Control with Delayed and Limited Information: A First Look

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Abstract

In a communication constrained environment, feedback information and feedback delay must necessarily be taken into consideration. In this paper we report some preliminary investigations on the relation between the information capacity of the feedback channel and the closed loop performance of a discrete time system.

1 Introduction

During the last decade, a flurry of activity has been going on in the intersection of controls and communications. (For a representative sample, see for example [1, 2, 4, 5, 10, 14, 15].) The emergence of sensor and actuator networks, teleoperated control systems, and embedded software systems has brought to the fore the need for a systematic study of control law design in the presence of communication constraints, such as bitrate (or bandwidth) limitations [2, 4], attention constraints in multi-process applications [6, 7, 8, 9], and channel induced delays [11].

In this paper we focus on two particular problems concerning control with delayed and limited information. In particular, we study the discrete time LTI-system

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where the state $x \in \mathbb{R}^n$, the control input $u \in \mathbb{R}^m$, and where w is an additive disturbance. We will, throughout most of the paper, furthermore assume that the control input takes on values in 2^β disjoint sets, i.e. that the communication channel restricts us to using β bits for coding the control signal. In the absence of noise or other signal errors, this problem has been studied when the control signal take on values in a countable or "large enough" set [3, 4], or when moving quantization levels can be exploited [2]. However, what is novel in this paper is that the system, evolving in discrete time (which is a very natural assumption in most communications applications), will be controlled using a small number of bits. In fact, we will show how the performance of the system very much depends on the size of β, and trade-offs between bitrates and system performance will be outlined.

Furthermore, in this paper we also investigate the situation when the control signal is based on delayed information, i.e. when the controller is of the form

$$u_k = f(x_{k-N}), \tag{2}$$

for some given delay time N.

The outline of this paper is as follows: In Section 2 we develop a framework for limited information control for scalar systems, similar to [5], and we introduce the notions of *trap regions* and *basins of attraction* for characterizing the proposed control strategy. In Section 3 we study the effect of delays on such systems. The problem of feedback control using delayed information has been extensively studied in the continuous time domain, but is poorly understood when the system dynamics evolve in discrete time. We will derive a collection of results that characterize the worst-case performance of such systems. An auxiliary result, of independent interest, is a perturbation approach to analyze the delay dependent stability of linear and a class of nonlinear discrete systems, which is presented in Section 5. These results are not to be thought of as a unified theory of delayed, discrete-time systems, but rather as a first look, i.e. as initial findings, in an emergent area of research.

2 Control with Limited Information

Given the scalar discrete-time system

$$x_{k+1} = f(x_k, u_k), \tag{3}$$

where the feedback from sensors to actuators is performed over a communication channel with a capacity of β bits per step. Then only 2^β different input levels can be used in the control algorithm. Suppose now that the inputs u_i (the subscript i refers to a particular control level in the set of 2^β control values) can be designed so that the function $f(\cdot, u_i)$ maps a closed interval $S_i \subset \mathbb{R}$ to the interval $S \subset \mathbb{R}$, where S satisfies

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the partitioning conditions

$$S = \bigcup_{i=1}^{2^\beta} S_i, \ \ S_i \cap S_j = \emptyset, i \neq j. \qquad (4)$$

We let $\chi_{S_i}(\cdot)$ denote the characteristic function for the set S_i. Then, the choice of

$$u(x) = \sum_{i=1}^{2^\beta} u_i \chi_{S_i}(x) = u_j \quad \text{if} \quad x \in S_j, \qquad (5)$$

clearly, by construction, leaves the future states in S, and we say that S is an *invariant set*, or a *trapping region* for the nonlinear system in (3).[1]

2.1 Linear Scalar Case

Let the trapping region be $S = [-1, 1]$ for the linear system

$$x_{i+1} = ax_i + bu_i, \qquad (6)$$

with $a > 1$. ($a < 1$ is not very interesting since $u = 0$ would globally stabilize the system.) For S to be a trapping region, we require a total number of $I(a)$ bits for describing the control law. (We use I to denote "information".) That number can be computed as follows: Let S_0 be the region from which the state remains in the interval $[-1, 1]$ after one step under the control action $u = 0$. This limits x to the interval $[-q, q]$, where $q = 1/a$.

The control level for x just larger than $x = q$ is $u = -2/b$, and at $x = 2q$, if $2q$ is smaller than 1, the control switches to $u = -4/b$, and so on. Hence, the number of control levels required in the interval $[-1, 1]$, using this methodology, is

$$2\lceil\frac{1-q}{2q}\rceil + 1 = 2\lceil\frac{a-1}{2}\rceil + 1, \qquad (7)$$

where $\lceil\cdot\rceil$ is the ceiling operator.

We thus require a total of

$$I(a) = \log_2(2\lceil\frac{a-1}{2}\rceil + 1) \qquad (8)$$

bits in order to make $[-1, 1]$ a trapping region, as shown in Figure 1.[2] We have thus established a computationally feasible framework relating the bitrates to the size of the trapping region.

Suppose now that we, in addition, want a domain of attraction to the trapping region $S = [-1, 1]$ that extends from $-A$ to A, and that in this domain, the system behaves *on average* as a linear system with behavior

$$\xi_{i+1} = \alpha\xi_i. \qquad (9)$$

[1]This definition can of course be extended to higher order systems, by replacing intervals by regions in state space.

[2]If we instead let $S = [-\gamma, \gamma]$, then (8) would change accordingly.

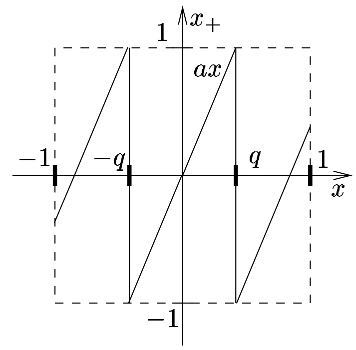

Figure 1: The trap region $[-1, 1]$ and the corresponding closed-loop system dynamics.

In order to accomplish this, take the closed loop characteristic that corresponds to an increasing amplitude saw-tooth. Let the switching to a new control level be performed when the value of $a\xi + bu$ reaches $\delta\xi$ for some δ, to be determined later. Since, for $x = 1$, we have the system response

$$f_{cl}(1) = a - 2b\lceil\frac{a-1}{2}\rceil, \qquad (10)$$

where the subscript *cl* denotes "closed-loop", the first switching occurs at $Q > 1$, where

$$\delta Q - a + 2b\lceil\frac{a-1}{2}\rceil = a(Q-1) \qquad (11)$$

if $a - 2b\lceil\frac{a-1}{2}\rceil < \delta$, and

$$1 - a + 2b\lceil\frac{a-1}{2}\rceil = a(Q-1) \qquad (12)$$

otherwise. The next switch occurs at Q_1, satisfying $\delta Q_1 = -\delta Q + a(Q_1 - Q)$, i.e. for

$$Q_1 = \frac{(a+\delta)}{(a-\delta)}Q. \qquad (13)$$

By repeating this argument we see that subsequent changes occur at Q_k, where

$$Q_k = \left(\frac{a+\delta}{a-\delta}\right)^k Q. \qquad (14)$$

Now, by setting

$$Q_{k-1} < A < Q_k \qquad (15)$$

we get that the additional number of control levels in the interval $[-A, A]$ is $2k$. Note that

$$2k < \frac{\log\left(\frac{A}{Q}\right)}{\log\frac{a+\delta}{a-\delta}}, \qquad (16)$$

which in turn requires a total of

$$b_A = \log_2\log\left(\frac{A}{Q}\right) - \log_2\log\frac{a+\delta}{a-\delta} \qquad (17)$$

bits.

Finally, observe that on average in the domain of attraction, the state decreases as follows: For $Q_j < x < Q_{j+1}$, the closed loop system is

$$x_+ = -\delta Q_j + a(x - Q_j). \tag{18}$$

Hence, the average next state can be approximated by assuming that x is uniformly distributed in $[Q_j, Q_{j+1}]$:

$$\begin{aligned} \mathbf{E}x_+ &= \frac{1}{Q_{j+1} - Q_j} \int_{Q_j}^{Q_{j+1}} (-\delta Q_j + a(x - Q_j))\, dx \\ &= \frac{\delta^2}{a - \delta} Q_j. \end{aligned} \tag{19}$$

A linear system with slope α would have the average next state for the interval $[Q_{j+1} - Q_j]$ equal to

$$\begin{aligned} \mathbf{E}x_+ &= \frac{1}{Q_{j+1} - Q_j} \int_{Q_j}^{Q_{j+1}} (\alpha x)\, dx \\ &= \frac{\alpha a}{a - \delta} Q_j. \end{aligned} \tag{20}$$

Equating both expressions gives the requisite slope

$$\delta^2 = \alpha a, \tag{21}$$

and thus, the requisite slope is the geometric mean between the open-loop and desired closed-loop behavior.[3] We therefore postulate that the limit performance in the domain of attraction is given by

$$\alpha_{\min} = \frac{1}{a}. \tag{22}$$

At any rate, in order to obtain the performance $\alpha \in [1/a, 1)$, over the region $[-A, -Q] \cup [Q, A]$, one needs approximately a total of

$$b_A(\alpha) = \log_2 \log \left(\frac{A}{Q}\right) - \log_2 \log \frac{1 + \sqrt{\frac{\alpha}{a}}}{1 - \sqrt{\frac{\alpha}{a}}} \tag{23}$$

additional bits. Denoting by ζ the *dynamic stabilization*

$$\zeta = \frac{\alpha}{a} \tag{24}$$

then

$$b_A(\zeta) = \log_2 \log \left(\frac{A}{Q}\right) - \log_2 \log \frac{1 + \sqrt{\zeta}}{1 - \sqrt{\zeta}}. \tag{25}$$

The information required (total number of bits) for a domain of attraction $[-A, A]$ with dynamic stabilization ζ, and standard trapping region $[-1, 1]$ is then

$$I(a, A, \zeta) = \log_2 \left(2\lceil \frac{a-1}{2} \rceil + 1\right) + b_A(\zeta), \tag{26}$$

as shown in Figure 2.

[3] This is only a rough approximation, and it makes no sense if $\alpha < 1/a$, since then the closed-loop behavior would be expanding over some subintervals.

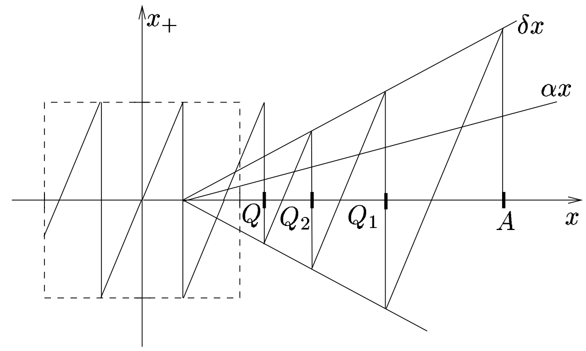

Figure 2: The trap region together with the basin of attraction.

2.2 Additive and Channel Noise

So far, the entire analysis has been focused on the deterministic case. There are several ways in which uncertainty or indeterminism may creep into the system. First of all, the system parameters a and b could be imprecisely known. This is the general form of parameter uncertainty and will not be considered here. However, as a second source of indeterminism, additive noise may enter the system through the actuators. Usually one attributes Gaussianity to such additive noise.[4] In the problem at hand, it is easy to see that with a Gaussian input, the state must eventually escape to infinity. This limits the usefulness of adhering to such a model. For instance, one may not be concerned with a long wait until escape. This motivates then the use of noise models with *bounded* amplitude. If indeed the additive noise term, w_k, is bounded by $W < 1$, then in the above analysis of the trapping region, one should simply replace the value q by $(1 - W)/a$ for positive a. It has the obvious effect of keeping [-1,1] as the trapping region, but the number of bits required may be increased accordingly. Likewise in the domain of attraction, one must now limit the value of δ in order to obtain the same dynamic stabilization factor. In turn this leads to an increase of the required number of bits.

Finally, a third source of error stems from the limited capacity of the channel in the feedback link. Assuming that binary bits are transmitted, occasionally a bit error may occur. This implies that an incorrect control value will be implemented by the actuator, perhaps forcing the resulting state into the unstable region. In order to minimize this effect, one should employ a Gray-encoding scheme, where adjacent codewords are mapped into adjacent control values. One could further guard for these larger control values by a reduction of the dynamic stabilization as mentioned above.

[4] This is partially due to the physically prevailing character of the system - in turn explained by the Central Limit Theorem - but also because Gaussianity is well suited to linear problems.

3 Delay Effects

In discrete time, systems with delayed feedback remain finite dimensional, unlike the continuous time case. Unfortunately, in the precense of delays, the time domain approach of Section 2 breaks down. However, in this section we outline an alternative route for investigating the effects of delays in the feedback controller for discrete time systems . We start off by a discussion concerning the worst-case performance that can be expected when we have a bounded disturbance.

3.1 Worst-Case Design

Consider the single input discrete-time system

$$x_{k+1} = Ax_k + b\nu_k, \tag{27}$$

where the input $\{\nu_k\}$ satisfies the bound $|\nu_k| \leq B$. What bound can be given on the scalar output $y_k = cx_k$? This bound can be found by solving for the maximum of $|y_N|$, for some N. Since

$$x_N = A^N x_0 + \sum_{i=1}^{N} A^{N-i} b\nu_{i-1} \tag{28}$$

the Hamiltonian for the problem is

$$H = \sum_{i=0}^{N-1} \lambda_{i+1}[Ax_i + b\nu_i], \tag{29}$$

from which the Euler-Lagrange equation can be derived as

$$\lambda_i = A'\lambda_{i+1}, \tag{30}$$

with the final condition

$$\lambda_N = c'. \tag{31}$$

The optimality condition gives

$$\max_{\nu} H \Rightarrow \nu_i = B \operatorname{sgn} \lambda_i' b, \tag{32}$$

giving

$$y_N = cA^N x_0 + B \sum_{i=1}^{N} |cA^{N-i}b|. \tag{33}$$

Clearly, for stable A and an absolutely summable pulse response $\{h_i\} = \{cA^i b\}$, we obtain

$$|y_N| \leq B\|h\|_1, \tag{34}$$

where $\|h\|_1$ is the ℓ_1 norm of the pulse response.

Consider now the system controlled with delay and limited information, i.e., let $u_i = f(x_{i-N})$:

$$\begin{aligned} x_{i+1} &= Ax_i - bf(x_{i-N}) \qquad (35)\\ &= Ax_i - bkx_{i-N} + b[kx_{i-N} - f(x_{i-N})]. \end{aligned}$$

This result can be interpreted as a finite precision emulation of a gain k, which we shall call the *virtual* gain. The term between the square brackets is the approximation error. Also we note that the feedback suffers a delay of N steps.

State augmentation of this system to an nN dimensional system ultimately yields a system of the form (27). Proper choice of the virtual state feedback gain k makes the undriven system stable. The nominal system is stable if the characteristic polynomial $\det(z^N - z^{N-1}A + bk)$ has all its roots inside the unit circle. Such an analysis for the case of scalar A and b was given in [10]. Also, the equivalent driving term $\nu_i = kx_{i-N} - f(x_{i-N})$ is bounded by B_k for an appropriate choice of the piecewise constant function $f(\cdot)$. It is intuitively clear that the more bits are available, the better the choice can be made to give a smaller bound B_k. On the other hand, the pulse response, and thus $\|h\|_1$, will also depend on the gain k used and on the delay time N.

4 Single Delays

Instead of focusing on the worst case performance, which was the case in the previous section, one could instead study what effect a single delay has on the system. Hence, let the system be controlled with a delay of one step:

$$x_{n+1} = Ax_n + bkx_{n-1}. \tag{36}$$

Then, the closed loop system is stable if the matrix

$$\begin{bmatrix} A & bk \\ I & 0 \end{bmatrix} \tag{37}$$

has all its eigenvalues within the unit circle. Note that

$$\begin{bmatrix} A & bk \\ I & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mu \begin{bmatrix} x \\ y \end{bmatrix} \tag{38}$$

implies that

$$(A + \frac{1}{\mu} bk)x = \mu x. \tag{39}$$

Hence, if we want all eigenvalues to be equal to λ for the closed loop without delay, and design the gain k accordingly, e.g. by the Bass-Gura formula, then, with delay, the gain to be implemented would be $k\lambda$. But the matrix in (37) has $2n$ eigenvalues. So what are the additional n eigenvalues? Of these $n-1$ are equal to zero. Indeed, let z be an arbitrary vector perpendicular to the gain k constructed above. Then

$$\begin{bmatrix} A & bk \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix} = 0 \begin{bmatrix} 0 \\ z \end{bmatrix}. \tag{40}$$

The last eigenvalue follows from the trace property:

$$\operatorname{Tr} A = n\lambda + \mu_{2n}. \tag{41}$$

Hence, choose the multiple closed loop eigenvalue λ such that it is inside the unit circle and such that

$$|\mu_{2n}| = |\mathrm{Tr}\, A - n\lambda| < 1. \tag{42}$$

Hence a necessary condition for *this* design method to work is

$$n - 1 < \mathrm{TrA} < n + 1.^5 \tag{43}$$

5 Lyapunov-Krasovskii Approach

A Lyapunov-Krasovskii approach has been proposed for obtaining robust stability conditions for delay systems, independent of the delay. This methodology was used successfully in the discrete time case in [13]. Here we adapt this result to the problem at hand. Consider the generic form

$$x_{k+1} = Ax_k + B\nu_k + w_k, \tag{44}$$

where $\{w_k\}$ is a disturbance, modeled as a stationary white noise sequence with zero mean and covariance matrix W. We assume that the delay is deterministic and fixed, although it may be unknown. The feedback is given by

$$\nu_k = f(x_{k-N}), \tag{45}$$

where N is the delay.

Now, take a Lyapunov functional of the form

$$V(\{x(i)\}_{i=k-N}^{k}) = x_k' P x_k + \sum_{i=k-N}^{k-1} f(x_i)' Q f(x_i). \tag{46}$$

Denote for simplicity the functional, evaluated along solutions of the closed loop system, by

$$V(\{x(i)\}_{i=k-N}^{k}) \stackrel{\mathrm{def}}{=} V_k. \tag{47}$$

Straightforward computation shows that

$$\begin{aligned} V_{k+1} - V_k &= x_k'[A'PA - P]x_k \\ &+ f(x_k)'Qf(x_k) \\ &+ f(x_{k-N})'[B'PB - Q]f(x_{k-N}) \\ &+ 2w_k'PAx_k - 2w_k'PBf(x_{k-N}) \\ &- 2f(x_{k-N})'B'PAx_k. \end{aligned} \tag{48}$$

Taking expectations with respect to w_k, and noting the independence of past and present states x_k with the noise w_k, gives

$$\begin{aligned} \mathbf{E}(V_{k+1} - V_k) &= \mathrm{Tr}\, PW \\ &+ x_k'[A'PA - P]x_k + f(x_k)'Qf(x_k) \\ &+ f(x_{k-N})'[B'PB - Q]f(x_{k-N}) \\ &- 2f(x_{k-N})'B'PAx_k. \end{aligned} \tag{49}$$

Let, for some positive definite matrix $S = S'$,

$$Q = B'PB + S, \tag{50}$$

[5] Of course this is, in general, not necessary for stabilizability.

then the expected decrease in V_k is *less* than

$$\begin{aligned} &x'[A'PA - P + A'PB'S^{-1}B'PA]x + \\ &+ f(x)'[B'PB + S]f(x) + \mathrm{Tr}\, PW, \end{aligned} \tag{51}$$

and this bound is *independent* of the transmission delay N. The second and third terms are nonnegative, and hence a necessary condition for V_k to be decreasing is that there exist positive definite matrices P and S such that

$$A'PA - P + A'PB'S^{-1}B'PA < 0. \tag{52}$$

This may also be written as

$$A'P[P^{-1} + B'S^{-1}B']PA - P \leq -rI, \tag{53}$$

for some positive scalar r.

With a finite number, β, of feedback bits, the value set $f(\mathbb{R})$ is finite. Let $\{\mathcal{P}_i\}$; $i = 1, \ldots, s^\beta$ be a partitioning of the state space $\mathbb{R}^n$ into 2^β disjoint sets. With this partitioning one thus encodes the state information using β bits.

Assume that $f(x) = \nu_i$ if $x \in \mathcal{P}_i$. Let furthermore

$$\nu_i'[B'PB + S]\nu_i = U_i. \tag{54}$$

Clearly, if $x \in \mathcal{P}_i$, then

$$\begin{aligned} &x'[A'PA - P + A'PB'S^{-1}B'PA]x \leq \\ &\leq -r\|x\|^2 + U_i + \mathrm{Tr}\, PW. \end{aligned} \tag{55}$$

If we define the region $\mathcal{P}_i$ by the set

$$\mathcal{P}_i = \left\{ x \mid \|x\|^2 \geq \frac{U_i + \mathrm{Tr}\, PW}{r} \right\}, \tag{56}$$

then V decreases on average as long as x is in $\mathcal{P}_i$.

However, the condition in Equation (52) is restrictive as it can only be satisfied if A is already Schur-Cohn stable. This is easily shown by contradiction: Since it involves stability for *all* delays, letting $N \to \infty$ means that the open loop system must be stable. A way out of this impasse is by rewriting the deterministic system with unstable A as

$$\begin{aligned} x_{i+1} &= (A - C)x_i + (B + C)x_{i-N} \\ &+ B[f(x_{i-N}) - x_{i-N}] + C[x_i - x_{i-N}]. \end{aligned} \tag{57}$$

Once the system is on this form, one should choose C such that the nominal delay system

$$x_{i+1} = (A - C)x_i + (B + C)x_{i-N} \tag{58}$$

is robustly stable (stable for all delays). Conditions for this to be possible are described in [13].

The next topic on the agenda is thus to analyze the stability of the *linear* system

$$x_{i+1} = (A - C)x_i + (B + C)x_{i-N} + C[(x_i) - x_{i-N}] \tag{59}$$

as a perturbation of the system (58), using Rouche's Theorem. This procedure was explained for continuous time delay systems in [12], and the method is analogous in the discrete time case. By taking the Z-transform of (59), the characteristic equation is equal to

$$\det[z^{N+1}I-(A-C)z^N-(B+C)]\det[I+P(z;N)], \quad (60)$$

where

$$P(z;N)=[z^{N+1}I-(A-C)z^N-(B+C)]^{-1}C(z^N-1). \quad (61)$$

If, along the unit circle ($z = e^{j\theta}$), the norm $\|\epsilon P(z;N)\| < 1$ for all $\epsilon \in [0,1]$, and the nominal system (58) is stable, then the system (59) is also stable for a delay of N steps, which is where the dependence on the delay enters. Finally, the nonlinear perturbation term $B[f(x_{i-N}) - x_{i-N}]$ and the stochastic perturbation w_i need to be taken into account for this prodecure to be complete.

6 Conclusions and Further Issues

In this paper we have outlined a suite of results about control design in the presence of delay times and limited information. However, we have not presented a final solution to the general case where β control bits are available to us and a general time delay (possibly depending on β, i.e. the longer the input string, the larger the delay) affects the system. Instead we have outlined a collection of potentially fruitful routes for addressing some relevant subproblems, such as finite information control of scalar systems, worst-case analysis of delayed systems, and the effect of one-step delays. What remains to be done to complete the picture is to design higher dimensional trap regions, and optimize the distribution of the available control levels over the trap region and the basin of attraction, and to get a tight coupling between the bitrates, the delay times, and the system performance in the presence of both additive noise and coding errors. A major drawback is the absence of a Lur'e type Lyapunov stability result in the discrete time setting. On the other hand, perturbation approaches may be fruitful in this analysis, at least when the delay is small, since each case needs to be tested individually. Only in the scalar case is a characterization of stability for all delays up to some number N known [11].

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