ON THE EXISTENCE OF EXECUTIONS OF HYBRID AUTOMATA

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ABSTRACT. Properties of executions of hybrid automata are studied. We provide necessary and sufficient conditions for an automaton to be non-blocking and deterministic, which corresponds to local existence and uniqueness of executions, respectively. We also study the problem of global existence of executions in the context of Zeno hybrid automata, that is, hybrid automata that can exhibit infinitely many discrete transitions in finite time. Extensions of Zeno executions are discussed and it is pointed out that regularizations of the automaton may suggest different extensions.

1. INTRODUCTION

Despite a great deal of recent activity in the area of hybrid systems, the study of fundamental properties, such as the existence and uniqueness of executions, has to a large extend been overlooked. Still, these properties are important for analysis, controller synthesis and simulation, which in the absence of general analysis methodologies plays an important role in the study of hybrid systems. Existence and uniqueness properties have only been formally studied for particular classes of systems, for example, switched systems [6] and complementarity systems [12]. For more general classes, existence and uniqueness is typically introduced as an assumption and not analyzed further.

In this paper we try to formalize some basic properties of executions for a quite general class of hybrid automata. In addition to the usual technical conditions associated with existence results for conventional continuous dynamical systems, new complications due to blocking and non-determinism need to be considered in the analysis. Yet another issue is the Zeno phenomenon [1, 2], where the executions of the hybrid system exhibit an infinite number of discrete transitions in finite time. We will see that Zeno hybrid automata are closely related to chattering, that may arise in some optimal control problems [4] and in variable structure systems [11].

We start by giving formal definitions of hybrid automata and their executions in Section 2. Results on existence and uniqueness of executions are derived in Section 3. Section 4 presents some examples of Zeno hybrid systems, highlighting various aspects of the Zeno phenomenon. The paper ends with conclusions in Section 5.

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2. Hybrid Automata

2.1. **Notation.** Consider a finite collection V of variables. Let $\mathbf{V}$ denote denote both a variable and its valuation. We refer to variables whose set of valuations is countable as *discrete* and to variables whose set of valuations is a subset of a Euclidean space as *continuous*. We assume that Euclidean spaces, $\mathbb{R}^n$ for $n \geq 0$, are given the Euclidean metric topology, whereas countable and finite sets are given the discrete topology (all subsets are open). Subsets of a topological space are given the subset topology and products of topological spaces are given the product topology. For a subset U of a topological space we use $\overline{U}$ to denote its closure, U^o its interior, ∂U its boundary, U^c its complement, $|U|$ its cardinality, and 2^U the set of all subsets of U. We use $\wedge$ to denote the logical "and", and $\vee$ to denote the logical "or".

2.2. **Hybrid Automata and Executions.** The following definitions are based on [7, 8].

Definition 1 (Hybrid Time Trajectory). *A hybrid time trajectory $\tau = \{I_i\}_{i=0}^N$ is a finite or infinite sequence of intervals of the real line, such that*

- $I_i = [\tau_i, \tau_i']$ *for* $i < N$;
- $I_N = [\tau_N, \tau_N']$ *or* $I_N = [\tau_N, \tau_N')$ *if* $N < \infty$; *and,*
- $\tau_i \leq \tau_i'$ *for* $i \geq 0$ *and* $\tau_i = \tau_{i-1}'$ *for* $i > 0$.

Note that hybrid time trajectories can extend to "infinity" if τ is an infinite sequence or if it is a finite sequence ending with an interval of the form $[\tau_N, \infty)$. We denote by $\mathcal{T}$ the set of all hybrid time trajectories. Each $\tau \in \mathcal{T}$ is fully ordered by the relation $\prec$, which for $t \in [\tau_i, \tau_i'] \in \tau$ and $t' \in [\tau_j, \tau_j'] \in \tau$ is defined by $t \prec t'$ if $i < j$ or $i = j$ and $t < t'$. For $t \in \mathbb{R}$ and $\tau \in \mathcal{T}$ we use $t \in \tau$ as a shorthand notation for "there exists a j such that $t \in [\tau_j, \tau_j'] \in \tau$". For a topological space K and a $\tau \in \mathcal{T}$, we use $k : \tau \to K$ as a shorthand notation for a map assigning values from K to all $t \in \tau$. For a collection of variables W, we denote the by $\mathrm{Hyb}(W)$ the set of all hybrid trajectories of W, defined as $\mathrm{Hyb}(W) = \{(\tau, w) \ : \ \tau \in \mathcal{T} \text{ and } w : \tau \to \mathbf{W}\}$. We say $\tau = \{I_i\}_{i=0}^N \in \mathcal{T}$ is a *prefix* of $\tau' = \{J_i\}_{i=0}^M \in \mathcal{T}$ and write $\tau \leq \tau'$ if either they are identical or τ is finite, $M \geq N$, $I_i = J_i$ for all $i = 0, \ldots, N-1$ and $I_N \subseteq J_N$. The prefix relation is a partial order on $\mathcal{T}$.

A hybrid automaton provides a formal way for restricting the set of hybrid trajectories of a collection of discrete and continuous variables.

Definition 2 (Hybrid Automaton). *A hybrid automaton H is a collection $H = (Q, X, \mathrm{Init}, f, I, E, G, R)$, where*

- Q *is a finite collection of discrete variables;*
- X *is a finite collection of continuous variables with* $\mathbf{X} = \mathbb{R}^n$;
- $\mathrm{Init} \subseteq \mathbf{Q} \times \mathbf{X}$ *is a set of initial states;*
- $f : \mathbf{Q} \times \mathbf{X} \to T\mathbf{X}$ *is a vector field, Lipschitz continuous in its second argument;*
- $I : \mathbf{Q} \to 2^{\mathbf{X}}$ *assigns to each* $q \in \mathbf{Q}$ *an invariant set;*
- $E \subset \mathbf{Q} \times \mathbf{Q}$ *is a collection of edges;*
- $G : E \to 2^{\mathbf{X}}$ *assigns to each edge* $e = (q, q') \in E$ *a guard; and*
- $R : E \times \mathbf{X} \to 2^{\mathbf{X}}$ *assigns to each edge* $e = (q, q') \in E$ *and* $x \in \mathbf{X}$ *a reset relation.*

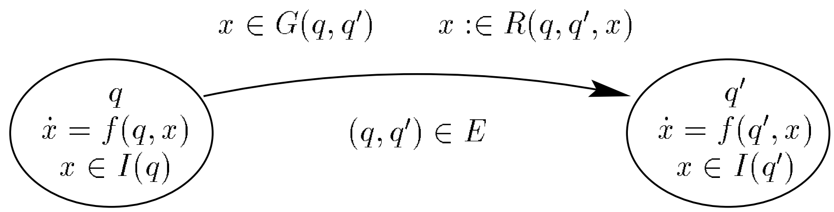

FIGURE 1. Graphical representation of a hybrid automaton.

Notice that $\chi \in \mathrm{Hyb}(Q \cup X)$. We refer to $(q, x) \in \mathbf{Q} \times \mathbf{X}$ as the state of H. Pictorially, a hybrid automaton is represented by a directed graph $(\mathbf{Q}, E)$ with vertices $\mathbf{Q}$ and edges E. With each vertex $q \in \mathbf{Q}$, we associate a vector field $f(q, x)$ and an invariant $I(q)$. With each edge $e \in E$, we associate a guard $G(e)$ and a reset relation $R(e, x)$. The directed graph notation is summarized in Figure 1.

Definition 3 (Execution). *An execution χ of a hybrid automaton H is a collection $\chi = (\tau, q, x)$ with $\tau \in \mathcal{T}$, $q : \tau \to \mathbf{Q}$, and $x : \tau \to \mathbf{X}$, satisfying*

- $\big(q(\tau_0), x(\tau_0)\big) \in \mathrm{Init}$ *(initial condition);*
- *for all i with $\tau_i < \tau_i'$, $x(t)$ is absolutely continuous and $q(t)$ is constant for $t \in [\tau_i, \tau_i']$, and $x(t) \in I\big(q(t)\big)$ and $dx(t)/dt = f\big(q(t), x(t)\big)$ for all $t \in [\tau_i, \tau_i')$ (continuous evolution); and*
- *for all i, $e = \big(q(\tau_i'), q(\tau_{i+1})\big) \in E$, $x(\tau_i') \in G(e)$, and $x(\tau_{i+1}) \in R\big(e, x(\tau_i')\big)$ (discrete evolution).*

For an execution $\chi = (\tau, q, x)$ we use $(q_0, x_0) = \big(q(\tau_0), x(\tau_0)\big)$ to denote the initial state of χ. We say $\chi = (\tau, q, x)$ is a prefix of $\chi' = (\tau', q', x')$ (write $\chi \leq \chi'$) if $\tau \leq \tau'$ and $\big(q(t), x)(t)\big)(t) = \big(q'(t), x'(t)\big)$ for all $t \in \tau$. We say χ is a strict prefix of χ' (write $\chi < \chi'$) if $\chi \leq \chi'$ and $\chi \neq \chi'$. Note that the prefix relation defines a partial order on the set of executions, and that the set of executions is prefix closed.

2.3. **Classification of Executions.** Unlike conventional continuous dynamical systems, the interpretation is that an automaton H *accepts* (as opposed to generates) an execution $\chi = (\tau, q, x)$. This difference in perspective allows one to consider hybrid automata that accept no executions for some initial states, accept multiple executions for the same initial state, or do not accept executions over arbitrarily long time horizons. To formalize these notions we introduce the following classification.

Definition 4 (Types of Execution). *An execution $\chi = (\tau, q, x)$ is called*

- *finite, if τ is a finite sequence ending with a closed interval;*
- *infinite, if τ is an infinite sequence, or if $\sum_{i=0}^{\infty}(\tau_i' - \tau_i) = \infty$;*
- *Zeno, if it is infinite but $\sum_{i=0}^{\infty}(\tau_i' - \tau_i) < \infty$; and*
- *maximal, if it is not a strict prefix of any other execution of H.*

For an infinite execution we define the Zeno time as $\tau_\infty = \sum_{i=0}^{\infty}(\tau_i' - \tau_i)$. Clearly, $\tau_\infty < \infty$ if the execution is Zeno and $\tau_\infty = \infty$ otherwise.

We use $\mathcal{H}_{(q_0,x_0)}$ to denote the set of all executions of H with initial condition $(q_0, x_0) \in \mathrm{Init}$, $\mathcal{H}^M_{(q_0,x_0)}$ the set of all maximal executions with initial condition $(q_0, x_0) \in \mathrm{Init}$, and $\mathcal{H}^\infty_{(q_0,x_0)}$ the set of all infinite executions with initial condition $(q_0, x_0) \in \mathrm{Init}$. We use $\mathcal{H}$ to denote the set of all executions of H, that is,

$$\mathcal{H} = \bigcup_{(q_0,x_0)\in \mathrm{Init}} \mathcal{H}_{(q_0,x_0)}.$$

Since an infinite execution can by definition not be a strict prefix of any other execution, we immediately have the following.

Proposition 1. *For a hybrid automaton H and for all $(q_0, x_0) \in \mathrm{Init}$, $\mathcal{H}^\infty_{(q_0,x_0)} \subseteq \mathcal{H}^M_{(q_0,x_0)}$.*

2.4. **Equivalent Hybrid Automata.** Definition 2 allows some redundancy when defining discrete transitions. A transition between two values of the discrete state $q, q' \in \mathbf{Q}$ can be prevented by ensuring that $(q, q') \notin E$, by setting $G(q, q') = \emptyset$, or by setting $R(q, q', x) = \emptyset$. As an aside, we show how this redundancy can be removed without reducing the expressiveness of the modeling formalism.

Definition 5 (Equivalent Hybrid Automata)**.** *Two hybrid automata H and $\widehat{H}$ are called equivalent if $Q = \widehat{Q}$, $X = \widehat{X}$, and $\mathcal{H} = \widehat{\mathcal{H}}$.*

Proposition 2. *For every hybrid automaton H there exists an equivalent hybrid automaton $\widehat{H}$, such that $(q, q') \in \widehat{E}$ if and only if $\widehat{G}(q, q') \neq \emptyset$ and $x \in \widehat{G}(q, q')$ if and only if $R(q, q', x) \neq \emptyset$.*

Proof. Consider the hybrid automaton $\widehat{H}$ defined by $\widehat{Q} = Q$, $\widehat{X} = X$, $\widehat{\mathrm{Init}} = \mathrm{Init}$, $\widehat{f}(q, x) = f(q, x)$ for all $q \in \mathbf{Q}$, $x \in \mathbf{X}$, $\widehat{I} = I$ for all $q \in \mathbf{Q}$ and

$$\begin{aligned}
\widehat{E} &= \{e \in E \;:\; \exists x \in G(e) \text{ with } R(e, x) \neq \emptyset\} \\
\widehat{G}(e) &= \{x \in \mathbf{X} \;:\; x \in G(e) \wedge R(e, x) \neq \emptyset\} \\
\widehat{R}(e, x) &= \{x' \in \mathbf{X} \;:\; x \in G(e) \wedge x' \in R(e, x)\}.
\end{aligned}$$

The components $\widehat{E}$, $\widehat{G}$, and $\widehat{R}$ satisfy the requirements of the proposition. It remains to show that $\widehat{H}$ is equivalent to H.

Since $\widehat{E} \subseteq E$, $\widehat{G}(e) \subseteq G(e)$ and $\widehat{R}(e, x) \subseteq R(e, x)$, $\widehat{\mathcal{H}} \subseteq \mathcal{H}$. Next consider an execution $\chi = (\tau, q, x) \in \mathcal{H}$. We inductively show that $\chi \in \widehat{\mathcal{H}}$. Since $\widehat{\mathrm{Init}} = \mathrm{Init}$, $(q_0, x_0) \in \widehat{\mathrm{Init}}$. Since $\widehat{f}(q, x) = f(q, x)$ for all (q, x), the finite prefix of χ defined over $[\tau_0, \tau_0'] \leq \tau$ belongs to $\widehat{\mathcal{H}}$. If $\tau = [\tau_0, \tau_0']$ (or if $\tau = [\tau_0, \tau_0')$) we are done. Otherwise, assume that the finite prefix of χ defined over $\{[\tau_i, \tau_i']\}_{i=1}^M < \tau$ belongs to $\widehat{\mathcal{H}}$. Since $\chi \in \mathcal{H}$, $\big(q(\tau_M'), q(\tau_{M+1})\big) \in E$, $x(\tau_M') \in G\big(q(\tau_M'), q(\tau_{M+1})\big)$ and $x(\tau_{M+1}) \in R\big(q(\tau_M'), q(\tau_{M+1}), x(\tau_M')\big)$. Therefore, by definition of $\widehat{E}$, $\widehat{G}$, and $\widehat{R}$, the finite prefix of χ defined over $\{[\tau_i, \tau_i']\}_{i=1}^M[\tau_{M+1}, \tau_{M+1}]$ belongs to $\widehat{\mathcal{H}}$. Moreover, since $\widehat{f}(q, x) = f(q, x)$ for all (q, x), the finite prefix of χ defined over $\{[\tau_i, \tau_i']\}_{i=1}^{M+1}$ also belongs to $\widehat{\mathcal{H}}$. Therefore, by induction, $\chi \in \widehat{\mathcal{H}}$, i.e. $\widehat{\mathcal{H}} \subseteq \mathcal{H}$. Overall, $\widehat{\mathcal{H}} = \mathcal{H}$, and thus H and $\widehat{H}$ are equivalent. $\square$

Proposition 2 indicates that when studying the executions of hybrid automata one can, without loss of generality, assume that a transition is included in the definition of an automaton if and only if it is enabled for at least one continuous state for which a "next state" exists. This in fact suggests that the definition of a hybrid automaton may be given equivalently without mentioning to the transitions E explicitly, but by appropriate definitions of G and R. Here we choose to tolerate the additional overhead in the notation, because it clarifies the presentation.

3. Local Existence and Uniqueness Conditions

In this section we derive conditions under which all executions of a hybrid automaton can be extended to infinite executions and conditions under which this extension can be done uniquely. More formally, we provide conditions to characterize the following classes of automata.

Definition 6 (Non-Blocking and Deterministic Automaton). *A hybrid automaton H is called non-blocking if $\mathcal{H}^{\infty}_{(q_0,x_0)}$ is non-empty for all $(q_0, x_0) \in \mathrm{Init}$. It is called deterministic if $\mathcal{H}^{M}_{(q_0,x_0)}$ contains at most one element for all $(q_0, x_0) \in \mathrm{Init}$.*

To state the local existence and uniqueness conditions we need to characterize the set of states that can be reached by H, as well as the set of states from which continuous evolution is possible.

Definition 7 (Reachable State). *A state $(\widehat{q}, \widehat{x}) \in \mathbf{Q} \times \mathbf{X}$ is called reachable by H if there exists a finite execution $\chi = (\tau, q, x)$ with $\tau = \{[\tau_i, \tau_i']\}_{i=0}^{N}$ and $(q(\tau_N'), x(\tau_N')) = (\widehat{q}, \widehat{x})$.*

We use $\mathrm{Reach}(H) \subseteq \mathbf{Q} \times \mathbf{X}$ to denote the set of states reachable by H.

Next, consider $q \in \mathbf{Q}$ and, for some $\epsilon > 0$ small enough[1], the solution, $x(\cdot) : [0, \epsilon) \to \mathbf{X}$ to the ordinary differential equation

$$\frac{dx}{dt}(t) = f(q, x(t)) \ \text{ with } \ x(0) = x^0. \tag{1}$$

The set of states from which continuous evolution is possible is then given by

$$\{(q^0, x^0) \in \mathbf{Q} \times \mathbf{X} \ : \ \exists \epsilon > 0 \text{ such that } \forall t \in [0, \epsilon), x(t) \in I(q^0)\},$$

where $x(\cdot)$ is the solution of (1). To characterize the states from which continuous evolution is impossible, we define the map $\mathrm{Out} : \mathbf{Q} \to 2^{\mathbf{X}}$ by

$$\mathrm{Out}(q) = \left\{x^0 \in \mathbf{X} \ : \ \forall \epsilon > 0 \ \exists t \in [0, \epsilon) \text{ such that } x(t) \not\in I(q)\right\}, \tag{2}$$

where again $x(\cdot)$ is the solution of (1). Note that $I(q)^c \subseteq \mathrm{Out}(q)$, since for all $x^0 \in I(q)^c$ and for all $\epsilon > 0$ there exists $t \in [0, \epsilon)$ (in particular $t = 0$) such that $x(t) \not\in I(q)$.

[1]Such ϵ exists by the Lipschitz assumption on f in Definition 2.

3.1. **Local Existence Conditions.**

Lemma 1. *A hybrid automaton H is non-blocking if for all $(q,x) \in \text{Reach}(H)$ with $x \in \text{Out}(q)$, there exists $(q,q') \in E$ such that*

- *$x \in G(q,q')$; and*
- *$R(q,q',x) \neq \emptyset$.*

Proof. Consider an initial state $(q_0,x_0) \in \text{Init}$ and assume, for the sake of contradiction, that there does not exist an infinite execution starting at (q_0,x_0). Let $\chi = (\tau,q,x)$ denote a maximal execution starting at (q_0,x_0), and note that τ is a finite sequence.

First consider the case $\tau = \{[\tau_i,\tau_i']\}_{i=0}^{N-1}[\tau_N,\tau_N')$. Let $(q_N,x_N) = \lim_{t\to\tau_N'}(q(t),x(t))$. Note that, by the definition of execution and a standard existence argument for continuous dynamical systems, the limit exists and χ can be extended to $\widehat{\chi} = (\widehat{\tau},\widehat{q},\widehat{x})$ with $\widehat{\tau} = \{[\tau_i,\tau_i']\}_{i=0}^{N}$, $\widehat{q}(\tau_N') = q_N$, and $\widehat{x}(\tau_N') = x_N$. This contradicts the maximality of χ.

Now consider the case $\tau = \{[\tau_i,\tau_i']\}_{i=0}^{N}$, and let $(q_N,x_N) = (q(\tau_N'),x(\tau_N'))$. Clearly, $(q_N,x_N) \in \text{Reach}(H)$. If $x_N \notin \text{Out}(q_N)$, then there exists $\epsilon > 0$ such that χ can be extended to $\widehat{\chi} = (\widehat{\tau},\widehat{q},\widehat{x})$ with $\widehat{\tau} = \{[\tau_i,\tau_i']\}_{i=0}^{N-1}[\tau_N,\tau_N'+\epsilon)$, by continuous evolution. If, on the other hand $x_N \in \text{Out}(q_N)$, then there exists $(q',x') \in \mathbf{Q} \times \mathbf{X}$ such that $(q_N,q') \in E$, $x_N \in G(q_N,q')$ and $x' \in R(q_N,q',x_N)$. Therefore, χ can be extended to $\widehat{\chi} = (\widehat{\tau},\widehat{q},\widehat{x})$ with $\widehat{\tau} = \{[\tau_i,\tau_i']\}_{i=0}^{N+1}$, $\tau_{N+1} = \tau_{N+1}' = \tau_N'$, $q(\tau_{N+1}) = q'$, $x(\tau_{N+1}) = x'$ by a discrete transition. In both cases the maximality of χ is contradicted. $\square$

Loosely speaking, the conditions of Lemma 1 indicate that a hybrid automaton is non-blocking if transitions with non-trivial reset relations are enabled whenever continuous evolution is impossible (outside the invariant sets and at points on the boundary where the continuous flow forces the state to exit the invariants).

The conditions of Lemma 1 are tight, in the sense that blocking automata that violate the conditions exist, but are not necessary, in the sense that not all automata that violate the conditions are blocking. However,

Lemma 2. *A deterministic hybrid automaton is non-blocking only if the conditions of Lemma 1 are satisfied.*

Proof. Consider a deterministic hybrid automaton H that violates the conditions of Lemma 1, that is, there exists $(q',x') \in \text{Reach}(H)$ such that $x' \in \text{Out}(q')$, but there is no $\widehat{q}' \in \mathbf{Q}$ with $(q',\widehat{q}') \in E$, $x' \in G(q',\widehat{q}')$ and $R(q',\widehat{q}',x') \neq \emptyset$. Since $(q',x') \in \text{Reach}(H)$, there exists $(q_0,x_0) \in \text{Init}$ and a finite execution, $\chi = (\tau,q,x) \in \mathcal{H}_{(q_0,x_0)}$ such that $\tau = \{[\tau_i,\tau_i']\}_{i=0}^{N}$ and $(q',x') = (q(\tau_N'),x(\tau_N'))$.

We first show that χ is maximal. Assume first that there exists $\widehat{\chi} = (\widehat{\tau},\widehat{q},\widehat{x})$ with $\widehat{\tau} = \{[\tau_i,\tau_i']\}_{i=0}^{N-1}[\tau_N,\tau_N+\epsilon)$ for some $\epsilon > 0$. This would violate the assumption that $x' \in \text{Out}(q')$. Next assume that there exists $\widehat{\chi} = (\widehat{\tau},\widehat{q},\widehat{x})$ with $\widehat{\tau} = \tau[\tau_{N+1},\tau_{N+1}']$ with $\tau_{N+1} = \tau_N'$. This requires that the execution can be extended beyond (q',x') by a discrete transition, that is there exists $(\widehat{q}', \widehat{x}') \in \mathbf{Q}$ such that $(q', \widehat{q}') \in E$, $x' \in G(q', \widehat{q}')$ and $\widehat{x}' \in R(q', \widehat{q}', x')$. This would also contradict our original assumptions. Overall, $\chi \in \mathcal{H}^M_{(q_0,x_0)}$.

Now assume, for the sake of contradiction that H is non-blocking. Then, there exists $\chi' \in \mathcal{H}^\infty_{(q_0,x_0)}$. By Proposition 1, $\chi' \in \mathcal{H}^M_{(q_0,x_0)}$. But $\chi \neq \chi'$ (as the former is finite and the latter infinite), therefore $\mathcal{H}^M_{(q_0,x_0)} \supset \{\chi, \chi'\}$. This contradicts the assumption that H is deterministic. $\square$

3.2. **Uniqueness Conditions.**

Lemma 3. *A hybrid automaton is deterministic if and only if for all* $(q, x) \in \mathrm{Reach}(H)$,

- $x \in \bigcup_{(q,q')\in E} G(q, q')$ *implies* $x \in \mathrm{Out}(q)$;
- $(q, q') \in E$ *and* $(q, q'') \in E$ *with* $q' \neq q''$ *imply* $x \notin G(q, q') \cap G(q, q'')$; *and*
- $(q, q') \in E$ *and* $x \in G(q, q')$ *imply* $|R(q, q', x)| \leq 1$.

Proof. For the "if" part, assume, for the sake of contradiction, that there exists an initial state $(q_0, x_0) \in \mathrm{Init}$ and two maximal executions $\chi = (\tau, q, x)$ and $\widehat{\chi} = (\widehat{\tau}, \widehat{q}, \widehat{x})$ starting at (q_0, x_0) with $\chi \neq \widehat{\chi}$. Let $\psi = (\rho, p, y) \in \mathcal{H}_{(q_0,x_0)}$ denote the maximal common prefix of χ and $\widehat{\chi}$. Such a prefix exists as the executions start at the same initial state. Moreover, ψ is not infinite, as $\chi \neq \widehat{\chi}$. Therefore, as in the proof of Lemma 1, ρ can be assumed to be of the form $\rho = \{[\rho_i, \rho_i']\}_{i=0}^N$, as otherwise the maximality of ψ would be contradicted by an existence and uniqueness argument of the continuous solution along f. Let $(q_N, x_N) = (q(\rho_N'), x(\rho_N')) = (\widehat{q}(\rho_N'), \widehat{x}(\rho_N'))$. Clearly, $(q_N, x_N) \in \mathrm{Reach}(H)$. We distinguish the following cases:

Case 1: $\rho_N' \notin \{\tau_i'\}$ and $\rho_N' \notin \{\widehat{\tau}_i'\}$, i.e., ρ_N' is not a time when a discrete transition takes place in either χ or $\widehat{\chi}$. Then, by the definition of execution and a standard existence and uniqueness argument for continuous dynamical systems, there exists $\epsilon > 0$ such that the prefixes of χ and $\widehat{\chi}$ are defined over $\widehat{\rho} = \{[\rho_i, \rho_i']\}_{i=0}^{N-1}[\rho_N, \rho_N' + \epsilon)$ and are identical. This contradicts the maximality of ψ.

Case 2: $\rho_N' \in \{\tau_i'\}$ and $\rho_N' \notin \{\widehat{\tau}_i'\}$, i.e., ρ_N' is a time when a discrete transition takes place in χ but not in $\widehat{\chi}$. The fact that a discrete transition takes place from (q_N, x_N) in χ indicates that there exists $q' \in \mathbf{Q}$ such that $(q_N, q') \in E$ and $x_N \in G(q_N, q')$. The fact that no discrete transition takes place from (q_N, x_N) in $\widehat{\chi}$ indicates that there exists $\epsilon > 0$ such that $\widehat{\chi}$ is defined over $\widehat{\rho} = \{[\rho_i, \rho_i']\}_{i=0}^{N-1}[\rho_N, \rho_N' + \epsilon)$. A necessary condition for this is that $x_N \notin \mathrm{Out}(q)$. This contradicts condition 1 of the lemma.

Case 3: $\rho_N' \notin \{\tau_i'\}$ and $\rho_N' \in \{\widehat{\tau}_i'\}$, symmetric to Case 2.

Case 4: $\rho_N' \in \{\tau_i'\}$ and $\rho_N' \in \{\widehat{\tau}_i'\}$, i.e., ρ_N' is a time when a discrete transition takes place in both χ and $\widehat{\chi}$. The fact that a discrete transition takes place from (q_N, x_N) in both χ and $\widehat{\chi}$ indicates that there exist (q', x') and $(\widehat{q}', \widehat{x}')$ such that $(q_N, q') \in E$, $(q_N, \widehat{q}') \in E$, $x_N \in G(q_N, q')$, $x_N \in G(q_N, \widehat{q}')$, $x' \in R(q_N, q', x_N)$, and $\widehat{x}' \in R(q_N, \widehat{q}', x_N)$. Note that by condition 2 of the lemma, $q' = \widehat{q}'$, hence, by condition 3, $x' = \widehat{x}'$. Therefore, the prefixes of χ and $\widehat{\chi}$ are defined over $\widehat{\rho} = \{[\rho_i, \rho_i']\}_{i=0}^N[\rho_{N+1}, \rho_{N+1}']$, with $\rho_{N+1} = \rho_{N+1}' = \rho_N'$, and are identical. This contradicts the maximality of ψ and concludes the proof of the "if" part.

For the "only if" part, assume that there exists $(q', x') \in \text{Reach}(H)$ such that at least one of the conditions of the lemma is violated. Since $(q', x') \in \text{Reach}(H)$, there exists $(q_0, x_0) \in \text{Init}$ and a finite execution, $\chi = (\tau, q, x) \in \mathcal{H}_{(q_0,x_0)}$ such that $\tau = \{[\tau_i, \tau_i']\}_{i=0}^{N}$ and $(q', x') = (q(\tau_N'), x(\tau_N'))$. If condition 1 is violated, then there exists $\widehat{\chi}$ and $\tilde{\chi}$ with $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{N-1}[\tau_N, \tau_N + \epsilon)$, $\epsilon > 0$, and $\tilde{\tau} = \tau[\tau_{N+1}, \tau_{N+1}']$, $\tau_{N+1} = \tau_N'$, such that $\chi < \widehat{\chi}$ and $\chi < \tilde{\chi}$. If condition 2 is violated, there exist $\widehat{\chi}$ and $\tilde{\chi}$ with $\widehat{\tau} = \tilde{\tau} = \tau[\tau_{N+1}, \tau_{N+1}]$, $\tau_{N+1} = \tau_N'$, and $\widehat{q}(\tau_{N+1}) \neq \tilde{q}(\tau_{N+1})$, such that $\chi < \widehat{\chi}$, $\chi < \tilde{\chi}$. Finally, if condition 3 is violated, then there exist $\widehat{\chi}$ and $\tilde{\chi}$ with $\widehat{\tau} = \tilde{\tau} = \tau[\tau_{N+1}, \tau_{N+1}']$, $\tau_{N+1} = \tau_N'$, and $\widehat{x}(\tau_{N+1}) \neq \tilde{x}(\tau_{N+1})$, such that $\chi < \widehat{\chi}$, $\chi < \tilde{\chi}$. In all three cases, let $\overline{\widehat{\chi}} \in \mathcal{H}_{(q_0,x_0)}^M$ and $\overline{\tilde{\chi}} \in \mathcal{H}_{(q_0,x_0)}^M$ denote maximal executions of which $\widehat{\chi}$ and $\tilde{\chi}$ are prefixes respectively. Since $\widehat{\chi} \neq \tilde{\chi}$, it follows that $\overline{\widehat{\chi}} \neq \overline{\tilde{\chi}}$. Therefore $|\mathcal{H}_{(q_0,x_0)}^M| \geq 2$ and thus H is non-deterministic. □

Loosely speaking, the conditions of Lemma 3 indicate that a hybrid automaton is deterministic if and only if discrete transitions have to be forced by the the continuous flow exiting the invariant set, no two discrete transitions can be enabled simultaneously, and no point can be mapped onto two different points by the reset map. Note that the conditions are both necessary and sufficient.

Summarizing Lemmas 1 and 3 give the following.

Theorem 1 (Existence and Uniqueness of Executions). *A hybrid automaton accepts a unique infinite execution for all* $(q_0, x_0) \in \text{Init}$ *if it satisfies the conditions of Lemmas 1 and 3.*

Proof. If the hybrid automaton satisfies the conditions of Lemma 1, then $|\mathcal{H}_{(q_0,x_0)}^\infty| \geq 1$, for all $(q_0, x_0) \in \text{Init}$. If it satisfies the conditions of Lemma 3, then $|\mathcal{H}_{(q_0,x_0)}^M| \leq 1$, for all $(q_0, x_0) \in \text{Init}$. By Proposition 1, $\mathcal{H}_{(q_0,x_0)}^\infty \subseteq \mathcal{H}_{(q_0,x_0)}^M$. Therefore, $1 \leq |\mathcal{H}_{(q_0,x_0)}^\infty| \leq |\mathcal{H}_{(q_0,x_0)}^M| \leq 1$, or in other words $|\mathcal{H}_{(q_0,x_0)}^\infty| = |\mathcal{H}_{(q_0,x_0)}^M| = 1$. □

The conditions of the theorem are sufficient and tight.

3.3. **Computation of** $\text{Out}(q)$ **and** $\text{Reach}(H)$**.** The existence and uniqueness conditions established above require one to compute the set $\text{Out}(q)$ for $q \in \mathbf{Q}$. In some cases this computation is rather straightforward, as is shown next.

Proposition 3 (Open Invariant). *If* $I(q)$ *is open, then* $\text{Out}(q) = I(q)^c$.

Proof. As argued earlier, $I(q)^c \subseteq \text{Out}(q)$. Conversely, consider $x^0 \in I(q)$. By the continuity of the solution of (1) with respect to time and the fact that $I(q)$ is open, there exists $\epsilon > 0$ such that for all $t \in [0, \epsilon)$, $x(t) \in I(q)$. Therefore, $I(q) \subseteq \text{Out}(q)^c$. Overall, $\text{Out}(q) = I(q)^c$. □

If the invariant set $I(q)$ is not open, the characterization of $\text{Out}(q)$ is slightly more involved. Assume f and $\sigma : \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$ are analytic functions in x. We inductively define the Lie derivatives of σ along f, $L_f^m \sigma : \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$, $m = 0, 1, \ldots$ by

$$L_f^0 \sigma(q, x) = \sigma(q, x) \quad \text{and} \quad L_f^m \sigma(q, x) = \left(\frac{\partial}{\partial x} L_f^{m-1} \sigma(q, x) \right) f(q, x), \quad \text{for} \quad m > 0.$$

The pointwise relative degree of σ with respect to f is defined as the function $n_{(\sigma,f)} : \mathbf{Q}\times\mathbf{X} \to \mathbb{N}$ given by

$$n_{(\sigma,f)}(q,x) := \min\left\{m \in \mathbb{N} :\ L_f^m \sigma(q,x) \neq 0\right\}.$$

Note that $n_{(\sigma,f)}(q,x) = 0$ for all (q,x) such that $\sigma(q,x) \neq 0$.

Proposition 4 (Analytic Invariant). *If $I(q) = \{x \in \mathbf{X} :\ \sigma(q,x) \geq 0\}$ and $f(q,x)$ and $\sigma(q,x)$ are analytic functions in x, then* $\mathrm{Out}(q) = \{x \in \mathbf{X} :\ L_f^{n_{(\sigma,f)}(q,x)} \sigma(q,x) < 0\}$.

Proof. Since f is analytic in x, the solution $x(t)$ of (1) is analytic as a function of t. Since σ is analytic in x, $\sigma\big(q, x(t)\big)$ is also analytic as a function of t. Consider the Taylor Series expansion of $\sigma\big(q,x(t)\big)$ about $t = 0$. By analyticity,

$$\begin{aligned}\sigma\big(q,x(t)\big) &= \sigma\big(q,x(0)\big) + \frac{d\sigma}{dt}\big(q,x(0)\big)t + \frac{d^2\sigma}{dt^2}\big(q,x(0)\big)\frac{t^2}{2!} + \dots \\ &= \sigma(q,x^0) + L_f\sigma(q,x^0)t + L_f^2\sigma(q,x^0)\frac{t^2}{2!} + \dots\end{aligned}$$

is convergent. If $n_{(\sigma,f)}(q,x^0) < \infty$, the first non-zero term in the expansion, $L_f^{n_{(\sigma,f)}(x)}\sigma(q,x^0)$, dominates the sum for t small enough. Therefore, if $L_f^{n_{(\sigma,f)}(q,x)}\sigma(q,x^0) < 0$, for all $\epsilon > 0$ there exists $t \in [0,\epsilon)$ such that $\sigma\big(q,x(t)\big) < 0$, and hence $x^0 \in \mathrm{Out}(q)$. Hence, $\{x \in \mathbf{X} :\ L_f^{n_{(\sigma,f)}(q,x)}\sigma(q,x) < 0\} \subseteq \mathrm{Out}(q)$.

Likewise, if $L_f^{n_{(\sigma,f)}(q,x)}\sigma(q,x^0) > 0$, then there exists $\epsilon > 0$ such that for all $t \in [0,\epsilon)$, $\sigma\big(q,x(t)\big) > 0$, and hence $x^0 \in \mathrm{Out}(q)^c$. Moreover, if $n_{(\sigma,f)}(Q,x^0) = \infty$, the Taylor series expansion is identically equal to zero. By analyticity, $\sigma\big(q,x(t)\big) = 0$ for all t, which implies that $x(t) \in I(q)$ for all t. Therefore, $\mathrm{Out}(q) \subseteq \{x \in \mathbf{X} :\ L_f^{n_{(\sigma,f)}(q,x)}\sigma(q,x) < 0\}$. □

The analyticity requirement may be too restrictive for some applications. It can be relaxed somewhat by noting that the definition of the Lie derivative and the relative degree do not require analyticity of f and σ, but only require the functions to be sufficiently differentiable. In fact, the only part of the proof of Proposition 4 where analyticity is crucial is when $n_{(\sigma,f)}(q,x^0) = \infty$. Finite $n_{(\sigma,f)}(q,x)$ indicates that the vector field f is in a sense "transverse" to the boundary of the invariant set given by $\sigma(q,x) = 0$.

Proposition 5 (Transverse Invariant). *Assume that $I(q) = \{x \in \mathbf{X} :\ \sigma(q,x) \geq 0\}$. If σ and f are $m \geq 1$ times differentiable functions in x and $n_{(\sigma,f)}(q,x) < m$ for all $(q,x) \in \mathbf{Q}\times\mathbf{X}$, then* $\mathrm{Out}(q) = \{x \in \mathbf{X} :\ L_f^{n_{(\sigma,f)}(q,x)}\sigma(q,x) < 0\}$.

Notice that there are still a number of cases that have not been covered, for example cases where the invariant set is closed and its boundary is piecewise differentiable. To determine the set $\mathrm{Out}(q)$ for these cases, one may require more powerful tools from generalized gradients [5] and viability theory [3]. Systems where the boundaries have "corners" are likely to be considerably more involved to analyze (refer to the analysis and classification of discontinuous differential equations in [6]).

The conditions of the existence and uniqueness lemmas also mention the set of reachable states, but do not necessarily require one to compute it explicitly. For the purposes of Theorem 1 it suffices to show that the conditions of the lemmas hold in a set of states that contains $\mathrm{Reach}(H)$.

Definition 8 (Invariant Set). *A set of states $S \subseteq \mathbf{Q} \times \mathbf{X}$ is called invariant if* $\mathrm{Reach}(H) \subseteq S$.

Note that the class of invariant sets is closed under union and intersection. Trivially $\mathbf{Q} \times \mathbf{X}$ is an invariant set. More interesting sets can be shown to be invariant by an induction argument on the length of the system executions. For example, assume f is analytic in x and let:

$$S = \{(q,x) \in \mathbf{Q} \times \mathbf{X} \ : \ s(q,x) \geq 0\}$$

for some $s : \mathbf{Q} \times \mathbf{X} \to \mathbb{R}$ also analytic in s. Set

$$\mathrm{Out}_S(q) = \{x \in \mathbf{X} \ : \ (q,x) \in S \wedge L_f^{n(\sigma,f)(q,x)} \sigma(q,x) < 0\}$$

Notice that $\mathrm{Out}_S(q) \subseteq \{x \in \mathbf{X} \ : \ s(q,x) = 0\}$.

Proposition 6. *S is invariant if:*

- $\mathrm{Init} \subseteq S$*;*
- *for all $(q,q') \in E$, and $(q,x) \in S \cap \mathrm{Reach}(H)$, if $x \in G(q,q')$, then $\{q'\} \times R((q,q'),x) \subseteq S$; and*
- *for all $(q,x) \in \mathrm{Reach}(H)$, if $x \in \mathrm{Out}_S(q)$ then $x \in \mathrm{Out}(q)$.*

Proof. Consider an arbitrary execution $\chi = (\tau, q, x) \in \mathcal{H}_{(q_0,x_0)}$. We show that for all $t \in \tau$, $(q(t), x(t)) \in S$, which implies that all reachable states are contained in S.

By the first condition of the proposition, $\mathrm{Init} \subseteq S$, therefore $(q_0, x_0) \in S$. Consider the finite prefix of χ defined over $\widehat{\tau} = \{[\tau_i, \tau_i']\}_{i=0}^{M-1}[\tau_M, r]$ and assume $(q(t), x(t)) \in S$ for all $t \in \widehat{\tau}$.

First consider the case $r < \tau_M'$. Assume for the sake of contradiction that there exists $r' \in (r, \tau_M']$ such that $(q(r'), x(r')) \notin S$. Then, by continuity of the execution along continuous evolution and by the assumption that S is closed, there must exist $r'' \in (r, r')$ such that $x(r'') \in \mathrm{Out}_S(q(r''))$ (i.e. $s(q(r''), x(r'')) = 0$ and the vector field pointing "outside" S). Since $(q(r''), x(r''))$ is reachable (in particular, by the finite prefix of χ defined over $\{[\tau_i, \tau_i']\}_{i=0}^{M-1}[\tau_M, r'']$), by the third condition of the proposition, $x(r'') \in \mathrm{Out}(q(r''))$, which contradicts the assumption that $r'' < \tau_M'$. Therefore, if $(q(t), x(t)) \in S$ for all $t \in \widehat{\tau}$, then $(q(t), x(t)) \in S$ for all $t \in \{[\tau_i, \tau_i']\}_{i=0}^{M}$.

Finally, consider the case $r = \tau_M'$. Clearly, $(q(r), x(r))$ is reachable and $x(r) \in G(q(r), q(\tau_{M+1}))$. Therefore, by the second condition of the proposition, $(q(\tau_{M+1}), x(\tau_{M+1})) \in S$. Therefore, if $(q(t), x(t)) \in S$ for all $t \in \widehat{\tau}$, then $(q(t), x(t)) \in S$ for all $t \in \widehat{\tau}[\tau_{M+1}, \tau_{M+1}]$. The claim follows by induction. ☐

Notice that $\mathrm{Reach}(H)$ again appears in the statement of Proposition 6. This (and the fact that invariant sets are closed under intersection) allows us to build chains of invariant sets, $S_0 \supseteq S_1 \supseteq S_2 \supseteq \ldots$, starting with $S_0 = \mathbf{Q} \times \mathbf{X}$ and using the fact that S_i is invariant in the proof that S_{i+1} is invariant.

4. Zeno Hybrid Automata

In this section we discuss global existence of executions. The conditions introduced in Section 3 determine whether the executions of a hybrid automaton are unique and whether they can be extended locally in time. We still need to determine whether the executions can be extended over arbitrarily long time horizons. The Lipschitz assumption on the vector field f excludes the possibility of finite escape time due to continuous evolution. There is still, however, the possibility of Zeno executions, i.e., executions that take an infinite number of transitions in a finite amount of time, thus preventing time from diverging. Recall from Definition 4 that a Zeno execution is an infinite execution such that $\sum_{i=0}^{\infty}(\tau_i' - \tau_i) < \infty$.

Definition 9 (Zeno Hybrid Automaton). *A hybrid automaton H is called Zeno, if there exists $(q_0, x_0) \in \text{Init}$ such that all executions in $\mathcal{H}_{(q_0,x_0)}$ are Zeno.*

We illustrate the Zeno property through a number of examples. These are further analyzed in [7].

Example 1 (Non-Analytic System). *Consider the hybrid automaton, H, defined by:*

- $\mathbf{Q} = \{q_1, q_2\}$ *and* $\mathbf{X} = \mathbb{R}$;
- $\text{Init} = \mathbf{Q} \times \mathbf{X}$;
- $f(q, x) = 1$ *for all* $(q, x) \in \mathbf{Q} \times \mathbf{X}$;
- $I(q_1) = \{x \in \mathbf{X} : e^{-1/|x|}\sin(1/x) \leq 0\}$ *and* $I(q_2) = \{x \in \mathbf{X} : e^{-1/|x|}\sin(1/x) \geq 0\}$;
- $E = \{(q_1, q_2), (q_2, q_1)\}$;
- $G(q_1, q_2) = \{x \in \mathbf{X} : e^{-1/|x|}\sin(1/x) \geq 0\}$ *and* $G(q_2, q_1) = \{x \in \mathbf{X} : e^{-1/|x|}\sin(1/x) \leq 0\}$; *and*
- $R(q_1, q_2, x) = R(q_2, q_1, x) = x$.

The execution of H with initial state $(q_1, -1)$ exhibits an infinite number of discrete transitions by $\tau_\infty = 1$. The reason is that the (non-analytic) function $e^{-1/|x|}\sin(1/x)$ has an infinite number of zeros in the finite interval $(-1, 0)$.

Example 2 (Chattering System). *Consider the hybrid automaton, H, defined by:*

- $\mathbf{Q} = \{q_1, q_2\}$ *and* $\mathbf{X} = \mathbb{R}$;
- $\text{Init} = \mathbf{Q} \times \mathbf{X}$;
- $f(q_1, x) = -1$ *and* $f(q_2, x) = 1$;
- $I(q_1) = \{x \in \mathbf{X} : x \geq 0\}$ *and* $I(q_2) = \{x \in \mathbf{X} : x \leq 0\}$;
- $E = \{(q_1, q_2), (q_2, q_1)\}$;
- $G(q_1, q_2) = \{x \in \mathbf{X} : x \leq 0\}$ *and* $G(q_2, q_1) = \{x \in \mathbf{X} : x \geq 0\}$; *and*
- $R(q_1, q_2, x) = R(q_2, q_1, x) = x$.

It is easy to show that all executions of this automaton are Zeno. In particular, an execution starting in x_0 at $\tau_0 = 0$ reaches $x = 0$ in finite time $\tau_\infty = |x_0|$ and takes an infinite number of transitions from then on, without any time progress.

Example 3 (Water Tank System). *Consider the water tank system of* [2]*, shown in Figure 2. For $i = 1, 2$, let x_i denote the volume of water in Tank i, and $v_i > 0$ denote the (constant)*

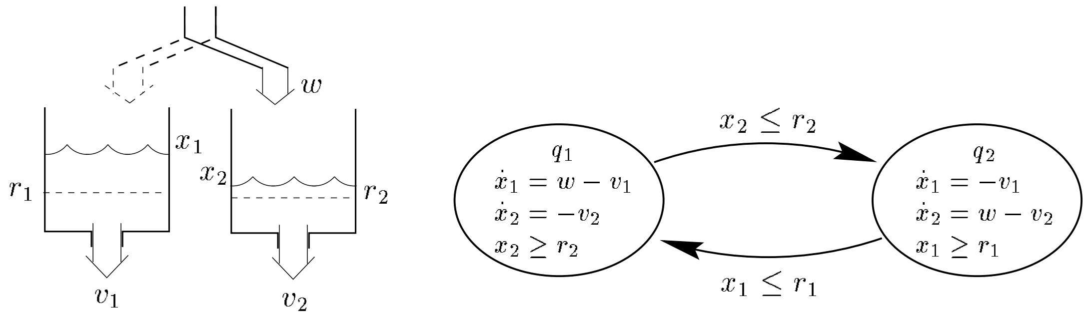

FIGURE 2. Water tank system and the corresponding hybrid automaton.

flow of water out of Tank i. Let w denote the constant flow of water into the system, dedicated exclusively to either Tank 1 or Tank 2 at each point in time. The control task is to keep the water volumes above r_1 and r_2, respectively (assuming that $x_1(0) > r_1$ and $x_2(0) > r_2$). This is to be achieved by a switched control strategy that switches the inflow to Tank 1 whenever $x_1 \le r_1$ and to Tank 2 whenever $x_2 \le r_2$. More formally, the water tank automaton is a hybrid automaton with

- $\mathbf{Q} = \{q_1, q_2\}$ *and* $\mathbf{X} = \mathbb{R}^2$;
- $\mathrm{Init} = \mathbf{Q} \times \{x \in \mathbf{X} : (x_1 > r_1) \wedge (x_2 > r_2)\}$, $r_1, r_2 > 0$;
- $f(q_1, x) = (w - v_1, -v_2)^T$ *and* $f(q_2, x) = (-v_1, w - v_2)^T$, $v_1, v_2, w > 0$;
- $I(q_1) = \{x \in \mathbf{X} : x_2 \ge r_2\}$ *and* $I(q_2) = \{x \in \mathbf{X} : x_1 \ge r_1\}$;
- $E = \{(q_1, q_2), (q_2, q_1)\}$;
- $G(q_1, q_2) = \{x \in \mathbf{X} : x_2 \le r_2\}$ *and* $G(q_2, q_1) = \{x \in \mathbf{X} : x_1 \le r_1\}$; *and*
- $R(q_1, q_2, x) = R(q_2, q_1, x) = x$.

The water tank automaton accepts a unique infinite execution for each initial state. It is also straightforward to show that if $\max\{v_1, v_2\} < w < v_1 + v_2$, *then the water tank automaton is Zeno. The Zeno time of the execution starting at* $\big(q_1, x(\tau_0)\big)$ *is*

$$\tau_\infty = \frac{x_1(\tau_0) + x_2(\tau_0) - r_1 - r_2}{v_1 + v_2 - w}.$$

Example 4 (Bouncing Ball System). *The bouncing ball automaton, shown in Figure 3, is a simplified model of an elastic ball that is bouncing and loosing a fraction of its energy with each bounce. Let x_1 denote the altitude of the ball and x_2 its vertical speed. More formally, the bouncing ball automaton is a hybrid automaton with*

- $\mathbf{Q} = \{q\}$ *and* $\mathbf{X} = \mathbb{R}^2$;
- $\mathrm{Init} = \{q\} \times \{x \in \mathbf{X} : x_1 \ge 0\}$;
- $f(q, x) = (x_2, -g)^T$ *with* $g > 0$;
- $I(q) = \{x \in \mathbf{X} : x_1 \ge 0\}$;
- $E = \{(q, q)\}$;
- $G(q, q) = \{x \in \mathbf{X} : [x_1 < 0] \vee [(x_1 = 0) \wedge (x_2 \le 0)]\}$; *and*
- $R(q, q, x) = (x_1, -cx_2)^T$ *with* $c \in (0, 1)$.

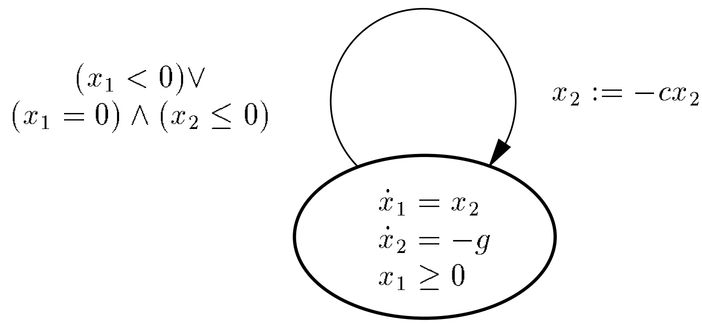

FIGURE 3. Hybrid automaton for the bouncing ball.

The bouncing ball automaton accepts a unique infinite execution for each initial state and the automaton is Zeno. The Zeno time of the execution starting at $\big(q, x(\tau_0)\big)$ *is*

$$\tau_\infty = \frac{x_2(\tau_0) + \sqrt{x_2(\tau_0)^2 + 2gx_1(\tau_0)}}{g} + \frac{2x_2(\tau_0)}{g(1-c)}.$$

The examples introduced above have some similarities, but shed light on different aspects of the Zeno phenomenon. The first example is more of a mathematical curiosity. The second example is an instance of the more conventional problem of differential equations with discontinuous right hand sides (see [6] for a thorough treatment). Note that in this example, there exists an interval $(\tau_\infty - \epsilon, \tau_\infty)$ with $\epsilon > 0$ that contains no discrete transitions. An infinite number of transitions takes place at τ_∞, whereas in the remaining examples there are infinitely many transitions on any interval $(\tau_\infty - \epsilon, \tau_\infty)$. In the water tank example, the state execution is continuous and approaches a limit as t approaches τ_∞. Finally, in the bouncing ball example the state execution is discontinuous (due to the non-trivial reset relation associated with the bounce) but still approaches a limit as t approaches τ_∞.

Regularization was proposed in [7] as a way of extending a Zeno execution beyond the Zeno time. It was, however, shown that in some cases different regularizations suggested different continuations. This is not a desirable property from a simulation point of view, since it shows that even if the simulation is reinitialized beyond the Zeno time the model does not give sufficient information to yield a unique solution. The water tank example was demonstrated to have such a non-unique extension: temporal regularization gave one extension, while spatial regularization gave another. This type of nonuniqueness is not very surprising. It has also been observed in the study of discontinuous differential equations and variable structure systems, where the sliding mode is known to be unique only in special cases [9, 10, 6].

5. CONCLUSIONS

Motivated by the lack of results on fundamental properties of executions of hybrid systems in the literature, some quite general results on local and global existence of executions were presented in this paper. Conditions were given for a hybrid automaton to be non-blocking and deterministic. When applying the conditions, it was shown that the structure of the invariant in each discrete state plays a crucial role. Comments were given on invariants with boundaries defined by analytic and smooth functions. Zenoness plays an important role in the global existence of executions. A Zeno execution is only defined up to a certain time instant, the Zeno time. This can be compared to finite escape time for ordinary differential equations. We illustrated various type of Zeno phenomena through examples. Particularly, the form of the invariants and the reset maps were shown to be important.

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References

[1] R Alur and D L Dill. Automata for modeling real-time systems. In *Proceedings of ICALP '90*, Lecture Notes in Computer Science 443, pages 322–335. Springer-Verlag, 1990.
[2] R Alur and T A Henzinger. Modularity for timed and hybrid systems. In A. Mazurkiewicz and J. Winkowski, editors, *CONCUR 97: Concurrency Theory*, Lecture Notes in Computer Science 1243, pages 74–88. Springer-Verlag, 1997.
[3] Jean-Pierre Aubin. *Viability Theory*. Birkhäuser, Boston, 1991.
[4] M Branicky, V Borkar, and S Mitter. A unified framework for hybrid control. In *IEEE Conference on Decision and Control*, pages 4228–4234, 1994.
[5] F H Clarke. *Optimization and nonsmooth analysis*. Classics in applied mathematics. SIAM, 1990.
[6] A F Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer Academic Publishers, 1988.
[7] K H Johansson, M Egerstedt, J Lygeros, and S Sastry. Regularization of Zeno hybrid automata. Submitted to Systems Control Letters, 1998.
[8] John Lygeros, Claire Tomlin, and Shankar Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, 1999. Accepted for publication, to appear in March 1999.
[9] Yu L Neimark. Remarks on the paper of A F Filippov. In *Proceedings of 1st IFAC World Congress*, Moscow, 1960.
[10] V I Utkin. Equations of the slipping regime in discontinuous systems. *Automation and Remote Control*, (2):211–219, 1972.
[11] V I Utkin. *Sliding Modes in Control Optimization*. Springer-Verlag, Berlin, 1992.
[12] A J van der Schaft and J M Schumacher. Complementarity modeling of hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):483–490, April 1998.

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