Distributed Coordination Control of Multiagent Systems While Preserving Connectedness

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***Abstract*—This paper addresses the connectedness issue in multiagent coordination, i.e., the problem of ensuring that a group of mobile agents stays connected while achieving some performance objective. In particular, we study the rendezvous and the formation control problems over dynamic interaction graphs, and by adding appropriate weights to the edges in the graphs, we guarantee that the graphs stay connected.**

***Index Terms*—Connected graphs, formation control, graph Laplacian, multiagent coordination.**

I. Introduction

THE HISTORY behind this work can be traced back to Reynolds' "boids" model [1], where each agent only reacts to its neighboring flock-mates following three *adhoc* protocols for autonomous agents, i.e., separation, alignment, and cohesion. A special case of the "boids" model was studied by Vicsek *et al.* [2], where all the agents move at the same constant speed and update their headings according to the nearest neighbor rule. Velocity cohesion and flocking behavior was observed in both cases, and a proof of convergence was provided by Jadbabaie *et al.* [3].

What makes the multiagent problem challenging is that the agents are subjected to limitations on the available information, which has made graph-based models useful and natural tools for encoding these limitations [4]–[12]. Among several important properties of such graph models, the graph Laplacian stands out, and it has been used for proving convergence and characterizing stability. Additional results that rely on such algebraic graph-theoretic tools for graph-based control of mobile agents involve edge-based control and Lyapunov functions over graphs [3], [13]–[15].

The agreement problem (or consensus problem) is concerned with finding decentralized strategies that achieve convergence to a common value. This problem arises in a number of applications such as swarming, schooling, flocking, or rendezvous. The agreement is typically achieved through a nearest-neighbor-like protocol

$$\dot{x}_i(t) = -k_i(t) \sum_{j \in nbhd_i(t)} \alpha_{ij}(t)(x_i(t) - x_j(t))$$

where x_i is the state vector of agent i, and $nbhd_i(t)$ denotes the neighborhood set (to be carefully defined later) of agent i at time t. Variations in this protocol consist mainly of different weight factors k_i and α_{ij}. For example, linear time-varying weights are used for α_{ij} in continuous time [6], [7] and discrete time [3], [9]. Nonlinear weights are proposed in [11] and [15]. In addition, a robust (in the sense of disturbance rejection) rendezvous algorithm is presented in [16].

The formation control problem has also been extensively studied. Generally speaking, there are two kinds of formation control approaches: the leader-follower approach and the leaderless approach. In the leader-follower approach, either an agent [17] or a virtual leader [18]–[21] is chosen as the leader, whose movement is constrained by a predefined trajectory. The remaining agents then track the leader, while obeying some coordination rules to keep the formation. In contrast, the other approach to formation control is the leaderless approach [22], [23]. Here, the controller is typically given by a mixture of formation-maintenance, obstacle-avoidance, and trajectory-following terms. Alternative approaches to this problem include, e.g., local navigation functions, and recent work along these lines can be found in [24] and [25].

In this paper, we will focus on providing solutions to the coordination problem that preserve connectedness in the presence of limited sensing and communication ranges. In particular, the rendezvous and formation control problems are investigated. It should be noted that these problems have already been solved if either connectedness is assumed [5], [7], [15], or connectedness is only required at distinct times [4], [3], [16], [26]–[28] in the sense that the agents sense their environment and then move in such a way that the network is connected at the sensing times, where the agents may be operating synchronously or asynchronously. In particular, the first solution to the connectedness preserving rendezvous problem was given by Ando *et al.* in [4]; a discrete-time control algorithm was proposed that evaluated and ensured connectivity, as well as other constraints, at each instant of (discrete time). An additional relevant contribution along these lines can be found in [29], where the connectivity-maintenance problem for *ad hoc* networks with discrete-time double-integrator dynamics is considered.

In this paper, we show how to make the graph stay connected for all times (thus, removing the separation of the movements into sensing and movement phases), and the outline of the paper is as follows: In Section II, we review some previous results and recall some basic notions from algebraic graph theory. In Section III, we show how to add weights in the static graph case in order to solve the rendezvous problem, followed by the dynamic case in Section IV. The connectedness preserving control law is extended to the formation control in Section V, followed by a collection of simulation results in Section VI.

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II. Background

The graph Laplacian can be thought of as an encoding of the discrete topology heat equation [30]–[32], where differential operators can be defined for functions over graphs. This interpretation of the graph Laplacian, defined as a mapping from graph nodes to graph nodes, tells us that the graph Laplacian defines a diffusion of information over the network in that a node value changes based on the values associated with its neighboring nodes. And, as already mentioned, pioneering work on consensus problems or agreement problems utilizing the graph Laplacian can be found in [6] and [11]–[13].

Since few mobile networks have a static network topology due to the movements of the individual nodes as well as to the temporal variations in the available communication channels, interest in networks with changing topologies has been growing rapidly. In [8] and [33], a dynamic extension of the static graph theory is proposed as a framework to address network problems with time-varying topologies. Ren and Beard [9] find that under a dynamically changing interaction topology, if the union of the interaction graphs across some time interval contains a spanning tree at a sufficient frequency as the system evolves, an information consensus is still achievable. An average consensus problem is solved for switching topology networks in [13], where a common Lyapunov function is obtained for directed balanced graphs, based on a so-called disagreement function. Moreover, similar Lyapunov function arguments were employed in [34], where the stability of coupled nonlinear oscillator networks was investigated. In this paper, we will draw inspiration from these results, and we first establish some notation and review some previous results.

N agents, whose positions $x_1, \ldots, x_N$ take on values in $\mathbb{R}^n$ are given. The problem that we are studying in this paper is that of limited information, decentralized control. As such, we are focusing our attention on interaction and high-level control strategies rather than on nonlinear vehicle models, and we assume that the dynamics of each individual agent is given by a single integrator

$$\dot{x}_i = u_i, \qquad i = 1, \ldots, N. \tag{1}$$

In order to establish what we mean by limited and decentralized control, we follow the standard procedure of associating an interaction graph with the available information flow in that the nodes correspond to agents, and edges to available interagent communication links. Such interaction graphs are, thus, reflective of the underlying network topology, and different graphs arise in different applications. Of particular importance to the development in this paper are the Δ-disk proximity graphs, of connectivity graphs, where edges are established between nodes v_i and v_j, if and only if the agents are within distance Δ of each other, i.e., when $|x_i - x_j| \leq \Delta$. It sould be noted at this point that such graphs are dynamic in nature, i.e., edges may appear or disappear as agents move in or out of sensing (or communication) distance of each other. Moreover, it is conceivable that agents are added or removed themselves, making not only the edge set but also the node set a dynamical structure. In this paper, we will not study this latter situation, and thus, restrict the node set to be static while allowing the edge-set to undergo dynamic changes.

Before we can study the dynamic situation, a few words should be said about the static case. In this situation, the agents have established communication links between the predefined agents, and these links are assumed to be available throughout the duration of the maneuver. In fact, by a static interaction graph (SIG) $\mathcal{G} = (V, E)$, we understand the graph where the nodes $V = \{v_1, \ldots, v_N\}$ are associated with the different agents, and the static edge set $E \subset V \times V$ is a set of unordered pairs of agents, with $(v_i, v_j) = (v_j, v_i) \in E$ if and only if a communication link exists between agents i and j. We will use the shorthand $V(\mathcal{G})$ and $E(\mathcal{G})$ to denote the edge and node sets associated with a graph $\mathcal{G}$.

Given an agent i, we will associate $\mathcal{N}_{\mathcal{G}}(i) = \{j | (v_i, v_j) \in E(\mathcal{G})\}$ with the neighborhood set to i, i.e., the set of agents adjacent to agent i. Using this terminology, what we understand by a limited-information time-invariant decentralized control law in (1) is that

$$u_i = \sum_{j \in \mathcal{N}_\sigma(i)} f(x_i - x_j) \tag{2}$$

where $\mathcal{N}_\sigma(i) \subseteq \mathcal{N}_{\mathcal{G}}(i)$. The symmetric indicator function $\sigma(i,j) = \sigma(j,i) \in \{0, 1\}$ determines whether or not the information available through edge (v_i, v_j) should be taken into account with

$$j \in \mathcal{N}_\sigma(i) \Leftrightarrow (v_i, v_j) \in E(\mathcal{G}) \wedge \sigma(i,j) = 1. \tag{3}$$

(Using the terminology in [35], just because two nodes are "neighbors" it does not follow that they are "friends.") Along the same lines, the decentralized control law $f(x_i - x_j)$ is assumed to be antisymmetric

$$f(x_i - x_j) = -f(x_j - x_i), \qquad \forall (v_i, v_j) \in E(\mathcal{G}). \tag{4}$$

A few remarks about these particular choices of control laws and indicator functions should be made. First of all, the fact that we only allow f to depend on the relative displacements between interacting agents is that this is, in general, the only type of information available to range-sensor-based information channels, where agent i simply measures the position of agent j relative to its current position. Secondly, we insist on having the agents be homogeneous in that the same control laws should govern the motion of all agents. This restriction is quite natural (and arguably necessary) when considering large-scale networks where it quickly becomes unmanageable to assign and keep track of individual control laws.

As a consequence of restricting the permissible control laws to those given in (2), we obain that the centroid of the system is static. This fact follows directly from the antisymmetry of f in (4), and will be elaborated further in the following sections.

The type of control terms presented in (2) have appeared repeatedly in the multiagent coordination community, and an intuitive linear control law for solving the rendezvous problem is given by

$$\begin{aligned} &\sigma(i,j) = 1, \\ &f(x_i - x_j) = -(x_i - x_j), \qquad \forall (v_i, v_j) \in E(\mathcal{G}) \end{aligned}$$

which gives that

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_{\mathcal{G}}(i)} (x_i - x_j), \qquad i = 1, \ldots, N. \tag{5}$$

Under the dynamics in (5), it has been shown that all the agents approach the same point asymptotically, provided that the SIG is connected. And, even though this is a well-established result (see, e.g., [7]), we will outline a proof in this paper in order to establish some needed notation and tools.

First, we need to associate an arbitrary orientation to the SIG, $\mathcal{G}$. An orientation is a declaration of direction to each edge $o : E(\mathcal{G}) \to \{-1, 1\}$such that if $(v_i, v_j) \in E(\mathcal{G})$, then $o(v_i, v_j) = -o(v_j, v_i)$. Using this orientation, we obtained the oriented graph $\mathcal{G}^o = (V, E, o)$.

Now, if the total number of edges is equal to M, and we associate an index with each edge such that $E(\mathcal{G}) = \{e_1, \ldots, e_M\}$, then the $N \times M$ incidence matrix of $\mathcal{G}^o$ is $\mathcal{I}(\mathcal{G}^o) = [\iota_{ij}]$, where

$$\iota_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is the head of } e_j \\ -1, & \text{if } v_i \text{ is the tail of } e_j \\ 0, & \text{otherwise} \end{cases}. \tag{6}$$

Through this incidence matrix, we can now define the graph Laplacian $\mathcal{L}(\mathcal{G}) \in \mathbb{R}^{N \times N}$ as

$$\mathcal{L}(\mathcal{G}) = \mathcal{I}(\mathcal{G}^o)\mathcal{I}(\mathcal{G}^o)^T \tag{7}$$

where we have removed the orientation dependence in the left-hand side of (7). The reason for this is that the Laplacian does not depend on the particular choice of orientation. In fact, one can easily define the Laplacian without any reference to orientation or incidence matrices, but we follow this definition to ease the notation in future sections.

The graph Laplacian has a number of well-studied properties (found, e.g., in [36]); the properties of importance to the developments in this paper are listed as follows.

1) $\mathcal{I}(\mathcal{G}^o)\mathcal{I}(\mathcal{G}^o)^T = \mathcal{I}(\mathcal{G}^{o'})\mathcal{I}(\mathcal{G}^{o'})^T$ for all orientation o, o', i.e., the Laplacian is orientation-independent.
2) $\mathcal{L}(\mathcal{G})$ is symmetric and positive semidefinite.
3) Let $\{\lambda_i\}_{i=1}^N$ be the ordered eigenvalues of $\mathcal{L}(\mathcal{G})$. Then $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$. Moreover, $\lambda_1 = 0$ and $\lambda_2, \ldots, \lambda_N > 0$ if $\mathcal{G}$ is connected.
4) If $\mathcal{G}$ is connected, then the set of eigenvectors $\nu_1, \ldots, \nu_N$ form an orthogonal basis in $\mathbb{R}^N$, and $\nu_1 = 1/\sqrt{N}\mathbf{1}$, where $\mathbf{1}$ denotes the vector with every entry equal to one. In other words, if $\mathcal{G}$ is connected then null$(\mathcal{G}) =$ span$\{\mathbf{1}\}$, where null$(\cdot)$ denotes the null space.

If we now let the n-dimensional position of agent i be given by $x_i = (x_{i,1}, \ldots, x_{i,n}), i = 1, \ldots, N$, and let $x = (x_1^T, \ldots, x_N^T)^T$, we can define the componentwise operator as

$$c(x, j) = (x_{1,j}, \ldots, x_{N,j})^T \in R^N, \qquad j = 1, \ldots, n.$$

Using this notation, together with the observation that (5) can be decoupled along each dimension, we can in fact rewrite (5) as

$$\frac{d}{dt} c(x, j) = -\mathcal{L}(\mathcal{G})c(x, j), \qquad j = 1, \ldots, n. \tag{8}$$

As pointed out in [7] and [36], if $\mathcal{G}$ is connected, then the eigenvector corresponding to the semisimple eigenvalue 0 is $\mathbf{1}$. This, together with the nonnegativity of $\mathcal{L}(\mathcal{G})$ and the fact that span$\{\mathbf{1}\}$ is $\mathcal{L}(\mathcal{G})$-invariant, is sufficient to show that $c(x, j)$ approaches $\mathrm{span}\{\mathbf{1}\}$ asymptotically.

This result, elegant in its simplicity, can in fact be extended to dynamic graphs as well. In fact, since $c(x, j)^T c(x, j)$ is a Lyapunov function to the system in (5), for any connected graph $\mathcal{G}$, the control law

$$\frac{d}{dt} c(x(t), j) = -\mathcal{L}(\mathcal{G}(t))c(x(t), j) \tag{9}$$

drives the system to span$\{\mathbf{1}\}$ asymptotically *as long as* $\mathcal{G}(t)$ *is connected for all* $t \ge 0$.

This well-known result is very promising, since dynamic network graphs are frequently occurring in which all real sensors and transmitters have finite range. This means that information exchange links may appear or be lost as the agents move around. In fact, if we focus our attention on Δ-disk proximity graphs, we get the dynamic interaction graph (DIG) $\mathcal{G}(t) = (V, E(t))$ where $(v_i, v_j) = (v_j, v_i) \in E(t)$, if and only if $|x_i(t) - x_j(t)| \le \Delta$.

By applying the control law in (5) to such DIGs, we get a system behavior that seemingly solves the rendezvous problem quite efficiently. However, the success of the control in (5) hinges on an assumption that it shares with most graph-based results (e.g., [3], [15]), i.e., on the connectedness assumption. Unfortunately, this property has to be assumed rather than proved, and in Fig. 1, an example is shown where connectedness is lost when using (9) to control a system whose network topology is a Δ-disk proximity DIG.

The remainder of this paper will show how this assumption can be overcome by modifying the control law in (5) in such a way that connectedness holds for all times, while ensuring that the control laws are still based solely on local information, in the sense of (2).

III. Weighted Graph-Based Feedback

In this section, we will restrict the interaction graphs to be static, i.e., we will only study the SIG-case in which the behavior of the multiagent system is defined through a fixed network topology. In particular, we will show how the introduction of nonlinear edge-weights can be used to establish certain invariance properties.

To arrive at the desired invariance properties, we will first investigate the decentralized control laws of the form

$$\begin{aligned} &\sigma(i, j) = 1, \\ &f(x_i - x_j) = -w(x_i - x_j)(x_i - x_j), \qquad \forall (v_i, v_j) \in E(\mathcal{G}) \end{aligned} \tag{10}$$

where $w : \mathbb{R}^n \to \mathbb{R}_+$ is a positive symmetric weight function that associates a strictly positive and bounded weight to each edge in the SIG, based solely on the displacement $x_i - x_j$. We will study in detail as to how to choose the weight w in order to maintain connectedness.

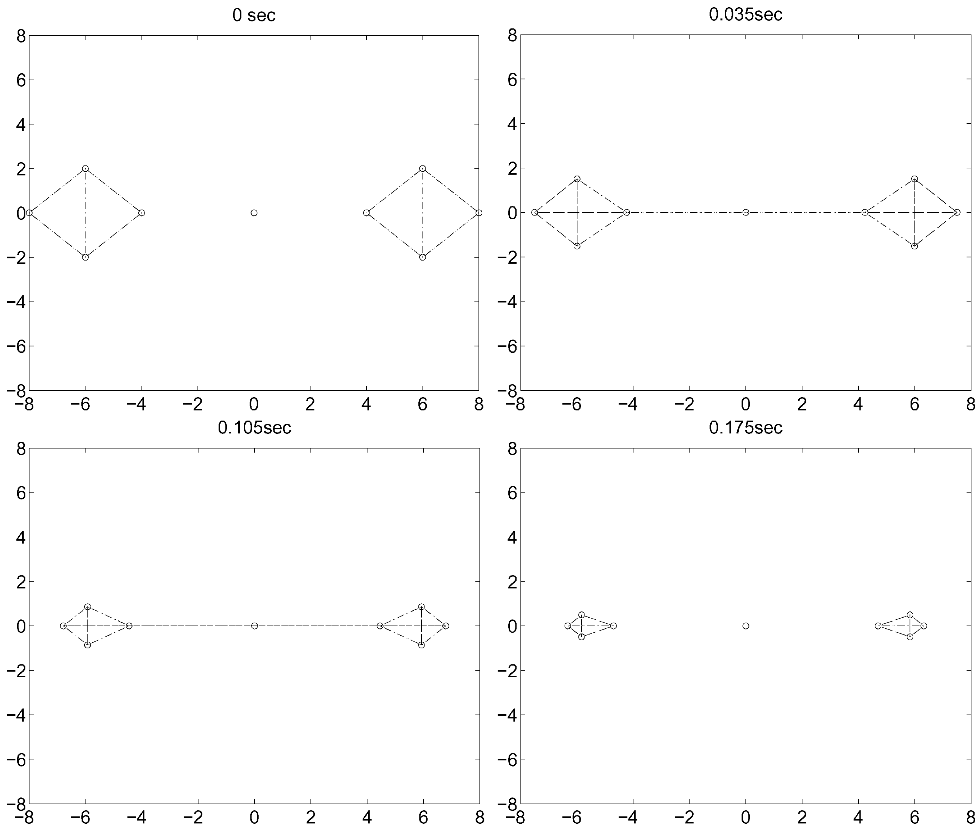

Fig. 1. Progression where connectedness is lost even though the initial graph is connected ($\Delta = 4.5$).

This choice of decentralized control law gives

$$\dot{x}_i = -\sum_{j\in\mathcal{N}_\mathcal{G}(i)} w(x_i - x_j)(x_i - x_j) \tag{11}$$

which can be rewritten as

$$\frac{d}{dt}c(x,j) = -\mathcal{I}^o\mathcal{W}(x)\mathcal{I}^{oT}c(x,j), \qquad j = 1,\ldots,n \tag{12}$$

where $\mathcal{W}(x) = \text{diag}(w_1(x),\ldots,w_M(x)) \in \mathbb{R}^{M\times M}$, where, as before, $M = |E(\mathcal{G})|$ is the total number of edges, and where we have associated a label in $\{1,\ldots,M\}$ with each of the edges.

We can then define the state-dependent weighted graph Laplacian as

$$\mathcal{L}_\mathcal{W}(x) = \mathcal{I}^o\mathcal{W}(x)\mathcal{I}^{oT} \tag{13}$$

where, as before, $\mathcal{W}(x) \in \mathbb{R}^{M\times M}$ is a diagonal matrix with each element corresponding to a strictly positive edge weight. It is moreover straightforward to establish that as long as the graph is connected, $\mathcal{L}_\mathcal{W}(x)$ is still positive semidefinite, with only one zero eigenvalue corresponding to the null-space span$\{\mathbf{1}\}$.

What we would like to show is that, given a critical distance δ, together with the appropriate edge-weights, the edge-lengths never go beyond δ if they start out being less than $\delta - \epsilon$, for some arbitrarily small $\epsilon \in (0,\delta)$. For this, we need to establish some additional notation, and, given an edge $(v_i, v_j) \in E(\mathcal{G})$, we let $\ell_{ij}(x)$ denote the edge vector between the agents i and j, i.e., $\ell_{ij}(x) = x_i - x_j$.

We, moreover, define the ϵ-interior of a δ-constrained realization of an SIG, $\mathcal{G}$ as

$$\mathcal{D}^\epsilon_{\mathcal{G},\delta} = \{x \in \mathbb{R}^{nN} \mid \|\ell_{ij}\| \le (\delta - \epsilon), \qquad \forall (v_i, v_j) \in E(\mathcal{G})\}.$$

An edge-tension function V_{ij}, can then be defined as

$$\mathcal{V}_{ij}(\delta, x) = \begin{cases} \frac{\|\ell_{ij}(x)\|^2}{\delta - \|\ell_{ij}(x)\|}, & \text{if } (v_i, v_j) \in E(\mathcal{G}) \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

with

$$\frac{\partial \mathcal{V}_{ij}(\delta, x)}{\partial x_i} = \begin{cases} \frac{2\delta - \|\ell_{ij}(x)\|}{(\delta - \|\ell_{ij}(x)\|)^2}(x_i - x_j), & \text{if} (v_i, v_j) \in E(\mathcal{G}) \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

it is to be noted that this edge-tension function (as well as its derivatives) is infinite when $\|\ell_{ij}(x)\| = \delta$ for some i, j, and, as such, it may seem like an odd choice. However, as we will see, we will actually be able to prevent the energy to reach infinity, and instead we will study its behavior on a compact set on which it is continuously differentiable.

The total tension energy of $\mathcal{G}$ can now be defined as

$$\mathcal{V}(\delta, x) = \frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N \mathcal{V}_{ij}(\delta, x). \tag{16}$$

Lemma 3.1: Given an initial position $x_0 \in \mathcal{D}^\epsilon_{\mathcal{G},\delta}$, for a given $\epsilon \in (0,\delta)$. If the SIG $\mathcal{G}$ is connected, then the set $\Omega(\delta, x_0) := \{x | \mathcal{V}(\delta, x) \le \mathcal{V}(\delta, x_0)\}$ is an invariant set to the system under the control law

$$\dot{x}_i = -\sum_{j\in\mathcal{N}_\mathcal{G}(i)} \frac{2\delta - \|\ell_{ij}(x)\|}{(\delta - \|\ell_{ij}(x)\|)^2}(x_i - x_j). \tag{17}$$

Proof: We first note that the control law in (17) can be rewritten as

$$\dot{x}_i = -\sum_{j\in\mathcal{N}_{\mathcal{G}}(i)} \frac{\partial \mathcal{V}_{ij}(\delta,x)}{\partial x_i} = -\frac{\partial \mathcal{V}(\delta,x)}{\partial x_i} = -\nabla_{x_i}\mathcal{V}(\delta,x).$$

This expression may be ill-defined, since it is conceivable that the edge-lengths approach δ and what will be shown is that this will not happen. In fact, assume that at time τ we have $x(\tau) \in \mathcal{D}^{\epsilon'}_{\mathcal{G},\delta}$ for some $\epsilon' > 0$. Then, the time derivative of $\mathcal{V}(\delta, x(\tau))$ is

$$\begin{aligned}\dot{\mathcal{V}}(\delta,x(\tau)) &= \nabla_x \mathcal{V}(\delta,x(\tau))^T \dot{x}(\tau)\\ &= -\sum_{i=1}^{N} \dot{x}_i(\tau)^T \dot{x}_i(\tau)\\ &= -\sum_{j=1}^{n} c(x(\tau),j)^T \mathcal{L}_{\mathcal{W}}(\delta,x(\tau))^2 c(x(\tau),j) \quad (18)\end{aligned}$$

where $\mathcal{L}_{\mathcal{W}}(\delta,x)$ is given in (13), with weight positive definite (on $\Omega(\delta,x_0)$) matrix $\mathcal{W}(\delta,x)$

$$\begin{aligned}\mathcal{W}(\delta,x) &= \mathrm{diag}(w_k(\delta,x)), \qquad k = 1,2,\ldots,M\\ w_k(\delta,x) &= \frac{2\delta - \|\ell_k(x)\|}{(\delta - \|\ell_k(x)\|)^2} \quad (19)\end{aligned}$$

where we have arranged the edges such that subscript k corresponds to edge k. We will use this notation interchangeably with w_{ij} and ℓ_{ij}, whenever it is clear from the context.

We note that for any ϵ' bounded away from 0 from below and δ from above, and for any $x \in \mathcal{D}^{\epsilon'}_{\mathcal{G},\delta}$, the time derivative of the total tension energy is well defined. Moreover, for any such x, $\mathcal{V}(\delta,x)$ is nonnegative and $\dot{\mathcal{V}}(\delta,x)$ is nonpositive (since $\mathcal{L}_{\mathcal{W}}(\delta,x)$ is positive semidefinite for all $x \in \Omega(\delta,x_0)$). Hence, in order to establish the invariance of $\Omega(\delta,x_0)$, all that needs to be shown is that as $\mathcal{V}$ decreases (or at lest does not increase), no edge-distances will tend to δ. In fact, since $\mathcal{D}^{\epsilon}_{\mathcal{G},\delta} \subset \mathcal{D}^{\epsilon'}_{\mathcal{G},\delta}$ if $\epsilon > \epsilon'$, we would have established the invariance of $\Omega(\delta,x_0)$ if we could find an $\epsilon' > 0$ such that whenever the system starts from $x_0 \in \mathcal{D}^{\epsilon}_{\mathcal{G},\delta}$, we can ensure that it never leaves the superset $\mathcal{D}^{\epsilon'}_{\mathcal{G},\delta}$.

Let

$$\hat{\mathcal{V}}_\epsilon := \max_{x\in\mathcal{D}^{\epsilon}_{\mathcal{G},\delta}} \mathcal{V}(\delta,x).$$

This maximum always exists, and is obtained when all edges are at the maximal allowed distance

$$\hat{\mathcal{V}}_\epsilon = \frac{M(\delta-\epsilon)^2}{\epsilon}$$

which is a monotonously decreasing function in ϵ over $(0,\delta)$.

We can bound the maximal edge distance that can generate this total tension energy, and the maximal edge-length $\hat{\ell}_\epsilon \geq \delta - \epsilon$ is one where the entire total energy is contributed from that one single edge. In other words, all other edges have length 0, and the maximal edge length satisfies

$$\hat{\mathcal{V}}_\epsilon = \frac{\hat{\ell}_\epsilon^2}{\delta - \hat{\ell}_\epsilon}$$

that is

$$\frac{M(\delta-\epsilon)^2}{\epsilon} = \frac{\ell_\epsilon^2}{\delta - \ell_\epsilon}$$

which implies that

$$\hat{\ell}_\epsilon \leq \delta - \frac{\epsilon}{M} < \delta.$$

Hence, ℓ_ϵ is bounded away from above from δ, and it is moreover bounded from above by a strictly decreasing function in ϵ on $(0,\delta)$. Hence, as $\mathcal{V}$ decreases (or at least is nonincreasing), no edge-distances will tend to δ, which completes the proof.

The invariance of $\Omega(\delta,x_0)$ now leads us to the main SIG theorem.

Theorem 3.2: Given a connected SIG $\mathcal{G}$ with initial condition $x_0 \in \mathcal{D}^{\epsilon}_{\mathcal{G},\delta}$, for a given $\epsilon > 0$. Then, the the multiagent system under the control law in (17) asymptotically converges to the static centroid $\bar{x}(x_0)$.

Proof: The proof of convergence is based on LaSalle's invariance theorem. Let $\mathcal{D}^{\epsilon}_{\mathcal{G},\delta}$ and $\Omega(\delta,x_0)$ be defined as before. From Lemma 3.1 , we know that $\Omega(\delta,x_0)$ is positively invariant with respect to the dynamics in (17). We also note that span$\{\mathbf{1}\}$ is $\mathcal{L}_{\mathcal{W}}(\delta,x)$-invariant for all $x \in \Omega(\delta,x_0)$. Hence, due to the fact that $\dot{\mathcal{V}}(\delta,x) \leq 0$, with equality only when $c(x(t),j) \in$ span$\{\mathbf{1}\}, \forall j \in \{1,\ldots,n\}$, convergence to $\mathrm{span}\{\mathbf{1}\}$ follows.

Next, we need to show that the agents converge to the centroid. The centroid is given by

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

and the component-wise dynamics of the centroid is

$$\frac{d}{dt}\overline{c(x,j)} = \frac{1}{N}\mathbf{1}^T \frac{d}{dt}c(x,j) = -\frac{1}{N}\mathbf{1}^T \mathcal{L}_{\mathcal{W}}(\delta,x)c(x,j).$$

Now, since $\mathbf{1}^T\mathcal{L}_{\mathcal{W}}(\delta,x) = (\mathcal{L}_{\mathcal{W}}(\delta,x)\mathbf{1})^T = 0, \forall x \in \Omega(x_0)$, we directly have $\dot{\bar{x}} = 0$, i.e., the centroid is static, determined entirely by the initial condition x_0. As such, we can denote the centroid by $\bar{x}(x_0)$. This is in fact just a special case of the observation that the centroid is static under any control law in (2).

Now, let $\bar{\xi} \in \mathbb{R}^N$ be any point on $\mathrm{span}\{\mathbf{1}\}$ (i.e., $\bar{\xi} = (\xi,\ldots,\xi)^T$ for some $\xi \in \mathbb{R}$) that is consistent with a static centroid. This implies that

$$\overline{c(x,j)} = \frac{1}{N}\sum_{i=1}^{N}\xi = \xi$$

and hence, ξ has to be equal to the centroid itself. As a consequence, if $x_i, i = 1,\ldots,n$, converged anywhere other than the centroid, we would have a contradiction, and the proof follows. ■

It is to be noted that the construction we have described corresponds to adding nonlinear state-dependent weights to the edges in the graph. One could conceivably also add weights to the nodes as well. Unless these weights were all equal, they would violate the general assumption in (2), but for the sake of completeness, we briefly discuss this situation in the next few paragraphs.

A node weight would be encoded in the dynamics of the system through the weight matrix $D(x)$ as

$$\frac{dc(x,j)}{dt} = -D(x)\mathcal{L}_{\mathcal{W}}(x)c(x,j), \qquad j = 1,\ldots,n.$$

As long as $D(x)$ is diagonal and positive definite for all x, (with the diagonal elements bounded away from 0), the null-space remains $\text{null}(D(x)\mathcal{L}_{\mathcal{W}}(x)) = \text{span}\{\mathbf{1}\}, \forall x \in \mathbb{R}^{nN}$, and the controller still drives the system to span$\{\mathbf{1}\}$. However, it is straightforward to show that in this case, the positions $x_i \in \mathbb{R}^n, i = 1,\ldots,N$ approach the same static point $\bar{x}_D(x_0) \in \mathbb{R}^n$, given by

$$\bar{x}_D(x_0) = \frac{1}{\text{tr}(D^{-1}(x_0))} \sum_{i=1}^{N} (d_i^{-1}(x_0))x_{0,i} \tag{20}$$

where $x_{0,i} \in \mathbb{R}^n, i = 1,\ldots,N$ is the initial location of agent i, $d_i(x)$ is the ith diagonal element of $D(x)$, and $\text{tr}(D(x))$ denotes the trace of matrix $D(x)$.

That concludes this section where an SIG was assumed. We will show ion Section IV that a similar strategy can be employed even if the graph is allowed to change as the agents move around in the environment.

IV. Dynamic Graphs

As already pointed out, during a maneuver, the interaction graph $\mathcal{G}$ may change as the different agents move in and out of each others sensory ranges. In this section, we focus on whether or not an argument, similar to the previous stability result, can be constructed for the case when $(v_i, v_j) \in E(\mathcal{G})$ if and only if $\|x_i - x_j\| \le \Delta$.

In fact, we intend to reuse the tension energy from the previous section, with the particular choice of $\delta = \Delta$. However, since in (19)

$$\lim_{\|\ell_k\|\uparrow\Delta} w_k(\Delta, \|\ell_k\|) = \infty$$

we can not directly let the interagent tension energy affect the dynamics as soon as two agents form edges in between them, i.e., as they move within distance Δ of each other. The reason for this is that we cannot allow infinite tension energies in the definition of the control laws. To overcome this problem, we chose to introduce a certain degree of hysteresis into the system through the indicator function σ. In particular, we let $\sigma(i,j)$ be given by the state machine in Fig. 2

To elaborate further on the state machine in Fig. 2, we let the total tension energy be affected by an edge (v_i, v_j) that was previously not contributing to the total energy when $\|\ell_{ij}\| \le (\Delta - \epsilon)$, where $\epsilon > 0$ is the predefined *switching threshold*. Once the edge is allowed to contribute to the total tension energy, it will keep doing so for all subsequent times. We note that the switching threshold can take on any arbitrary value in $(0, \Delta)$. The interpretation is simply that a smaller ϵ-value corresponds to a faster inclusion of the inter-robot information into the decentralized control law.

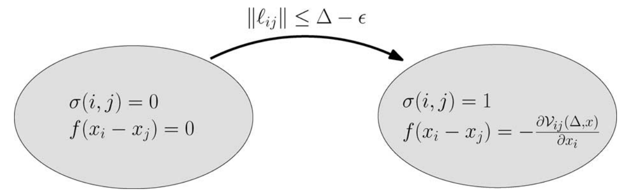

Fig. 2. Hysteresis protocol for adding interagent tension functions to the total tension function only when agents get within a distance $\Delta - \epsilon$ of each other, rather than when they first encouter each other at a distance Δ.

In other words, for the Δ-disk proximity DIGs, we propose to let

$$\begin{aligned} \sigma(i,j)[t^+] &= \begin{cases} 0 & \text{if } \sigma(i,j)[t^-] = 0 \wedge \|\ell_{ij}\| > \Delta - \epsilon \\ 1 & \text{otherwise} \end{cases} \\ f(x_i - x_j) &= \begin{cases} 0 & \text{if } \sigma(i,j) = 0 \\ -\frac{\partial \mathcal{V}_{ij}(\Delta,x)}{\partial x_i} & \text{otherwise} \end{cases} \end{aligned} \tag{21}$$

where we have used the notation $\sigma(i,j)[t^+]$ and $\sigma(i,j)[t^-]$ to denote the value of $\sigma(i,j)$ before and after the state transition in Fig. 2. It is worth noting that if $\sigma(i,j)[t_0] = 1$ for some t_0, then $\sigma(i,j)[t_0] = 1$ for all $t < t_0$.

Before we can state the rendezvous theorem for dynamic graphs, we also need to introduce the subgraph $\mathcal{G}_\sigma \subset \mathcal{G}$, induced by the indicator function σ

$$\mathcal{G}_\sigma = (V(\mathcal{G}), E(\mathcal{G}_\sigma))$$

where

$$E(\mathcal{G}_\sigma) = \{(v_i, v_j) \in E(\mathcal{G}) |\ \sigma(i,j) = 1\}.$$

Theorem 4.1: Given an initial position $x_0 \in D^{\epsilon}_{g^0,\Delta}$, where $\epsilon > 0$ is the switching threshold in (21), and $\mathcal{G}^0$ is the initial Δ-disk DIG. Assume that the graph $\mathcal{G}^0_\sigma$ is connected, where $\mathcal{G}^0_\sigma$ is the graph induced by the initial indicator function value. Then, by using the control law

$$u_i = -\sum_{j \in \mathcal{N}_\sigma(i)} \frac{\partial \mathcal{V}_{ij}(\Delta, x)}{\partial x_i} \tag{22}$$

where $\sigma(i,j)$ is given in (21), the group of agents asymptotically converges to span$\{\mathbf{1}\}$.

Proof: Since, from Lemma 3.1, we know that no edges in $\mathcal{G}^0_\sigma$ will be lost, only two possibilities remain, i.e., that no new edges will be added to the graph during the maneuver, or new edges will in fact be added. If no edges are added, then we know from Theorem 3.2 that the system will converge to span$\{\mathbf{1}\}$ asymptotically. However, the only graph consistent with $x \in \text{span}\{\mathbf{1}\}$ is $\mathcal{G}^0_\sigma = K_N$ (the complete graph over N nodes), and hence, no new edges will be added only if the initial, indicator-induced graph is complete. If it is not complete, at least one new edge will be added. But, since $\mathcal{G}^0_\sigma$ is an arbitrary connected graph, and connectivity can never be lost by adding new edges, we obtain that new edges will be added until the indicator-induced graph is complete, at which point the system converges asymptotically to span$\{\mathbf{1}\}$. ■

V. Formation Control

In the previous sections, it was shown that the connectedness-preserving control method solves the rendezvous problem. In this section, we will follow the same methodology to solve the distributed formation control problem.

A. Graph Formation

By formation control, we understand the problem of driving the collection of mobile agents to some translationally invariant target geometry, i.e., the control objective is to drive the collection of autonomous mobile agents to a specific configuration such that their relative positions satisfy some desired topological and physical constraints. These constraints can be described by a connected edge-labeled graph $\mathcal{G}_d = (V, E_d, d)$, where the subscript d denotes "desired." Here, E_d encodes the desired robot interconnections, i.e., whether or not a desired interagent distance is specified between two agents or not, and the edge-labels $d : E_d \to \mathbb{R}^n$ defines the desired relative interagent displacements, with $\|d_{ij}\| > \Delta$ for all i, j such that $(v_i, v_j) \in E_d$. In other words, what we would like is that $x_i - x_j \to d_{ij}, \forall i, j$ such that $(v_i, v_j) \in E_d$.

One may notice that it is possible that the assignment of general edge-labels to a DIG may result in conflicting constraints. This is addressed in [37] as the realization problem of connectivity graphs. We will not discuss this problem here and simply assume that the constraints are compatible. Another issue concerning the target formation is that of rigidity, which has been discussed in [10], [38], and [39], and will not be discussed further in this paper. Instead, we assume that the target formation is chosen in such a way that rigidity is obtained if, in fact, this is a desired characteristic of the target formation graph-topology.

Given a desired formation, the goal of the distributed formation control is to find a feedback law such that:

- F1 dynamic interaction graph $\mathcal{G}(t)$ converges to a graph that is a supergraph of the desired graph $\mathcal{G}_d$ (without labels) in finite time. In other words, what we want is that $E_d \subset E(t)$ for all $t \geq T$, for some finite $T \geq 0$;
- F2 $\|\ell_{ij}(t)\| = \|x_i(t) - x_j(t)\|$ converges asymptotically to $\|d_{ij}\|$ for all i, j such that $(v_i, v_j) \in E_d$; and
- F3 feedback law utilizes only local information.

Here, "F" stands for "formation" and it will be established that these properties hold for a particular choice of decentralized control law.

B. Graph-Based Formation Control

Analogous to the treatment of the rendezvous problem, we first propose a solution to the formation control problem, and then show that this solution does, in fact, preserve connectedness as well as guarantee convergence in the sense of F1 and F2 above. The solution will be based on a variation of the previously derived rendezvous controller. In fact, assume that we have established a set of arbitrary targets $\tau_i \in \mathbb{R}^n$ that are consistent with the desired interagent displacement

$$d_{ij} = \tau_i - \tau_j, \, \forall i, j$$

such that

$$(v_i, v_j) \in E_d.$$

We can then define the displacement from τ_i at time t as

$$y_i(t) = x_i(t) - \tau_i.$$

We let $\ell_{ij}(t) = x_i(t) - x_j(t)$ and $\lambda_{ij}(t) = y_i(t) - y_j(t)$, implying that

$$\lambda_{ij}(t) = \ell_{ij}(t) - d_{ij}.$$

Now, under the assumption that $\mathcal{G}_d$ is a connected *spanning graph* of the initial interaction graph, $\mathcal{G}$ i.e., $V(\mathcal{G}_d) = V(\mathcal{G})$ and $E_d \subseteq E(\mathcal{G})$, we propose the following control law:

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_{\mathcal{G}_d}(i)} \frac{2(\Delta - \|d_{ij}\|) - \|\ell_{ij} - d_{ij}\|}{(\Delta - \|d_{ij}\| - \|\ell_{ij} - d_{ij}\|)^2}(x_i - x_j - d_{ij}). \tag{23}$$

The reason why this seemingly odd choice makes sense is because we can again use the edge-tension function $\mathcal{V}$ to describe this control law. In particular, using the following parameters in the edge-tension function

$$\mathcal{V}_{ij}(\Delta - \|d_{ij}\|, y) = \begin{cases} \frac{\|\lambda_{ij}\|^2}{\Delta - \|d_{ij}\| - \|\lambda_{ij}\|}, & \text{if } (v_i, v_j) \in E_d \\ 0, & \text{otherwise} \end{cases} \tag{24}$$

we obtain the decentralized control law

$$\sigma(i,j) = 1 \quad f(x_i - x_j) = -\frac{\partial \mathcal{V}_{ij}(\Delta - \|d_{ij}\|, y)}{\partial y_i}, \quad \forall (v_i, v_j) \in E_d. \tag{25}$$

Note that this control law, in fact, implies something stronger than just measurements of displacement. Instead, the agents must also share a common coordinate system. However, they do not need to know their exact location in this coordinate system.

Now, along each individual dimension, the dynamics in (25) becomes

$$\frac{dc(x,j)}{dt} = \frac{dc(y,j)}{dt} = -\mathcal{L}_{\mathcal{W}}(\Delta - \|d\|, y) c(y, j), \qquad j = 1, 2, \ldots n$$

where $\mathcal{L}_W(\Delta - \|d\|, y)$ is the graph Laplacian associated with $\mathcal{G}_d$, weighted by $\mathcal{W}(\Delta - \|d\|, y)$, and where we have used the convention that the term $\Delta - \|d\|$ should be interpreted as

$$\mathcal{W}(\Delta - \|d\|, y) = \mathrm{diag}(w_k(\Delta - \|d_k\|, y)), \quad k \in \{1, |E_d|\}$$
$$w_k(\Delta, -\|d_k\|, y) = \frac{2(\Delta - \|d_k\|) - \|\lambda_k\|}{(\Delta - \|d_k\| - \|\lambda_k\|)^2}). \tag{26}$$

Here again, the index k runs over the edge set E_d. This construction allows us to study the evolution of y_i, rather than $x_i, i = 1, \ldots, N$, and we formalize this in the following lemma for static interaction graphs.

Corollary 5.1: Let the total tension energy function be

$$\mathcal{V}(\delta - \|d\|, y) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathcal{V}_{ij}(\Delta - \|d_{ij}\|, y). \tag{27}$$

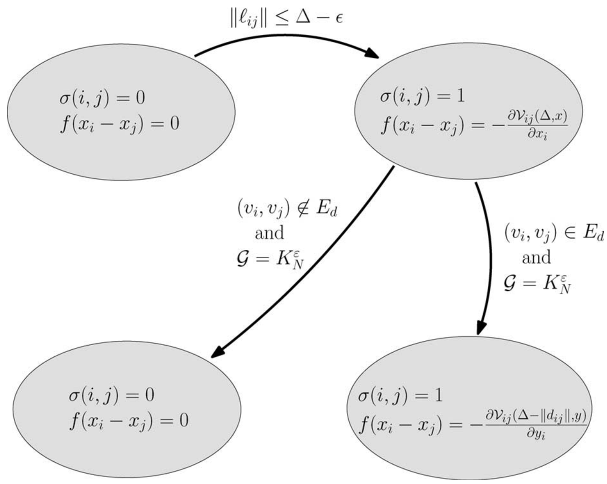

Fig. 3. State machine describing how the system undergoes transitions from rendezvous (collection of the agents to a tight complete graph) to formation control.

Given $y_0 \in \mathcal{D}^{\epsilon}_{\mathcal{G}_d, \Delta - \|d\|}$, with $\mathcal{G}_d$ being a connected spanning graph, then the set $\Omega(\Delta - \|d\|, y_0) := \{y | \mathcal{V}(\delta - \|d\|, y) \leq \mathcal{V}_0\}$, where $\mathcal{V}_0$ denotes the initial value of the total tension energy function, is an invariant set under the control law in (23), under the assumption that the interaction graph is static.

Proof: By the proposed control law in (23),

$$\begin{aligned}\dot{y}_i &= -\sum_{j \in \mathcal{N}_{\mathcal{G}_d}(i)} \frac{\partial \mathcal{V}_{ij}(\Delta - \|d_{ij}\|, y)}{\partial y_i} \\ &= -\frac{\partial \mathcal{V}(\Delta - \|d\|, y)}{\partial y_i} = -\nabla_{y_i} \mathcal{V}(\delta - \|d\|, y).\end{aligned}$$

The nonpositivity of $\dot{\mathcal{V}}$ now follows the same argument as in (18) in the proof of Lemma 3.1. Moreover, for each initial $y_0 \in \mathcal{D}^{\epsilon}_{\mathcal{G}_d, \Delta - \|d\|}$, the corresponding maximal total tension energy induces a maximal possible edge length. Following the same line of reasoning as in the proof of Lemma 3.1, the invariance of $\Omega(\Delta - \|d\|, y_0)$ thus follows.

It is to be noted that Lemma 5.1 says that if we could use $\mathcal{G}_d$ as an SIG, $\Omega(\Delta - \|d\|, y_0)$ is an invariant set. In fact, it is straightforward to show that if $\mathcal{G}_d$ is a spanning graph to the initial proximity Δ-disk DIG, then it remains a spanning graph to $\mathcal{G}(x(t)) \forall t \geq 0$.

Corollary 5.2: Given an initial condition x_0 such that $y_0 = (x_0 - \tau_0) \in \mathcal{D}^{\epsilon}_{\mathcal{G}_d, \Delta - \|d\|}$, with $\mathcal{G}_d$ being a connected spanning graph of $\mathcal{G}(x_0)$, the group of autonomous mobile agents adopting the decentralized control law in (23) can guarantee that $\|x_i(t) - x_j(t)\| = \|\mathcal{L}_{ij}(t)\| < \Delta, \forall t > 0$ and $(v_i, v_j) \in E_d$.

Proof: Given two agents i, j that are adjacent in $\mathcal{G}_d$, and suppose that $\|\lambda_{ij}\| = \|y_i - y_j\|$ approaches $\Delta - \|d_{ij}\|$. Since $\mathcal{V}_{ij} \geq 0, \forall i, j$ and $t > 0$, as well as

$$\lim_{\|\lambda_{ij}\| \uparrow (\Delta - \|d_{ij}\|)} \mathcal{V}_{ij} = \infty$$

this would imply that $\mathcal{V} \to \infty$, which contradicts Lemma 5.1. As a consequence, $\|\lambda_{ij}\|$ is bounded away from $\Delta - \|d_{ij}\|$. This means that

$$\|\ell_{ij}\| = \|\lambda_{ij} + d_{ij}\| \leq \|\lambda_{ij}\| + \|d_{ij}\| < \Delta - \|d_{ij}\| + \|d_{ij}\| = \Delta$$

and hence, edges in E_d are never lost under the control law in (2.3). In other words, $\|\mathcal{L}_{ij}(t)\| < \Delta, \forall t \geq 0$, which in turn implies that connectedness is preserved. ■

We have, thus, established that if $\mathcal{G}_d$ is a spanning graph of $\mathcal{G}(x_0)$, then it remains a spanning graph of $\mathcal{G}(x(t)), \forall t > 0$ (under certain assumptions on x_0), even if $\mathcal{G}(x(t))$ is given by a Δ-disk DIG. And, since the control law in (23) only takes pairwise interactions in E_d into account, we can view this dynamic situation as a static situation, with the SIG being given by $\mathcal{G}_d$. It still remains to be shown that the system in fact converges in the sense of the formation control properties F1, F2, and F3, as previously defined. That F3 (decentralized control) is satisfied follows trivially from the definition of the control law in (23). Moreover, we have already established that F1 (finite time convergence to the appropriate graph) holds trivially as long as it holds initially, and what remains to be shown here is that we can drive the system in finite time to a configuration in which F1 holds, after which Lemma 5.2 applies. Moreover, we need to establish that the inter-robot displacements (defined for edges in E_d) converge asymptotically to the desired relative displacements (F3), which is the topic of the next theorem.

Theorem 5.3: Under the same assumptions as in Lemma 5.2 $\|\ell_{ij}(t)\| = \|x_i(t)\| - \|x_j(t)\|$ converges asymptotically to $\|d_{ij}\|$ for all i, j such that $(v_i, v_j) \in E_d$.

Proof: Based on the observation that $\mathcal{G}_d$ remains a spanning graph to the DIG, together with the observation that

$$\frac{dc(y,j)}{dt} = -\mathcal{L}_W(\Delta - \|d\|, y) c(y, j), \quad j = 1, 2, \ldots n,$$

Theorem 3.2 ensures that $c(y, j)$ will converge to span$\{\mathbf{1}\}, \forall j \in \{1, \ldots, n\}$. This implies that all displacements must be the same, i.e., that $y_i = \zeta, \forall i \in \{1, \ldots, N\}$ for some constant $\zeta \in \mathbb{R}^n$. But, this simply means that the system converges asymptotically to a fixed translation away from the target points $\tau_i, i = 1, \ldots, N$, as

$$\lim_{t\to\infty} y_i(t) = \lim_{t\to\infty} \big(x_i(t) - \tau_i\big) = \zeta, \qquad i = 1, \ldots, N$$

which, in turn, implies that

$$\begin{aligned}\lim_{t\to\infty} \ell_{ij}(t) &= \lim_{t\to\infty} \big(\underline{(}x_i(t) - x_j(t)\big) \\ &= \lim_{t\to\infty} (y_i(t) + \tau_i - y_j(t) - \tau_j) \\ &= \zeta + \tau_i - \zeta - \tau_j = d_{ij} \qquad \forall i, j\end{aligned}$$

such that$(v_i, v_j) \in E_d$, which completes the proof. ■

C. Hybrid, Rendezvous-To-Formation Control Strategies

The last property that must be established is that it is possible to satisfy F1, i.e., that the initial Δ-disk proximity DIG does, in fact, converge to a graph that has $\mathcal{G}_d$ as a spanning graph in finite time. If this was achieved, then Theorem 5.3 would be applicable and F2 (asymptotic convergence to the correct interagent displacements) would follow. To achieve this, we propose to use the rendezvous control law developed in Section

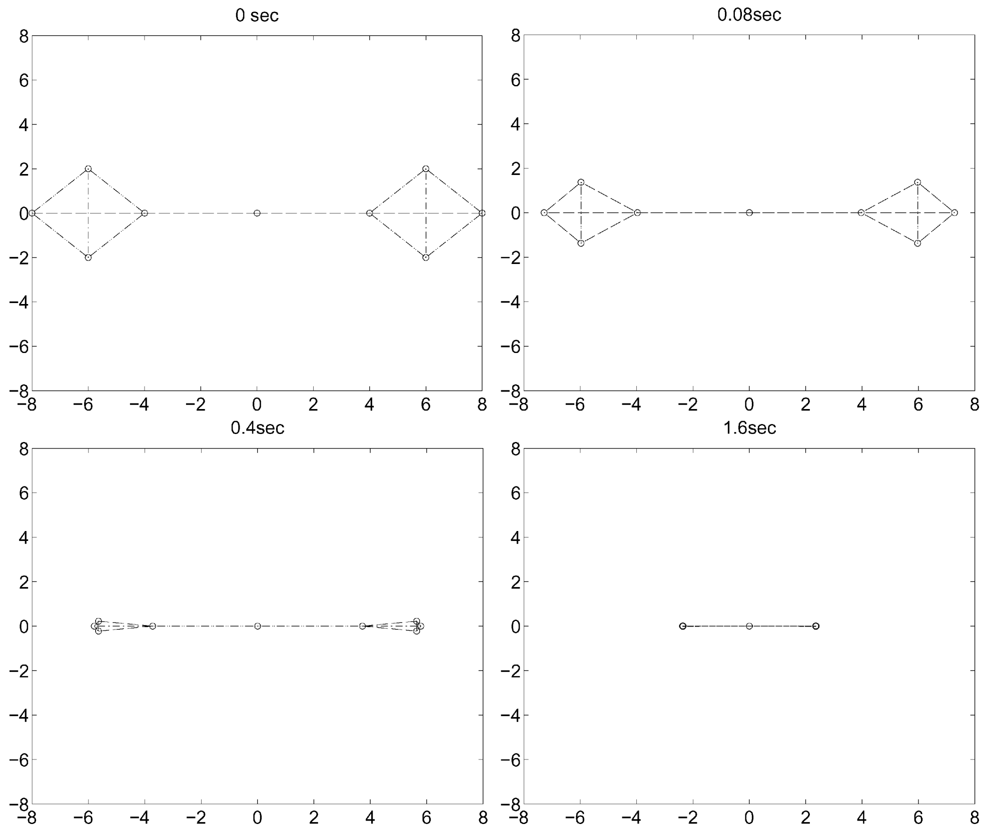

Fig. 4. Progression where connectedness is maintained during the rendezvous maneuver, with $D = I$. Positions of the agents and the edges in the DIG as a function of time are depicted.

V-B for gathering all agents into a complete graph, of which any desired graph is a subgraph. Moreover, we need to achieve this in such a manner that the assumptions in Theorem 5.3 are satisfied.

Let K_N denote the complete graph over N agents. Moreover, we will use K_N^ε to denote the situation in which the ε-disk proximity graph is in fact a complete graph, i.e., a DIG that is a complete graph in which no interagent distances are greater than ε. This notation is slightly incorrect in that graphs are inherently combinatorial objects, while interagent distances are geometric, and, to be more precise, we will use the notation $\mathcal{G} = K_N^\varepsilon$ to denote the fact that

$$\begin{cases} \mathcal{G} = K_N \\ \|\ell_{ij}\| \leq \varepsilon, & \forall (i,j), i \neq j. \end{cases}$$

The reason for this construction is that, in order for Theorem 5.3 to be applicable, the initial condition has to satisfy $y_0 = (x_0 - \tau_0) \in \mathcal{D}^\epsilon_{\mathcal{G}_d, \Delta - \|d\|}$, which is ensured by making ε small enough. Moreover, since the rendezvous controller in (22) asymptotically achieves rendezvous, it will consequently drive the system to K_N^ε in finite time, for all ε bounded below by 0 and above by Δ.

After K_N^ε is achieved, the controller switches to the controller in (23), as depicted in Fig. 3. However, this hybrid control strategy is only viable if the condition that $\mathcal{G} = K_N^\varepsilon$ is locally verifiable in the sense that the agents can decide for themselves that a synchronous mode switch is triggered [35]. In fact, if an agent has $N-1$ neighbors, i.e., degree $N-1$, all of which are within a distance $\varepsilon/2$, this implies that the maximal separation between two of those neighbors is ε. (This occurs when the agents are polar opposites on an n-sphere of radius $\varepsilon/2$.) Hence, when one agent detects this condition, it will trigger a switching signal (involving a 1-bit broadcast communication to all its neighbors), and the transition in Fig. 3 occurs. It is to be noted that first, this argument hinges on the fact that the total number of agents N is known to each agent. This could arguably be a concern for graphs with time-varying node sets. Second, transition in Fig. 3 might actually not occur at the exact moment when $\mathcal{G}$ becomes K_N^ε, but rather at a later point. Regardless of the kind of transition, we know that this transition will, in fact, occur in finite time in such a way that the initial condition assumptions of Theorem 5.3 are satisfied.

VI. Examples

In this section, we will show some simulation results that illustrate the proposed coordination control strategies for different problems. In all of these simulations, the cutoff distance for interagent sensing and communication is $\Delta = 4$, and the switching threshold dictating when to add edges is $\epsilon = 0.05$, i.e., a new edge is added only when the corresponding interagent distance is $\Delta - \epsilon$.

The first two simulations show the rendezvous behavior under slightly different control laws. In fact, Fig. 4 shows

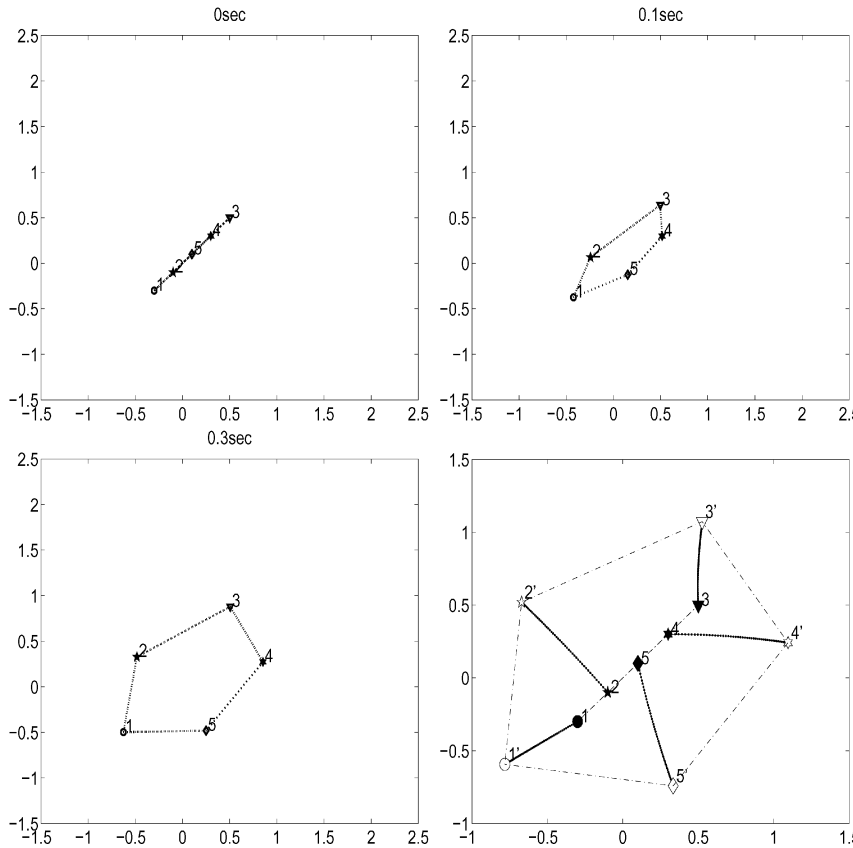

Fig. 5. Evolution of the formation process. The last graph shows the trajectory of the formation process from t = 0 s to t = 0.5 s, starting from 1, 2, 3, 4, 5 and ending at 1', 2', 3', 4', 5'.

the movement of the collection of agents under the weighted Laplacian control law given in (22), under exactly the same initial position as in Fig. 1. What is different here is, as could be expected, that no links are broken.

The third simulation highlights the proposed formation control strategy, and is implemented based on the formation control law in (23). In the simulation, five agents starting from a straight line are to form a pentagonal formation, with $\mathcal{G}_d = C_5$ (the cyclic graph with 5 nodes), and the desired interagent distances being $\|d_{ij}\| = 3.2$ for all $(v_i, v_j) \in E_d$. The movement of the group during the first 0.5 s and the trajectories corresponding to the same time period are shown in Fig. 5.

VII. Conclusion

In this paper, a collection of graph-based nonlinear feedback control laws are studied for distributed multiagent systems. The nonlinear feedback laws are based on weighted graph Laplacians and they are proved to be able to solve the rendezvous and formation-control problems while ensuring connectedness.

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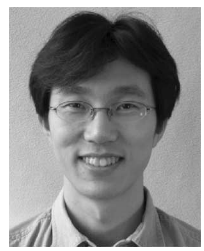

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