OPTIMAL IMPULSIVE CONTROL FOR POINT DELAY SYSTEMS WITH REFRACTORY PERIOD

Erik I. Verriest, Florent Delmotte, Magnus Egerstedt

School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, Georgia, USA
{erik.verriest,florent,magnus}@ece.gatech.edu

Abstract: The optimal impulsive control problem for a system with a single discrete delay is studied. In such systems the control consists only of a sequence of modulated impulses, the control variables being the impulse times and their magnitudes. It is assumed that the systems considered all have a refractory period, in the sense that once an action is taken, it takes a non-infinitesimal amount of time before a subsequent action can be taken. Necessary conditions for a stationary solution are derived and shown to extend those of the delay free case.

Keywords: Optimal control, impulses, time delay.

1. INTRODUCTION

Systems with impulsive inputs have been studied by Bainov and Simeonov (1989,1995,1996). For more recent work, see (Yang 1999). The optimal control problem has recently received some attention in the delay free case: (Chudoung *et al.* 2001) discuss a timing problem for the delay-free case. In the present paper, optimization with respect to the strengths of the impulses as well as their timing, is considered, and necessary conditions are established. This optimal impulsive control problem is not unrelated to the optimal switching problem (Branicky *et al.* 1998, Egerstedt *et al.* 2003, Verriest 2003).

We assume that the systems considered all have a refractory period, in the sense that once an action is taken, it takes a non-infinitesimal amount of time before a subsequent action can be taken. Refractory periods are ubiquitous in many physiological systems, and many technological systems, (e.g. time required to recharge a capacitor). The paper extends the results of (Verriest 2003) to systems with delays, but derives the optimality conditions via a classical variational approach. The presence of delays adds a nontrivial twist to the original problem posed in (Xu *et al.* 2002). Necessary conditions for the optimal impulsive control are determined in Section 2, and an illustrative example is given in Section 3.

2. VARIATIONAL APPROACH TO OPTIMAL IMPULSIVE CONTROL

Breaking with the standard notation (Hale *et al.* 1993), denote the data $\{\, x(t+\theta) \mid -\tau \le \theta \le 0 \,\}$ by $\{x_t\}$, and let instead $x_\tau(t) \stackrel{\text{def}}{=} x(t-\tau)$. Let a finite set of autonomous vector fields, $\{f^{(a)}(\{x_t\})\}$, be given. The dynamical system discussed in this paper is modelled by an autonomous point-delay system, i.e., a system with a discrete delay, where we assume that the control consists of a sequence of modulated impulses at discrete instants. In addition, we model a *refractory time* in the problem. This means that once an impulsive input is applied, a certain recovery period is required before the next impulse can be applied. Such recovery times seem to occur in many physiological applications, e.g., neural spike propagation, and in technological processes (refurbishing, restocking, etc.). As the system dynamics may change because of the impulsive inputs (e.g., due to loss of mass in spacecraft trajectory applications), we let f and g also depend on the number of impulses that have been applied. To fix the ideas, let the the autonomous system be modelled by

$$\dot{x} = f(x, \xi) + g(x_\tau, \xi), \tag{1}$$

with $x(\theta)$ given for $-\tau < \theta < 0$, and and let the effect of the impulsive inputs be given by

$$x(T_i^+) = x(T_i^-) + G(x(T_i^-), u_i, T_i) \tag{2}$$

The amplitudes, u_i, and instants, T_i, are the control variables to be chosen such that a performance index

$$J = \int_0^T L(x, \xi)\, dt + \sum_{i=1}^{N-1} K(x(T_i^-), u_i, T_i) + \Phi(x(T)) \tag{3}$$

is optimized. Here, ξ is a *discrete state*, counting the number of impulses. In some applications, the added generality of a running cost, L, depending on the number of past impulses, may be of interest. Thus, $\xi(t) = i$, if $T_{i-1} < t < T_i$. We shall assume that the vector fields $f_i(x) = f(x, i)$ and $g_i(x) = g(x, i)$ and the functions $L_i(x) = L(x, i)$ are smooth. For simplicity we shall assume that the *number* of impulses, $N-1$, and the initial time $T_0 = 0$, as well as final time, $T = T_N$, are fixed. Φ is the terminal cost at the fixed terminal time, and K is the cost associated with the control. Note that, in view of the above, we may set

$$\Phi(x(T)) = K(x(\tau_N), 0, \tau_N), \tag{4}$$

thus including the terminal cost in the sum of the control costs. This is useful in the more general problem of a free endpoint.

As stated, the problem is a *parameter optimization* problem. However, solving it as such requires the explicit solution of the state equations (and their dependencies on the u_i and T_i. We therefore solve the problem using classical variational methods instead (Bryson *et al.* 1975). Consider arbitrary, independent perturbations of the instants T_i and of the strengths u_i, with scale parameter ϵ; i.e., perturb $T_i \to T_i + \epsilon\theta_i$, and $u_i \to u_i + \epsilon\nu_i$. Since at the jump times the function $f(x, \xi)$ jumps, the equation (1) cannot be satisfied at these times. Hence, one must take care in adjoining the dynamical constraints only in the *open* subintervals defined by the jump times. This will be done with *different* Lagrange multipliers, λ_i, defined in the subintervals between the state jumps. In addition, the jump constraints (2) are adjoined with Lagrange multipliers, μ_i, to the discrete summation in (3).

$$\begin{aligned}\overline{J}_0 = &\sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} [L_i(x) + \lambda_i'(f_i(x) + g_i(x_\tau) - \dot{x})]\, dt \\ &+ \sum_{i=1}^{N} [K_i + \mu_i'(G_i - \Delta x|_{T_i})].\end{aligned} \tag{5}$$

For simplicity, we set $K_i = K(x(T_i^-), u_i, T_i)$, $G_i = G(x(T_i^-, u_i, T_i)$ and $\Delta x|_{T_i} = x(T_i^+) - x(T_i^-)$. So, defining the *Hamiltonian functionals*, $H_i(\{x_t\}, \lambda_i) \stackrel{\text{def}}{=} L_i(x) + \lambda_i'[f_i(x) + g_i(x_\tau)]$, and the *Lagrangean constraints*, $M_i \stackrel{\text{def}}{=} K_i + \mu_i G_i$, we find for a neighboring solution

$$\begin{aligned}\overline{J}_\epsilon = &\sum_{i=1}^{N} \int_{T_{i-1}+\epsilon\theta_{i-1}}^{T_i+\epsilon\theta_i} [H_i(\{x_t + \epsilon\eta_t\}, \lambda_i) - \lambda_i'(\dot{x} + \epsilon\dot{\eta})]\, dt \\ &+ \sum_{i=1}^{N} M_i([x + \epsilon\eta]((T_i + \epsilon\theta_i)^-), u_i + \epsilon\nu_i, T_i + \epsilon\theta_i) \\ &- \sum_{i=1}^{N} \mu_i'\, \Delta(x + \epsilon\eta)|_{T_i+\epsilon\theta_i}\end{aligned} \tag{6}$$

Note now that $f(x)$ and $L(x)$, and therefore H jumps whenever x jumps; i.e., at the times $\{T_i + \epsilon\theta_i\}$. This however is not the end of the story. Due to the delay, also $g(x_\tau)$ jumps at the times $\{T_i + \epsilon\theta_i + \tau\}$. This yields an additional discontinuity for the Hamiltonian *inside* the subintervals. In general, many subsequent impulse times may occur before the time $T_i + \tau$ so that $T_j < T_i + \tau < T_{j+1}$ for some $j \geq i+1$. However, the presence of a refractory time ensures that once a switch occurred, the next impulse cannot occur until after some minimal duration of time, thus avoiding infinitely many switches (the Zeno effect) (Johansson *et al.* 1999). We shall assume that the refractory time is *at least* equal to the delay time τ. In this case the mathematics simplifies, as only two adjacent intervals need to be considered ($j = i+1$). The first term in (6) expands then to

$$\begin{aligned}\overline{J}_\epsilon^{(1)} = &\sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} [H_i(\{x_t\}, \lambda_i) - \lambda_i'\dot{x})]\, dt + \\ &+ \epsilon \sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} [D_x H_i - \lambda_i'\dot{\eta}]\ dt + \\ &+ \sum_{i=1}^{N} \Big(\int_{T_i}^{T_i+\epsilon\theta_i} [H_i - \lambda_i'\dot{x}]\, dt - \int_{T_{i-1}}^{T_{i-1}+\epsilon\theta_{i-1}} [H_i - \lambda_i'\dot{x}]\, dt \Big) \\ &+ \sum_{i=1}^{N} \Big(\int_{T_i+\tau}^{T_i+\tau+\epsilon\theta_i} H_i^-\, dt - \int_{T_{i-1}+\tau}^{T_{i-1}+\tau+\epsilon\theta_{i-1}} H_i^+\, dt \Big)\end{aligned} \tag{7}$$

Here DH_i is the functional derivative of H_i.

$$D_x H_i = \lim_{\epsilon \to 0} \frac{H_i(\{x_t + \epsilon\eta_t\}, \lambda) - H_i(\{x_t\}, \lambda)}{\epsilon}, \tag{8}$$

The integrals in the next to the last last line of (7) both involve the dynamics in the i-th interval. The first integrals are evaluated with the dynamical characterization *to the left* of T_i, while the second ones with dynamics characterized *to the right* of T_{i-1}. This observation is important for what follows. These integrals over the ϵ-intervals evaluate, up to first order in ϵ, to

$$\epsilon \sum_{i=1}^{N-1} [L_i(x(T_i^-))\theta_i - L_i(x(T_{i-1}^+)\theta_{i-1}] + \epsilon \sum_{i=1}^{N-1} \lambda'_{i+1}(T_i + \tau)[g(X(T_i^+) - g(X(T_i^-)]\,\theta_i. \tag{9}$$

The nonintegral terms in (6) expand to

$$\begin{aligned}\overline{J}_\epsilon^{(2)} &= \sum_{i=1}^{N} [M_i - \mu'_i\,\Delta x|_{T_i}] \\ &+\epsilon \sum_{i=1}^{N} \Big(\frac{\partial M_i}{\partial x}\,[\dot{x}(T_i^-)\theta_i + \eta(T_i^-)] + \frac{\partial M_i}{\partial u}\nu_i + \frac{\partial M_i}{\partial T}\theta_i \Big) + \\ &- \sum_{i=1}^{N} \mu'_i[\Delta x|_{T_i+\epsilon\theta_i} - \Delta x|_{T_i}] - \epsilon \sum_{i=1}^{N} \mu'_i \Delta\eta|_{T_i}\end{aligned} \tag{10}$$

Here, $\frac{\partial M_i}{\partial \alpha}$ is the partial derivative of $M_i(x, u, T, \mu)$ taken at $(x(T_i^-), u_i, T_i, \mu_i)$.

Note that the expression of the partial of M with respect to x in (10) involves the variation of x, i.e., η, as well as the differential $dx = \dot{x}\,dt$ for $dt = \epsilon\theta$. Moreover, as this variation may also be discontinuous (see below), the value $\eta(T_i^-)$ is required. The last terms in (10) equal

$$-\epsilon \sum_{i=1}^{N} \mu'_i \Big[\dot{x}(T_i^+)\theta_i - \dot{x}(T_i^-)\theta_i + \eta(T_i^+) - \eta(T_i^-)\Big]$$

Likewise, we find for $T_{i-1} < t < T_i$:

$$D_x H_i(x(\cdot), \lambda, \eta(\cdot)) = \frac{\partial L_i(x(t))}{\partial x}\eta(t) + +\lambda'_i(t) \left(\frac{\partial f_i(x(t))}{\partial x}\eta(t) + \frac{\partial g_i(x(t-\tau))}{\partial x}\,\eta(t-\tau) \right) \tag{11}$$

Assume that an optimizing control policy and associated trajectory exists, resulting in the nominal $\overline{J}_0$. Subtracting this $\overline{J}_0$ from $\overline{J}_\epsilon = \overline{J}_\epsilon^{(1)} + \overline{J}_\epsilon^{(2)}$, and using (11) gives the first order variation of J due to the independent perturbations θ_i and ν_i.

$$\begin{aligned}\frac{\overline{J}_\epsilon - \overline{J}_0}{\epsilon} &= \sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} \Big[\left(\frac{\partial L_i}{\partial x} + \lambda'_i(t)\frac{\partial f_i}{\partial x} \right)\eta(t) \\ &\quad +\lambda'_i(t)\frac{\partial g_i}{\partial x}\eta(t-\tau) - \lambda'_i(t)\dot{\eta}(t) \Big]\, dt \\ &+ \sum_{i=1}^{N} \Big(\frac{\partial M_i}{\partial x}\,[\dot{x}(T_i^-)\theta_i + \eta(T_i^-)] + \frac{\partial M_i}{\partial u}\nu_i + \frac{\partial M_i}{\partial T}\theta_i \Big) \\ &+ \sum_{i=1}^{N} [L_i(x(T_i^-))\theta_i - L_i(x(T_{i-1}^+))\theta_{i-1}] \\ &+ \sum_{i=1}^{N-1} \lambda'_{i+1}(T_i+\tau)[g(X(T_i^+) - g(X(T_i^-)]\theta_i + \\ &- \sum_{i=1}^{N} \mu'_i \Big[\dot{x}(T_i^+)\theta_i - \dot{x}(T_i^-)\theta_i + \eta(T_i^+) - \eta(T_i^-)\Big]\end{aligned}$$

Rearrange the integral of the delayed term as follows:

$$\begin{aligned}&\int_{T_{i-1}}^{T_i} \lambda'_i(t)\frac{\partial g_i(x(t-\tau))}{\partial x}\eta(t-\tau)\,dt \\ &= \int_{T_{i-1}-\tau}^{T_i-\tau} \lambda'_i(t+\tau)\frac{\partial g_i(x(t))}{\partial x}\eta(t)\,dt \\ &= \int_{T_{i-2}}^{T_{i-1}} \chi_{[T_{i-1}-\tau, T_{i-1}]}(t)\lambda'_i(t+\tau)\frac{\partial g_i(x(t))}{\partial x}\eta(t)\,dt \\ &\quad + \int_{T_{i-1}}^{T_i} \chi_{[T_{i-1}, T_i-\tau]}(t)\lambda'_i(t+\tau)\frac{\partial g_i(x(t))}{\partial x}\eta(t)\,dt.\end{aligned} \tag{12}$$

In these expressions, $\chi_{[a,b]}$ is the *indicator function* of the interval $[a, b]$. Integrating by parts and recollecting terms in the summations gives then, noting that $\eta(t) = 0$ for $t \leq 0$ if the initial data is specified

$$\begin{aligned}\frac{\overline{J}_\epsilon - \overline{J}_0}{\epsilon} &= \sum_{i=1}^{N-1} \int_{T_{i-1}}^{T_i} \Big[\frac{\partial L_i}{\partial x} + \lambda'_i(t)\frac{\partial f_i(x(t))}{\partial x} + \dot{\lambda}_i(t) \\ &\quad + \chi_{[T_{i-1}, T_i-\tau]}(t)\lambda'_i(t+\tau)\frac{\partial g_i(x(t))}{\partial x} \\ &\quad + \chi_{[T_i-\tau, T_i]}(t)\lambda'_{i+1}(t+\tau)\frac{\partial g_{i+1}(x(t))}{\partial x} \Big]\eta(t)\,dt \\ &+ \int_{T_{N-1}}^{T_N} \Big[\frac{\partial L_N}{\partial x} + \lambda'_N\frac{\partial f_N(x(t))}{\partial x} + \dot{\lambda}_N(t) \\ &\quad + \chi_{[T_{N-1}, T_{N-\tau}]}(t)\lambda'_N(t+\tau)\frac{\partial g_N(x(t))}{\partial x} \Big]\eta(t)\,dt \\ &+ \sum_{i=1}^{N} \lambda'_i(T_{i-1}^+)\eta(T_{i-1}^+) - \sum_{i=1}^{N} \lambda'_i(T_i^-)\eta(T_i^-) + \\ &+ \sum_{i=1}^{N} \Big(\frac{\partial M_i}{\partial x}\,[\dot{x}(T_i^-)\theta_i + \eta(T_i^-)] + \frac{\partial M_i}{\partial u}\nu_i + \frac{\partial M_i}{\partial T}\theta_i \Big) + \\ &+ \sum_{i=1}^{N-1} [L_i(x(T_i^-))\theta_i - L_i(x(T_{i-1}^+))\theta_{i-1}] + \\ &+ \theta_N[L_N]_T - \theta_0[L_1]_0 + \\ &+ \sum_{i=1}^{N-1} \lambda'_{i+1}(T_i+\tau)[g(X(T_i^+) - g(X(T_i^-)]\theta_i \\ &- \sum_{i=1}^{N} \mu'_i \Big[\dot{x}(T_i^+)\theta_i - \dot{x}(T_i^-)\theta_i + \eta(T_i^+) - \eta(T_i^-)\Big]\end{aligned}$$

$$
\begin{aligned}
&= \lambda_1(0)'\eta(0) + \left(\frac{\partial\Phi}{\partial x}\bigg|_T - \lambda'_N(T)\right)\eta(T) \\
&+ \int_{-\tau}^{0} \lambda_1(t+\tau)\frac{\partial g_1(x(t))}{\partial x}\eta(t)\,dt \\
&+ \int_{T_{N-1}}^{T_N} \Big[\frac{\partial L_N}{\partial x} + \lambda'_N\frac{\partial f_N(x(t))}{\partial x} + \dot\lambda_N(t) \\
&\quad +\chi_{[T_{N-1},T_{N-\tau}]}(t)\lambda'_N(t+\tau)\frac{\partial g_N(x(t))}{\partial x}\Big]\eta(t)\,dt \\
&+ \sum_{i=1}^{N-1}\int_{T_{i-1}}^{T_i}\Big[\frac{\partial L_i}{\partial x} + \lambda'_i(t)\frac{\partial f_i(x(t))}{\partial x} + \dot\lambda_i(t) \\
&\quad +\chi_{[T_{i-1},T_i-\tau]}(t)\lambda'_i(t+\tau)\frac{\partial g_i(x(t))}{\partial x} \\
&\quad +\chi_{[T_i-\tau,T_i]}(t)\lambda'_{i+1}(t+\tau)\frac{\partial g_{i+1}(x(t))}{\partial x}\Big]\eta(t)\,dt \\
&+ \sum_{i=1}^{N-1}\lambda'_{i+1}(T_i^+)\eta(T_i^+) - \sum_{i=1}^{N-1}\lambda'_i(T_i^-)\eta(T_i^-) + \\
&+ \sum_{i=1}^{N}\Big(\frac{\partial M_i}{\partial x}\left[\dot x(T_i^-)\theta_i + \eta(T_i^-)\right] + \frac{\partial M_i}{\partial u}\nu_i + \frac{\partial M_i}{\partial T}\theta_i\Big) + \\
&+ \sum_{i=1}^{N-1}\theta_i\Big[L_i(T_i^-) - L_{i+1}(T_i^+)\Big] + \\
&+ \sum_{i=1}^{N-1}\lambda'_{i+1}(T_i+\tau)[g(X(T_i^+) - g(X(T_i^-)]\theta_i + \\
&- \sum_{i=1}^{N}\mu'_i\Big[\dot x(T_i^+)\theta_i - \dot x(T_i^-)\theta_i + \eta(T_i^+) - \eta(T_i^-)\Big].
\end{aligned}
$$

For simplicity of notation, denote by $\lambda^\tau(t)$, the advanced value $\lambda(t+\tau)$. In addition, we denote the indicator functions by $\chi_i^+ = \chi_{[T_{i-1},T_i-\tau]}$ and $\chi_{i+1}^- = \chi_{[T_i-\tau,T_i]}$.

Since we assume that the initial data is fixed, we have $\eta(t) = 0$ for $t \le 0$. Consequently $\lambda_1(t)$ is not constrained for nonpositive t.

In order to avoid laborious computation of unknown quantities, *choose* the Lagrange multiplier functions to satisfy

$$
\begin{aligned}
\dot\lambda_i &= -\left(\frac{\partial L_i}{\partial x}\right)' - \left(\frac{\partial f_i}{\partial x}\right)'\lambda_i - \chi_i^+\left(\frac{\partial g_i}{\partial x}\right)'\lambda_i^\tau \\
&\quad -\chi_{i+1}^-\left(\frac{\partial g_{i+1}}{\partial x}\right)'\lambda_{i+1}^\tau \\
&\text{with}\quad T_{i-1} < t < T_i, \quad i = 1,\dots,N-1 \\
\dot\lambda_N &= -\left(\frac{\partial L}{\partial x}\right)' - \left(\frac{\partial f_N}{\partial x}\right)'\lambda_N - \chi_N^+(t)\left(\frac{\partial g_N}{\partial x}\right)'\lambda_N^\tau
\end{aligned}
\tag{13}
$$

These are coupled *advanced* equations, coupled also with the forward delay equations. Choose also a final condition

$$
\lambda(T) = \left(\frac{\partial\Phi}{\partial x}\right)' \tag{14}
$$

This zeroes the integral terms in the variation of the performance index. The resulting variation $\delta\overline{J}$ involves then only terms in the ν_i, θ_i and both $\eta(T_i^-)$ and $\eta(T_i^+)$.

Since for given a initial condition, the system follows a fixed trajectory until the first impulse, we also have

$$
\eta(T_1^-) = 0, \tag{15}
$$

These choices reduce the variation $\delta J = \frac{(\overline{J}_\epsilon - \overline{J}_0)}{\epsilon}$ to

$$
\begin{aligned}
\delta J = \sum_{i=1}^{N-1}&\left[\frac{\partial M_i}{\partial u}\right]\nu_i + \\
&+\Big[\lambda'_{i+1}(T_i^+) - \mu'_i\Big]\eta(T_i^+) + \\
&+\Big[\frac{\partial M_i}{\partial x} - \lambda'_i(T_i^-) + \mu'_i\Big]\eta(T_i^-) + \\
&\Big\{\frac{\partial M_i}{\partial x}\dot x(T_i^-) + \frac{\partial M_i}{\partial T} + [L_i(T_i^-) - L_{i+1}(T_i^+)] \\
&+\lambda'_{i+1}(T_i+\tau)[g(X(T_i^+) - g(X(T_i^-)] + \\
&-\mu'_i[\dot x(T_i^+) - \dot x(T_i^-)]\Big\}\theta_i + \\
+&\Big[\frac{\partial M_N}{\partial x} - \lambda'_N(T_N^-) + \mu'_N\Big]\eta(T_N^-)
\end{aligned}
\tag{16}
$$

If we *choose* the boundary conditions on the Lagrange multipliers so that

$$
\lambda_i(T_i^-) = \lambda_{i+1}(T_i^+) + \left(\frac{\partial M_i}{\partial x}\right)' \tag{17}
$$

and

$$
\mu_i = \lambda_{i+1}(T_i^+), \tag{18}
$$

for $i = 1, \dots N-1$ with

$$
\mu_N = \lambda_N(T_N^-) - \left(\frac{\partial M_N}{\partial x}\right)' \tag{19}
$$

then a moments reflection shows that δJ reduces to

$$
\begin{aligned}
\delta J = \sum_{i=1}^{N-1}&\Big[\frac{\partial M_i}{\partial T} + H_i(T_i^-) - H_{i+1}(T_i^+) \\
&+\lambda_{i+1}(T_i+\tau)[g(X(T_i^+) - g(X(T_i^-)]\Big]\theta_i \\
+&\sum_{i=1}^{N-1}\frac{\partial M_i}{\partial u}\nu_i.
\end{aligned}
\tag{20}
$$

Since the variations θ_i and ν_i are independent, necessary conditions for optimality are their vanishing.

We summarize these results in the theorem below

Theorem:
The impulsive system (2) minimizes the performance index J if the magnitudes u_i, and times T_i are chosen as follows:
Define:

$$
H_i = L_i + \lambda_i[f(x) + g(x_\tau)] \tag{21}
$$

$$
M_i = K_i + \mu_i G_i \tag{22}
$$

Euler-Lagrange Equations

$$\dot{\lambda}_i = -\left(\frac{\partial L_i}{\partial x}\right)' - \left(\frac{\partial f_i}{\partial x}\right)' \lambda_i - \chi_i^+ \left(\frac{\partial g_i}{\partial x}\right)' \lambda_i^\tau$$
$$-\chi_{i+1}^- \left(\frac{\partial g_{i+1}}{\partial x}\right)' \lambda_{i+1}^\tau$$
$$\textit{with} \quad T_{i-1} < t < T_i, \quad i = 1, \ldots, N-1$$
$$\dot{\lambda}_N = -\left(\frac{\partial L}{\partial x}\right)' - \left(\frac{\partial f_N}{\partial x}\right)' \lambda_N - \chi_N^+(t) \left(\frac{\partial g_N}{\partial x}\right)' \lambda_N^\tau \tag{23}$$

Boundary Conditions

$$\lambda_N(T) = 0, \tag{24}$$

$$\lambda_i(T_i^-) = \lambda_{i+1}(T_i^+) + \left(\frac{\partial M_i}{\partial x}\right)'. \tag{25}$$

Multipliers

$$\mu_i = \lambda_{i+1}(T_i^+), \quad i = 1 \ldots N-1, \tag{26}$$

$$\mu_N = \lambda_N(T_N^-) - \left(\frac{\partial M_N}{\partial x}\right)' \tag{27}$$

Optimality Condition

$$\frac{\partial M_i}{\partial u} = 0 \tag{28}$$

Transversality Conditions

$$H_i(T_i^+) = H_i(T_i^-) + \frac{\partial M_i}{\partial T}$$
$$+\lambda_{i+1}(T_i + \tau)[g(X(T_i^+)) - g(X(T_i^-))] \tag{29}$$

Remark: The calculus of variations leads to necessary conditions for optimality of smooth unconstrained controls. Replacing the small variation principle by a large one is the object of the *Pontryagin minimum principle* (Bryson *et al.* 1975). It postulates the same costate equations as the one obtained above, but states that optimality is achieved if the u_i are chosen to globally minimize the M_i over the set of *admissible* control values $u_i \in U$.

Note that in the case of a delay free system, we get the requisite necessary conditions by letting $g_i = 0$ in the above theorem. See also (Chudoung *et al.* 2001, Gilbert *et al.* 1971, Luo *et al.* 1998, Rishel 1966, Silva *et al.* 1993, Sussmann 1999). An approach to the minimization of a functional on a Banach space by steepest descent methods is reviewed in (Gollwitzer 1965). A related problem was treated in (Egerstedt *et al.* 2003).

3. ILLUSTRATIVE EXAMPLE

Consider the simple scalar impulsive delay system

$$\dot{x}(t) = x(t-1) \tag{30}$$

$$x(T^+) = x(T_-) + u \tag{31}$$

with performance index

$$J = \frac{1}{2}\int_0^2 (x(t)-1)^2\, dt + \frac{u^2}{T} \tag{32}$$

Let the initial data be $x(t) = 0$ for $t \le 0$.
Application of the theorem yields the optimality condition: $u = -3/2\frac{(\tau^2-4\tau+5)\tau}{\tau^4-6\tau^3+12\tau^2-10\tau-6}$ and the transversality condition: $u = -2\frac{\tau^2(\tau-2)}{\tau^4-4\tau^3+4\tau^2+2}$ from which

$$T^* = 0.47835 \quad \text{and} \quad u^* = 0.27527.$$

The figure shows the evolution of the optimal state x (solid), the co-state λ (dotted) and its time derivative $\dot{\lambda}$ (dashed). With these, the Hamilto-

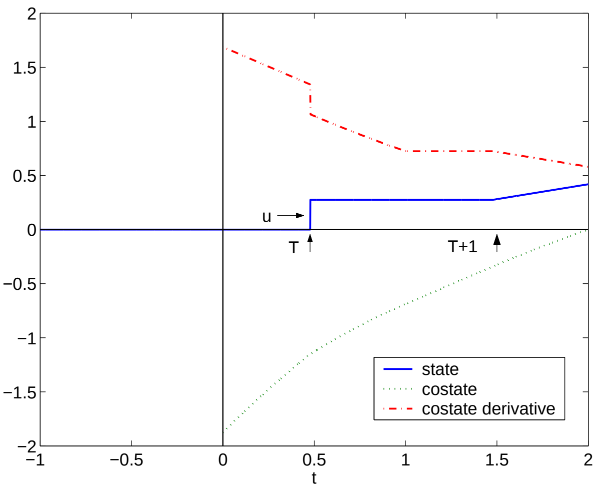

Fig. 1. The optimal solution (solid) to the example in (30) - (32) is shown.

nian $\{H_i(t)\}$ is piecewise constant. A brute force computation of the optimum is easily performed as verification for this system.

4. CONCLUSIONS

We derived necessary conditions for stationarity of the performance index of an impulsively controlled system with refractory period and a prespecified number of pulses. To the best of this author's knowledge, this problem has not been treated before in the literature on optimal control for systems with delays. In fact, a great deal of research is still going on in the delay free case (Chudoung *et al.* 2001, Luo *et al.* 1998, Sussmann 1999, Verriest 2003). The problem is also related to the optimal control problem for hybrid and switched systems (Branicky *et al.* 1998, Egerstedt *et al.* 2003, Verriest 2003). This is a first step in the complete optimal control of such a system, where also the *optimal number* of impulses needs to be found. In principle all possible N between zero and T/τ should be optimized for the impulsive controls, and the value of the performance index compared to find the global optimum. Whereas this quickly leads to large number of problems to be solved, regularization methods as for instance presented in (Verriest 2003) could be invoked to obtain a first approximation and thus narrow down the search.

As an illustration, we have set up the optimal control problem for a simple scalar system. A more realistic application is in optimal control through pulse vaccination of an epidemic(Behncke 2000, Nokes *et al.* 1997). The equations for disease dynamics are inherently nonlinear (Anderson *et al.* 1982), and the delay enters through an incubation period (Cooke *et al.* 1996, Wendi *et al.* 2002). Space limitations did not allow the inclusion of this model.

REFERENCES

Anderson, R.M. and R.M. May (1982). Directly Transmitted Infectious Diseases: Control by Vaccination. *Science*, **215**, pp. 1053-1060.

Bainov, D.D. and P.S. Simeonov, (1995). *Impulsive Differential Equations: Asymptotic Properties of the Solutions.* World Scientific.

Bainov D.D. and P.S. Simeonov, (1996). *Impulsive Differential Equations: Periodic Solutions and Applications.* Pitman.

Bainov, D.D. and P.S. Simeonov, (1989). *Systems with impulse effect: stability, theory, and applications.* Halsted Press.

Behncke, H., (2000). Optimal Control of Deterministic Epidemics. *Optimal Control Applications and Methods*, **21**, 269-285.

Branicky, M.S., V.S. Borkar and S. Mitter, (1998). A Unified Framework for Hybrid Control Theory: Model and Optimal Control Theory. *IEEE Trans. Auto. Control*, **43**, No. 1, 31-45.

Bryson, A.E., and Y.C. Ho, (1975). *Applied Optimal Control.* Hemisphere.

Cooke K.L., and P. van den Driessche, (1996). Analysis of an SEIRS epidemic model with two delays. *J. Math. Biol.* **35** 240-269.

Chudoung, J. and C. Beck, (2001). The Minimum Principle for Deterministic Impulsive Control Systems. *Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, FL, 3569-3574.

Chukwu, E.N., (1992). *Stability and Time-Optimal Control of Hereditary Systems.* Academic Press.

Egerstedt, M., Y. Wardi, and F. Delmotte, (2003). Optimal Control of Switching Times in Switched Dynamical Systems. *Proc. 42nd Conference on Decision and Control*, Maui, HI, 2138-2143.

Gilbert E.G. and G.A. Harasty, (1971). A Class of Fixed-Time Fuel-Optimal Impulsive Control Problems and an Efficient Algorithm for Their Solution. *IEEE Transactions on Automatic Control*, Vol. **AC-16**, No. 1, 1-11.

Gollwitzer, H.E., (1965). *Applications of the Method of Steepest Descent to Optimal Control Problems.* MS Thesis, University of Minnesota.

Hale, J.K. and S. M. Verduyn Lunel (1993). *Introduction to functional differential equations*, Applied Math. Sciences Series, **99**, Springer-Verlag, New York.

Johansson, K.J., M. Egerstedt, J. Lygeros and S. Sastry, (1999). On the Regulation of Zeno Hybrid Automata. *Systems & Control Letters*, **38**, 141-150.

Kolmanovskii, V. and A. Myshkis, (1992). *Applied Theory of Functional Differential Equations.* Kluwer Academic Publishers.

Luo, J.C. and E.B. Lee, (1998). Time-optimal Control of the Swing using Impulse Control Actions. *Proceedings of the American Control Conference*, Philadelphia, PA, 200-204.

Nokes D.J. and J. Swinton, (1997). Vaccination in Pulses: A Strategy for Global Eradication of Measles and Polio? *Trends in Microbiology*, Vol. **5**, 14-19.

Rishel, R.W., (1966). Application of an Extended Pontryagin Principle, *IEEE Transactions on Automatic Control*. Vol. **AC-11**, No. 2, 167-170.

Silva, G.N., and R.B. Vinter, (1993). Optimal Impulsive Control Problems with State Constraints. *Proceedings of the 32nd IEEE Conference on Decision and Control*, San Antonio, TX, 3811-3812.

Sussmann, H., (1999). A Maximum Principle for Hybrid Optimal Control Problems. *Proc. 38th Conference on Decision and Control*, Phoenix, AZ, 425-430.

Verriest, E.I., (1999). Robust Stability, Adjoints and LQ Control of Scale-Delay Systems. *Proceedings of the 38th IEEE Conference on Decision and Control*, Phoenix, AZ, 209-214.

Verriest, E.I., (2003). Regularization Method for Optimally Switched and Impulsive Systems with Biomedical Applications. *Proceedings of the 42th IEEE Conference on Decision and Control*, Maui, HI, 2156-2161.

Wendi W. and M. Zhien, (2002). Global dynamics of an epidemic model with time delay. *Nonlinear Analysis: Real World Applications*, **3**, 365-373.

Xu, X. and P. Antsaklis, (2002). Optimal Control of Switched Autonomous Systems. *Proc. 41st Conference on Decision and Control*, Las Vegas, NV, 4401-4406.

Yang, T., (1999). Impulsive Control. *IEEE Transactions on Automatic Control*, **AC-44**, No. 5, 1081-1083.