let P be the characteristic matrix of an equitable partition π of the graph $\mathcal{G}$, and let $\hat{\mathcal{A}} = \mathcal{A}(\mathcal{G}/\pi)$. Then $\mathcal{A}P = P\hat{\mathcal{A}}$ and $\hat{\mathcal{A}} = P^+\mathcal{A}P$, where $P^+ = (P^TP)^{-1}P^T$ is the pseudo-inverse of P.

As an example, the graph in Figure 3 has a nontrivial cell $(2,3)$. The adjacency matrix of original graph is

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The adjacency matrix of the quotient is

$$\hat{\mathcal{A}} = P^+\mathcal{A}P = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Lemma 3.4: ([4] Lemma 9.3.2) Let $\mathcal{G}$ be a graph with adjacency matrix $\mathcal{A}$, and let π be a partition of $\mathcal{V}(\mathcal{G})$ with characteristic matrix P, then π is equitable if and only if the column space of P is $\mathcal{A}$-invariant.

Lemma 3.5: Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$, and let S be a subspace of $\mathbb{R}^n$. Then $S^\perp$ is A-invariant if and only if S is A-invariant.
The proof of this well-known fact can for example be found in [16].

Remark 3.6: Let $\mathcal{R}(\cdot)$ denote the range space. Suppose $|\mathcal{V}(\mathcal{G})| = n$, $|C_i| = n_i$ and $|\pi| = r$, then we can find an orthogonal decomposition for $\mathbb{R}^n$ as

$$\mathbb{R}^n = \mathcal{R}(P) \oplus \mathcal{R}(Q), \tag{3}$$

where the matrix Q satisfies $\mathcal{R}(Q) = \mathcal{R}(P)^\perp$, such that its columns together with those of P form a basis for $\mathbb{R}^n$. Following Lemma 3.5, $\mathcal{R}(Q)$ is also $\mathcal{A}-$invariant.

Unlike matrix P, Q is derived from the nullspace of P and can be constructed in different ways. One possible choice of such a Q is the $n \times n - r$ matrix with r column blocks $Q = [Q_1, Q_2, \ldots, Q_r]$, where $Q_i \in \mathbb{R}^{n\times n_i - 1}$ corresponds to C_i. Moreover, each column sums to zero in the positions associated with C_i and has zeros in the other positions. In other words,

$$Q_i = \begin{bmatrix} \mathbf{0} \\ \tilde{Q}_i \\ \mathbf{0} \end{bmatrix}_{n\times(n_i-1)}.$$

In Q_i, the upper and lower parts are zero matrices with appropriate dimensions (possibly empty). One possible choice of $\tilde{Q}_i$ would be

$$\tilde{Q}_i = \begin{bmatrix} I_{n_i-1} \\ -\mathbf{1}^T \end{bmatrix}_{n_i\times(n_i-1)}, \tag{4}$$

where $\mathbf{1} \in \mathbb{R}^{n_i-1}$ is a vector with ones in each position. Based on this method, the Q matrix for the equitable partition in Figure 3(a) can be given by $Q = Q_2 = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \end{bmatrix}^T$, and thus, $\tilde{Q}_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$.

The other choice of Q matrix is using the orthonormal basis of $\mathcal{R}(P)^{\perp}$. We denote this matrix as $\bar{Q}$. If we define

$$\bar{P} = P(P^T P)^{-\frac{1}{2}}. \tag{5}$$

Note that the invertibility of $P^T P$ follows from the fact that the cells of the partition are nonempty [3]. Moreover, it satisfies that $\bar{P}^T \bar{Q} = \mathbf{0}$ and $\bar{Q}^T \bar{Q} = I_{n-r}$. In other words,

$$T = [\bar{P} \mid \bar{Q}] \tag{6}$$

is a matrix whose columns are defined on an orthonormal basis of $\mathbb{R}^n$ based on the equitable partition π, and $\bar{P}$ and $\bar{Q}$ have the same column spaces as P and Q respectively.

Theorem 3.7: ([4] Theorem 9.3.3) If π is an equitable partition of a graph $\mathcal{G}$, then the characteristic polynomial of $\hat{\mathcal{A}} = \mathcal{A}(\mathcal{G}/\pi)$ divides the characteristic polynomial of $\mathcal{A}(\mathcal{G})$.

Lemma 3.8: ([4] Theorem 9.5.1) Let $\Phi \in \mathbb{R}^{n\times n}$ be a real symmetric matrix and let $R \in \mathbb{R}^{n\times m}$ be such that $R^T R = I_m$. Set $\Psi = R^T \Phi R$ and let $v_1, v_2, \ldots, v_m$ be an orthogonal set of eigenvectors for Ψ such that $\Psi v_i = \theta_i(\Psi) v_i$, where $\theta_i(\Psi) \in \mathbb{R}$ is an eigenvalue of Ψ. Then

1) The eigenvalues of Ψ interlace the eigenvalues of Φ.
2) If $\theta_i(\Psi) = \theta_i(\Phi)$ then there is an eigenvector v of Ψ with eigenvalue $\theta_i(\Psi)$ such that Rv is an eigenvector of Φ with eigenvalue $\theta_i(\Phi)$.
3) If $\theta_i(\Psi) = \theta_i(\Phi)$ for $i = i, \ldots, l$, then Rv_i is an eigenvector for A with eigenvalue $\theta_i(\Phi)$ for $i = i, \ldots, l$.
4) If the interlacing is tight, then $\Phi R = R\Psi$.

Now we are in the position to prove the sufficiency of Lemma 2.2, i.e. if $\mathcal{L}$ and $\mathcal{L}_f$ share a common eigenvalue, the system $(\mathcal{L}, l_{fl})$ is not completely controllable.

Proof of Sufficiency of Lemma 2.2:
Since $\mathcal{L}_f$ is a principle sub-matrix of $\mathcal{L}$, it can be given by

$$\mathcal{L}_f = R^T \mathcal{L} R,$$

where $R = [I_{n_f}, 0]^T \in \mathbb{R}^{n\times n_f}$. Following Lemma 3.8(2), if $\mathcal{L}_f$ and $\mathcal{L}$ share a common eigenvalue, say λ, then the corresponding eigenvector satisfies

$$v = Rv_f = \begin{bmatrix} v_f \\ \mathbf{0} \end{bmatrix},$$

where v is λ's eigenvector of $\mathcal{L}$ and v_f is that of $\mathcal{L}_f$. Moreover, we know that

$$\mathcal{L}v = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{bmatrix} \begin{bmatrix} v_f \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} v_f \\ \mathbf{0} \end{bmatrix},$$

which gives us $l_{fl}^T v_f = \mathbf{0}$, and thus the system is uncontrollable. ∎

[3] In fact, $P^T P$ is a diagonal matrix with $(P^T P)_{ii} = |C_i|$.

Now, we have shown that the existence of a common eigenvalue shared by $\mathcal{L}$ and $\mathcal{L}_f$ is a necessary and sufficient condition for the leader-follower network to be uncontrollable.

IV. CONTROLLABILITY

In this section, we will utilize a graph theoretic approach to characterize the necessary condition for a multiple-leader networked system to be controllable. The way we approach this necessary condition is through Lemma 2.2. In what follows we will show first that both $\mathcal{L}$ and $\mathcal{L}_f$ are both similarity to some block diagonal matrices. Then we will, furthermore, show that under some circumstances, the diagonal block matrices, resulted from diagonalize $\mathcal{L}$ and $\mathcal{L}_f$, have some diagonal block(s) in common.

Lemma 4.1: If a graph $\mathcal{G}$ has a nontrivial equitable partition (NEP) π with characteristic matrix P, the adjacency matrix $\mathcal{A}(\mathcal{G})$ of the graph is similar to a diagonal matrix

$$\bar{\mathcal{A}} = \begin{bmatrix} \mathcal{A}_P & \mathbf{0} \\ \mathbf{0} & \mathcal{A}_Q \end{bmatrix},$$

where $\mathcal{A}_P$ is similar to the adjacency matrix $\hat{\mathcal{A}} = \mathcal{A}(\mathcal{G}/\pi)$ of the quotient.

Proof: Let the matrix $T = [\bar{P} \mid \bar{Q}]$ be the orthonormal matrix with respect to π, as what we have defined in (6).

Now, let

$$\bar{\mathcal{A}} = T^T \mathcal{A} T = \begin{bmatrix} \bar{P}^T \mathcal{A} \bar{P} & \bar{P}^T \mathcal{A} \bar{Q} \\ \bar{Q}^T \mathcal{A} \bar{P} & \bar{Q}^T \mathcal{A} \bar{Q} \end{bmatrix}. \tag{7}$$

Since $\bar{P}$ and $\bar{Q}$ have the same column spaces as P and Q respectively, they inherit the $\mathcal{A}-$invariance property, i.e.

$$\mathcal{A}\bar{P} = \bar{P}B \quad \text{and} \quad \mathcal{A}\bar{Q} = \bar{Q}C.$$

for some matrices B and C. Since their column spaces are orthogonal complements to each other, we get

$$\bar{P}^T \mathcal{A} \bar{Q} = \bar{P}^T \bar{Q} C = \mathbf{0}$$

and

$$\bar{Q}^T \mathcal{A} \bar{P} = \bar{Q}^T \bar{P} B = \mathbf{0}.$$

In addition, let $D_p^2 = P^T P$, we get

$$\begin{aligned} \bar{P}^T \mathcal{A} \bar{P} &= D_P^{-1} P^T \mathcal{A} P D_P^{-1} \\ &= D_P (D_P^{-2} P^T \mathcal{A} P) D_P^{-1} \\ &= D_P \hat{\mathcal{A}} D_P^{-1}, \end{aligned} \tag{8}$$

and therefore the first diagonal block is similar to $\hat{\mathcal{A}}$. ∎

Lemma 4.2: Let P be the characteristic matrix of a NEP in $\mathcal{G}$. $\mathcal{R}(P)$ is K-invariant, where K is any diagonal block matrix of the form

$$K = \mathrm{diag}(k_i I_{n_i})_{i=1}^r,$$

where $k_i \in \mathbb{R}$, $n_i = |C_i|$ is the cardinality of the cell, and $r = |\pi|$ is the cardinality of the partition. Consequently,

$$\bar{Q}^T K \bar{P} = \mathbf{0},$$

where $\bar{P} = P(P^T P)^{-\frac{1}{2}}$ and $\bar{Q}$ is chosen in such a way that $T = [\bar{P} \mid \bar{Q}]$ is a orthonormal matrix.

Proof: Since

$$P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_r \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & \dots & p_r \end{bmatrix},$$

where $P_i \in \mathbb{R}^{n_i \times r}$ is a row block which has 1's in column i and 0's elsewhere. On the other hand p_i is a characteristic vector representing C_i, which has 1's in the positions associated with C_i and zeros otherwise. Recall the example given in (2)

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{9}$$

and we can find

$$P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

while $p_2 = [0\ 1\ 1\ 0\ 0]^T$.

With a little bit calculation we can find

$$KP = \begin{bmatrix} k_1 P_1 \\ k_2 P_2 \\ \vdots \\ k_r P_r \end{bmatrix} = \begin{bmatrix} k_1 p_1 & k_2 p_2 & \dots & k_r p_r \end{bmatrix} = P\hat{K},$$

where $\hat{K} = \mathrm{diag}(k_i)_{i=1}^r$, which shows that $\mathcal{R}(P)$ is K-invariant. Since $\mathcal{R}(\bar{Q}) = \mathcal{R}(P)^\perp$, it is K-invariant as well by Lemma 3.5, and

$$\bar{Q}^T K \bar{P} = \bar{Q}^T \bar{P} \hat{K} = \mathbf{0}.$$

■

By the definition of the equitable partition, the subgraph induced by a cell is regular and every node in the same cell has the same number of neighbors outside the cell. Therefore, the nodes belonging to the same cell have the same degree, and thus by Lemma 4.2, $\mathcal{R}(\bar{Q})$ and $\mathcal{R}(P)$ are $\mathcal{D}$-invariant, where $\mathcal{D}$ is the degree matrix given by

$$\mathcal{D} = \mathrm{diag}(d_i I_{n_i})_{i=1}^r,$$

where $d_i \in \mathbb{R}$ denotes the degree of each nodes in cell.

Since the graph Laplacian satisfy $\mathcal{L}(\mathcal{G}) = \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$, Lemma 4.1 and Lemma 4.2 together can show that $\mathcal{R}(\bar{Q})$ and $\mathcal{R}(P)$ are $\mathcal{L}$-invariant, and thus, we have following corollary

Corollary 4.3: Given the same condition as in Lemma 4.1 $\mathcal{L}$ is similar to a diagonal block matrix

$$\bar{\mathcal{L}} = T^T \mathcal{L} T = \begin{bmatrix} \mathcal{L}_P & \mathbf{0} \\ \mathbf{0} & \mathcal{L}_Q \end{bmatrix}, \tag{10}$$

where $\mathcal{L}_P = \bar{P}^T \mathcal{L} \bar{P}$ and $\mathcal{L}_Q = \bar{Q}^T \mathcal{L} \bar{Q}$, and $T = [\bar{P} \mid \bar{Q}]$ defines a orthonormal basis for $\mathbb{R}^n$ with respect to π.

As (10) defines a similarity transformation, it follows that $\mathcal{L}_P$ and $\mathcal{L}_Q$ carry all the spectrum information of $\mathcal{L}$, i.e. they share eigenvalues with $\mathcal{L}$.

Now that, as we have show in Section II, in a leader-follower network, the graph Laplacian can be partitioned as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{bmatrix}$$

according to the leader assigning scheme. Transformations similar to (10) can be found for $\mathcal{L}_f$ in the presence of NEPs in the follower graph $\mathcal{G}_f$.

Corollary 4.4: Let $\mathcal{G}_f$ be a follower graph, and let $\mathcal{L}_f$ be the diagonal sub-matrix of $\mathcal{L}$ related to $\mathcal{G}_f$. If there is a NEP π_f in $\mathcal{G}_f$ and a π in $\mathcal{G}$, such that all the nontrivial cells in π_f are also cells in π, there exists an orthonormal matrix T_f such that

$$\bar{\mathcal{L}}_f = T_f^T \mathcal{L}_f T_f = \begin{bmatrix} \mathcal{L}_{fP} & \mathbf{0} \\ \mathbf{0} & \mathcal{L}_{fQ} \end{bmatrix}. \tag{11}$$

Proof: Let $\bar{P}_f = P_f(P_f^T P_f)^{\frac{1}{2}}$, where P_f is the characteristic matrix for π_f, and let $\bar{Q}_f$ be defined on a orthonormal basis of $\mathcal{R}(P_f)^\perp$. In the above way, we have obtained an orthonormal basis for $\mathbb{R}^{n_f}$ with respect to π_f. Moreover, $\mathcal{L}_f = \mathcal{D}_f + \mathcal{A}_f$ where $\mathcal{A}_f$ denotes the adjacency matrix of $\mathcal{G}_f$ while $\mathcal{D}_f$ is the degree matrix corresponding to the original graph $\mathcal{G}$. Since all the nontrivial cells in π_f are also cells in π, $\mathcal{D}_f$ satisfies the condition in Lemma 4.2, i.e. nodes from an identical cell in π_f have the same degree. Hence from Lemma 4.1 and Lemma 4.2, $\mathcal{R}(P)$ and $\mathcal{R}(\bar{Q})$ are $\mathcal{L}_f$-invariant and thus

$$\bar{\mathcal{L}}_f = T_f^T \mathcal{L}_f T_f = \begin{bmatrix} \mathcal{L}_{fP} & \mathbf{0} \\ \mathbf{0} & \mathcal{L}_{fQ} \end{bmatrix}, \tag{12}$$

where $T_f = [\bar{P}_f \mid \bar{Q}_f]$, $\mathcal{L}_{fP} = \bar{P}_f^T \mathcal{L}_f \bar{P}_f$ and $\mathcal{L}_{fQ} = \bar{Q}_f^T \mathcal{L}_f \bar{Q}_f$, . ■

Again, the diagonal blocks of $\bar{\mathcal{L}}_f$ share all the spectrum information with $\mathcal{L}_f$. Now, we are in the position to prove our main result.

Theorem 4.5: Given a connected graph $\mathcal{G}$ and the induced follower graph $\mathcal{G}_f$, the system $(\mathcal{L}_f, l_{fl})$ is not complete controllable if there exist NEPs on $\mathcal{G}$ and $\mathcal{G}_f$, say π and π_f, such that all the nontrivial cells of π are contained in π_f, i.e. $\exists\ \pi$ and π_f, such that $|C_i| = 1,\ \forall\ C_i \in \pi \backslash \pi_f$.

Proof: In Corollary 4.3 and Corollary 4.4, we have already shown that $\mathcal{L}$ and $\mathcal{L}_f$ are both similar to some diagonal block matrices. Here we want to show the relation ship between these diagonal block matrices.

Assume $\pi \cap \pi_f = \{C_1, C_2, \dots, C_{r_1}\}$. According to the given condition, $|C_i| \geq 2,\ i = 1, 2, \dots, r_1$. Without loss of generality, we can index the nodes in such a way that the nontrivial cells comprise the first n_1 nodes [4] such that

$$n_1 = \sum_{i=1}^{r_1} |C_i| \leq n_f < n.$$

Since all the nontrivial cells of π are in π_f, their characteristic matrices have similar structures

$$P = \begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & I_{n-n_1} \end{bmatrix}_{n \times r} \text{ and } P_f = \begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & I_{n_f-n_1} \end{bmatrix}_{n_f \times r_f},$$

where P_1 is a $n_1 \times r_1$ matrix that contains the nontrivial part of the characteristic matrices. Since $\bar{P}$ and $\bar{P}_f$ are only normalized P and P_f respectively, they have the same block structures. Consequently $\bar{Q}$ and $\bar{Q}_f$, the matrices containing orthonormal basis of $\mathcal{R}(P)$ and $\mathcal{R}(P_f)$, have following structures

$$\bar{Q} = \begin{bmatrix} Q_1 \\ \mathbf{0} \end{bmatrix}_{n \times (n_1 - r_1)} \text{ and } \bar{Q}_f = \begin{bmatrix} Q_1 \\ \mathbf{0} \end{bmatrix}_{n_f \times (n_1 - r_1)},$$

where Q_1 is a $n_1 \times (n_1 - r_1)$ matrix that satisfies

$$Q_1^T P_1 = \mathbf{0}.$$

As one can observe, $\bar{Q}_f$ is different from $\bar{Q}$ only by $n - n_f$ rows of zeros. In other words, the special structures of $\bar{Q}$ and $\bar{Q}_f$ gives us the relationship

$$Q_f = R^T Q,$$

where $R = [I_{n_f}, 0]^T$.

Now, recall the definition of $\mathcal{L}_Q$ and $\mathcal{L}_{Qf}$ from (10) and (12), which gives us

$$\begin{aligned} \mathcal{L}_Q &= \bar{Q}^T \mathcal{L} \bar{Q} \\ &= \bar{Q}_f^T R^T \mathcal{L} R \bar{Q}_f \\ &= \bar{Q}_f^T \mathcal{L}_f \bar{Q}_f = \mathcal{L}_{fQ}. \end{aligned} \tag{13}$$

Therefore $\mathcal{L}_f$ and $\mathcal{L}$ share the same eigenvalues associated with $\mathcal{L}_Q$, and by Lemma 2.2, the system is not completely controllable. ■

In the situation described in Theorem 4.5, the system is not completely controllable. This theorem thus gives us a method to identify uncontrollable situations in a leader-follower system. Intuitively speaking, vertices in the same cell of a NEP that satisfy the condition in Theorem 4.5 is not distinguishable from the leaders' point of view. In other words, if the agents belong to the same cell shared by π and π_f and they start from the same point, it is impossible for the leaders to pull them apart. Thus the controllable subspace can be obtained by collapsing all the nodes in the same cell into a single meta-node. However, since the NEPs may not be unique, as we have seen in the case of the Peterson graph, more work is required before a complete understanding of this issue is obtained.

[4] We introduce n_1 for convenience. It is easy to verify that $n_1 - r_1 = n - r = n_f - r_f$

Note that our sufficient condition for a graph to be uncontrollable immediately produces a necessary condition for a graph to be controllable and this states as a corollary.

Corollary 4.6: Given a connected graph $\mathcal{G}$ with the induced follower graph $\mathcal{G}_f$, a necessary for $(\mathcal{L}_f, l_{fl})$ to be controllable is that no NEPs π and π_f on $\mathcal{G}$ and $\mathcal{G}_f$ exist such that π and π_f share all nontrivial cells.

Corollary 4.7: If $\mathcal{G}$ is disconnected, a necessary condition for $(\mathcal{L}_f, l_{fl})$ to be controllable is that all of its connected components are controllable.

V. Examples and Discussions

In this part, we will show some uncontrollable situations that are identifiable by our method, and discuss the relationship among some existing results, our result and our ultimate goal.

a) Single Leader with Symmetric Followers. In Figure 3, if we choose node '5' as the leader, the symmetric pair (2,3) in the follower graph renders the network uncontrollable as stated in [3] . The dimension of the controllable subspace is three, while there are four nodes in the follower group. This result can also be interpreted by Theorem 4.5, since all the automorphism groups introduce equitable partitions.

b) Single Leader with Equal Distance Partitions. We have shown in Figure 2 that the Peterson graph has two NEPs. One is introduced by the automorphism groups and the other (π_2) is introduced by the equal distance groups. Based on π_2, if we choose node '1' as the leader, the leader-follower group ends up with a controllable subspace with dimension of two. Since there are four orbits[5] in the automorphism groups, this dimension can only be interpreted by the two-cell equal distance partitions[6].

c) Multiple Leaders. The last example is a modified leader graph based on the peterson graph. In Figure 4, we add another node ('11') connected to $\{3, 4, 7, 8, 9, 10\}$ as the second leader in addition to node '1'. In this network, there is an equal distance partition with four cells, $\{1\}$, $\{2, 5, 6\}$ $\{3, 4, 7, 8, 9, 10\}$ and $\{11\}$. In this situation, the dimension of the controllable subspace is still two, which is consistent with example b).

The examples above can be put into a graph described in Figure 5, which shows the relationship among previous theorems, our theorem, and the ultimate goal of this line of work. This paper provides a tool, which can identify a subset of the uncontrollable follower graph represented by 'II', which represents the graphs satisfy the condition described in Theorem 4.5. Since that condition is based on NEPs, and is able to deal with multiple leaders, we can identify a larger set of graphs than the set of graphs in 'I', which has been revealed by some previous works. Our ultimate goal is to achieve a necessary and sufficient condition which can gives us a topological insight of the network. This condition should

[5] They are $\{2, 5, 6\}, \{7, 10\}, \{8, 9\}, \{3, 4\}$

[6] They are $\{2, 5, 6\}$ and $\{3, 4, 7, 8, 9, 10\}$

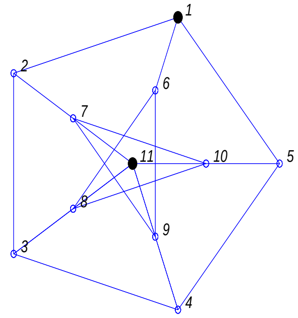


Fig. 4. A 2-leader network based on the Peterson graph. The second leader '11' is connected to '3','4','7','8','9' and '10'.

be able to identify all the uncontrollable follower graph in 'III'.

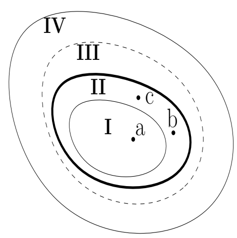


Fig. 5. The relationship among some previous work, our theorem, and the desired ultimate result: I. The follower graphs containing automorphism groups (single leader); II. The follower graphs identifiable by Theorem 4.5 of this paper; III. The follower graphs that are not completely controllable; IV. All the follower graphs.

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