Decentralized Coordination with Local Interactions: Some New Directions

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Summary. Due to the limited effective range of every physical sensor, as well as potential bandwidth limitations on the communication channels, there is a need for modeling and analysis tools when studying multi-agent robot systems that take these local interactions into account. In this paper, we review some of the work done on graphs and configuration spaces, and introduce the *connectivity graph* as a bridge between these two areas. We give sufficient conditions for a graph to be a connectivity graph in the sense that it can be realized as a formation in the configuration space. Moreover, the topological shape of a given connectivity graph is captured using triangulation techniques.

1 Introduction

The problem of controlling multiple, mobile robots in a coordinated fashion, i.e. to enforce desired formations, has received considerable attention during the last decade. The underlying driver of these research efforts is the implicit assumption that there is strength in numbers, which has been exploited when exploring and negotiating unknown or hostile environments [2, 26]. Of particular concern has been to defined formations and to develop control laws that guarantee formation stability when global information is available (e.g. [3, 6, 10, 11, 24, 30, 35]). In other words, the underlying assumption has been that each individual robot has complete knowledge of the whereabouts of the other robots. However, this is not always the case. In particular, if the number of robots is large, bandwidth limitations as well as range constraints on the available sensing capabilities, imply that the global knowledge assumption has to be abandoned. Hence, recent work has studied what information the individual robots need to communicate in order to maintain the formation [7, 8, 9, 32, 36], by viewing the interactions between the different robots as edges in a directed graph.

However, as the shift is made from global to local interactions, there are a number of issues that need to be resolved, stemming from the inherent global nature of a formation. For instance, if a number of robots have to decide which individual roles to take on in the formation, a distributed decision making mechanism has to be employed. Similarly, different formations are potentially beneficial in different situations. For example, when exploring unknown terrains, maximally spread formations may be to prefer, but as obstacles are encountered new formations must be used, e.g. for negotiating narrow passages. As of yet, little work has been done on how to choose formations in a decentralized and autonomous fashion as a reaction to changes in the environment.

The problem of decentralized control has been successfully addressed when investigating swarm behaviors, where the individual robots are moving according to limited range potential fields (e.g. [12, 34]), or according to some averaging orientation rules [19, 37]. However, these results are not constructive in the sense that one can not specify or change desired formations in any direct manner. For this, it is inevitable that the robots are allowed to communicate, which is the main topic to be investigated in this paper.

The outline of this paper is as follows: In Section 2, we introduce some of the key assumptions made when trying to model multi-agent formations in a precise manner. These assumptions include limited perception and communication capabilities. Moreover, it is shown how these assumptions lead to configuration space formulations as well as graph-based models in a natural way. In Section 3 the notation of a connectivity graph is introduced as an object that bridges graph theory and configuration space descriptions of multi-agent formations. We derive sufficient conditions for a graph to be a connectivity graph and show how the topology of a given formation can be analyzed within the context of connectivity graphs. Finally, in Section 4, we discuss how local rules and communication strategies can be exploited to arrive at the desired, global formations.

2 Models of Decentralized Coordination

Before we can properly define what we mean by formations or distributed control, a number of modeling issues need to be resolved. These involve how the local nature of the interactions (e.g. due to inter-robot communication and/or perception) should be captured, or how graph theoretic and topological tools can be put to work when characterizing multi-agent formations.

2.1 Limited Communication and Sensory Capabilities

In most multi-agent applications, the individual robots can collect information about their environment and neighboring robots in the formation by either peer to peer communication or by relying on sensory information. Since any physical sensor is always limited by its range and resolution, or by calibration errors, the information available to each agent by direct observation or state estimation is always limited and uncertain. In non-omnidirectional sensors, the limitations may also arise due to the directivity patterns of sensors. e.g. the conic field of view of a camera or the radiation patterns of antennas and sonars [25].

Instead, if we let the robots share information using peer to peer communication strategies, we may overcome this problem as long as the number of robots in the formation is relatively small. However, as the formation size increases, both in cardinality and spatial dimension, bandwidth limitations as well as large spatial distances, or the absence of feasible communication channels altogether, severely limits the possibility to convey and use global information. In this paper, we thus take the point of view that multi-agent solutions should be scalable in terms of cardinality as well as along spatial dimensions. Hence no individual robot can be assumed to have complete knowledge about the states of every other robot in the formation. This assumption directly leads to the question about how the local interactions should be represented. An obvious choice is to let the existence of such interactions be represented by edges in graph-based models, which is the topic of the next paragraphs.

2.2 Spatial Relationships in Formations: Graph Theoretic Models

One natural way in which local interactions can be expressed is to let this aspect of the formation be represented as graph, in which nodes correspond to individual agents and the presence of an edge between two nodes (robots) signifies that an interaction exists between them. In other words, an edge between two nodes means that the corresponding robots are within sensory range of each other, or that a communication channel is established between them.

If we let V be the set of nodes representing the agents, and S be the set of states associated with each node, then we say that a relationship $\mathcal{P} : S \times S \to Bool$ exists between two nodes if $\mathcal{P}$ is `TRUE`. We can then define a graph $G = (V, E)$, where $e_{ij} \in E$ if $\mathcal{P}(S_i, S_j) = \texttt{TRUE}$. These types of graphs have been proposed in the literature mainly to represent spatial or geometric relationships between agents. Examples include [9, 19, 29, 31, 37, 38].

As an example, consider [31], in which Olifati-Saber *et. al.* defines a *spatial adjacency matrix* for formations of agents, equipped with sensors with limited range, as follows: Let $x = (x_1, x_2, \ldots, x_N)$ denote the position vector of N robots in an ambient space $\mathbb{R}^k$. If agent i has an omnidirectional sensor of range δ_i then, the spatial adjacency matrix $A(x) = [a_{ij}]$ is defined as:

$$a_{ij}(x) = \begin{cases} 1 \text{ if } x_j \in \mathcal{B}(x_i, \delta_i),\ j \neq i \\ 0 \text{ otherwise,} \end{cases} \tag{1}$$

where $\mathcal{B}(x_i, \delta_i)$ is the closed ball in $\mathbb{R}^k$ with radius δ_i, centered at x_i.

Now, if the sensors are directional, or if the agents themselves have an orientation, conic neighborhoods can be used to define the spatial adjacency matrix as:

$$a_{ij}(x) = \begin{cases} 1 \text{ if } \|x_j - x_i\| \leq \delta_i,\ |\theta_j - \theta_i| \leq \phi_i,\ j \neq i \\ 0 \text{ otherwise,} \end{cases} \tag{2}$$

where θ_i denotes either the orientation of robot i or the directionality of its sensor. Moreover, ϕ_i defines the conic neighborhood in which robot i can effectively acquire information from neighboring robots. It should be noted that these models (especially in the conic case) imply directed rather than undirected graphs. In other words the relationships that model the graphs need not be commutative.

These types of constructions capture the local interactions in a straight forward manner. However, little work has been done on exploring and characterizing the graph theoretic properties associated with these graphs, which will be the topic of Section 3. However, some striking results have been obtained using *Algebraic Graph Theory* [14]. E.g. in [9], the stability of a formation has been shown to be closely related with the *Laplacian* L of the underlying graph. If we define the $N \times N$ *degree matrix* Δ of a graph as:

$$\Delta_{ij} = \begin{cases} deg(v_i), & i = j \\ 0, & i \neq j, \end{cases} \tag{3}$$

then the *Laplacian* L is defined by:

$$L = \Delta - A, \tag{4}$$

where A is the adjacency matrix. It can be noted that the rank of L can be related to the connectivity of the graph [14], and in [33], similar results have been obtained for the construction of agreement protocols between the robots. Moreover, in [19, 37, 38], the Laplacian is used for studying the behavior of certain classes of graphs when the alignment of the individual robots is based on nearest neighbor rules.

The main contribution of this paper is based on [29], and the idea is to complement this algebraic view of graphs by studying the geometric structure of so-called *connectivity graphs* of formations. For example, results on realizable and non-realizable formations have been derived, as will be explained in Section 3.1.

2.3 Configuration Spaces: Lessons from Robotic Manipulators

So far, we have seen how graph theoretic models lend themselves well to capturing certain aspects of spatially induced relations between robots in a multi-agent context. However, another possibility when studying multi-agent formations that obey spatial relationships is to obtain the configuration space of the formation. Even if the agents are fully actuated and living in Euclidean spaces, the configuration space of formations is non-trivial. In the multi-agent robotics literature, it is quite standard to (explicitly or implicitly) assume that the robots are evolving on the simplified configuration space, i.e. the product space $\mathbb{R}^k \times \mathbb{R}^k \times \ldots \times \mathbb{R}^k = (\mathbb{R}^k)^N$. However, as pointed out by Ghrist in [13], the configuration space of N robots, even without any inter-agent constraints, is:

$$C^N(\mathbb{R}^k) = (\mathbb{R}^k \times \mathbb{R}^k \times \ldots \mathbb{R}^k) - \Delta, \tag{5}$$

where $\Delta = \{(x_1, x_2, \ldots, x_N) : x_i = x_j \text{ for some } i \neq j\}$. When inter-agent constrains, e.g. represented as a relationship graph, are present, one could for instance ask how the configuration can evolve while the graph is preserved, as discussed in [29]. Since the movements of the individual agents make the graph itself a dynamic structure, a characterization of when a fixed relationship can be assumed between the robots would be useful. It is clear that such a characterization can not be obtained using graph theory alone.

Since many of the geometric constraints are polynomial, the resulting configurations spaces can in many cases be described as *semi-algebraic sets*[29]. Such configuration spaces have been extensively studied in the literature on robotic manipulators, where mechanical linkages provide the constraints. For example, in [27] and [20], the configuration space of a weighted graph (corresponding to the mechanical linkages between joints in the manipulator) is described as all possible realizations of the graph, with the given constraints satisfied. A rank test on a quadratic form has been given in [27] to test if some particular degenerate realizations are valid. Recently, some *universality theorems* have also appeared in the literature [21] that answer the converse problem of whether a mechanical linkage exists for a given algebraic variety. However, the problem of determining the configuration space of a graph defined by inequality constraints (instead of equality constraints in rigid mechanical linkages) is a more difficult question to answer. For example, in [28], the authors characterize the configuration space for a closed chain with a prismatic joint. Since the prismatic joint represents an inequality constraint in the linkage, a similar approach may be useful for characterizing the configuration space in the case of multiple inequality constraints.

3 Formations and Connectivity Graphs

As discussed in Section 2.2, spatial relationships between robots in a formation can be captured in a natural way using graphs. In particular, graphs provide an immediate formalism in which local interactions between agents can be modeled when individual agents have limited knowledge of other agents. In this section, we present a graph theoretic formalism for describing formations in which the primary limitation of perception for each agent is the effective range of its sensors.

Suppose the team is given by N robots with identical dynamics, living in an ambient space $\mathbb{R}^2$, each of which carries a unique identification tag $n \in \{1, 2, \ldots, N\}$. Each robot is also equipped with an identical range sensor, by which it can sense the position of other robots within a certain distance δ.

Definition 1 (Formations and Their Configuration Spaces). *The configuration space $\mathcal{C}^N(\mathbb{R}^2)$ of the robot formation is made up of all ordered N-tuples in $\mathbb{R}^2$, with the property that no two points coincide. Formally,*

$$\mathcal{C}^N(\mathbb{R}^2) = (\mathbb{R}^2 \times \mathbb{R}^2 \times \ldots \mathbb{R}^2) - \Delta, \tag{6}$$

where $\Delta = \{(x_1, x_2, \ldots, x_N) : x_i = x_j \text{ for some } i \neq j\}$.

Note that the evolution of the formation can be represented as a trajectory $\mathcal{F} : \mathbb{R}_+ \rightarrow \mathcal{C}^N(\mathbb{R}^2)$, usually written as $\mathcal{F}(t)$ to signify time evolution.

The spatial relationships between robots can now be represented as a graph in which the vertices represent the robots and the pair of vertices on each edge tell us that the corresponding robots are within sensor range δ of each other. However, it is clear that several formations may produce the same graph. We make these ideas precise as follows:

Definition 2 (Connectivity Graph of a Formation). *Let $\mathcal{G}_N$ denote the set of all possible graphs that can be formed on N vertices $V = \{v_1, v_2, \ldots, v_N\}$. Then we can define a function $\Phi_N : C^N(\mathbb{R}^2) \rightarrow \mathcal{G}_N$, with $\Phi_N(\mathcal{F}(t)) = \mathcal{G}(t)$. Here $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t)) \in \mathcal{G}_N$ is the connectivity graph of the formation $\mathcal{F}(t)$. The vertex, $v_i \in \mathcal{V}$, represents robot i at position x_i, and $\mathcal{E}(t)$ denotes the edges of the graph. We have that $e_{ij}(t) = e_{ji}(t) \in \mathcal{E}(t)$ if and only if $\|x_i(t) - x_j(t)\| \leq \delta,\ i \neq j$. In other words,*

$$\Phi_N(\mathcal{F}(t)) = (\{v_1, \ldots v_N\}, \{(v_i, v_j) \mid i \neq j \text{ and } \|x_i(t) - x_j(t)\| \leq \delta\}). \tag{7}$$

Some comments about these connectivity graphs should be made:

- The graphs are *simple* by construction i.e. there are no loops or parallel edges;
- The graphs are always undirected since each robot's sensor range is assumed to be equal;
- The motion of individual robots in the formation may result in the removal or addition of edges in the graph, and therefore $\mathcal{G}(t)$ is a dynamic structure; and
- Every graph in $\mathcal{G}_N$ is not a connectivity graph.

The last observation, due to the special way in which the connectivity graphs are defined, means that not all graphs can be realized in $\mathcal{C}^N(\mathbb{R}^2)$, which is the topic of investigation in the next paragraphs.

3.1 Realizations of Connectivity Graphs

Proposition 1. *Let the "star" connectivity graph $\mathcal{S}_N$ in $\mathcal{G}_N$ be the graph which has $N-1$ vertices $v_1, v_2 \dots v_{N-1}$ of degree 1, and one vertex v_N with degree $N-1$. Then $\mathcal{S}_N$ does not belong to $\mathcal{G}_{N,\delta}$ for $N \geq 6$.*

The proof of this is given in [29] and it relies on the fact that as more robots are placed in the star graph, sooner or later two robots in the "periphery" have to be within a distance less than or equal to δ of each other in order to be close enough to the "center" robot. It turns out that this is the case for $N \geq 6$, and hence, as a consequence, the star graph in Figure 1 (the graph to the right) is not a connectivity graph. In a similar manner it can be shown that the graph to the left in Figure 1 is not a connectivity graph either. Note that this graph has 5 vertices, which means that the set of connectivity graphs is a proper subset of the set of graphs of 5 vertices, which is a fact that we will make use of further on.

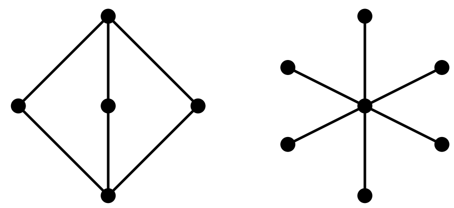

Fig. 1. Graphs that are not connectivity graphs.

Definition 3 (Realization of a Graph in $\mathcal{C}^N(\mathbb{R}^2)$). *A connectivity graph $\mathcal{G} \in \mathcal{G}_N$ can be realized in $\mathcal{C}^N(\mathbb{R}^2)$ if $\Phi_N^{-1}(\mathcal{G})$ is nonempty. In other words, a realization of $\mathcal{G}$ is some $\mathcal{F} \in \mathcal{C}^N(\mathbb{R}^2)$, such that $\Phi_N(\mathcal{F}) = \mathcal{G}$.*

A consequence of this definition is thus that $\mathcal{G}_{N,\delta}$ is the space of all connectivity graphs in $\mathcal{G}_N$ that can be realized in $\mathcal{C}^N(R^2)$.

There are many interesting examples of realizable and non-realizable connectivity graphs. If a graph is completely disconnected, it means that the distance between any two robots in the formation are separated by more than the distance δ. This can easily be achieved by placing the robots, one by one, in such a way that x_i does not belong to $\bigcup_{j=1}^{i-1} \mathcal{B}_\delta(x_j)$. An example of such a formation is given in Figure 2

This observation can be further generalized as follows:

Lemma 1. *A graph $\mathcal{G} \in \mathcal{G}_{N,\delta}$ if and only if each of its connected component $\mathcal{G}_i \in \mathcal{G}_{M_i}$ is realizable in some $\mathcal{G}_{M_i,\delta}$, $M_i < N$.*

We refer to [29] for details of the proofs, but the concept can be easily conveyed. We saw that completely disconnected graphs are trivially realizable

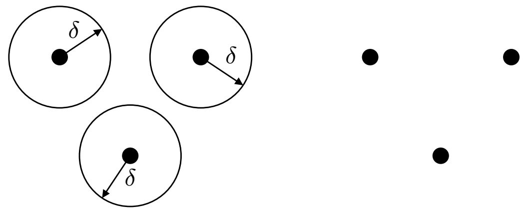

Fig. 2. A completely disconnected formation and its connectivity graph.

by placing the robots further than δ from one another, as in fig 2. If $\mathcal{G} \in \mathcal{G}_N$ has many disjoint connected components, say $\{\mathcal{G}_i\}$, we can place each connected component "far away" from all other components so that none of the robots in one component can sense robots in other connected component. By this construction, we may have a realization for G if and only if all G_i have realizations in their respective spaces $G_{M_i,\delta}$. An example can be seen in Figure 3.

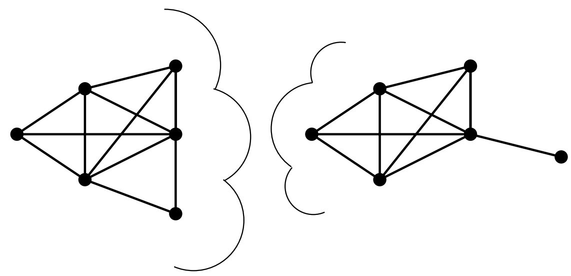

Fig. 3. Realization of a connectivity graph with 2 disconnected components.

Theorem 1. *$\mathcal{G}_{N,\delta}$ is a proper subset of $\mathcal{G}_N$ if and only if $N \geq 5$.*

Proof: In order to prove that $\mathcal{G}_{N,\delta}$ is a proper subset of $\mathcal{G}_N$ for some N, it is enough to show that $\Phi : C^N(\mathbb{R}^2) \to \mathcal{G}_N$ is not onto. Therefore we need to provide a graph $\mathcal{G} \in \mathcal{G}_N$ such that $\Phi^{-1}(\mathcal{G}) = \emptyset$. For $N \geq 6$ the star graphs $\mathcal{S}_N$ of the above proposition provide the examples of graphs that cannot be realized as connectivity graphs in $\mathcal{G}_{N,\delta}$. For $N = 5$, consider the first graph in Figure 1. It is easy to see that this graph is also not realizable as a connectivity graph in $\mathcal{G}_{5,\delta}$. (See [29] for details). This proves that $\mathcal{G}_{N,\delta}$ is a proper subspace of $\mathcal{G}_N$ if $N \geq 5$.

To prove that every graph in $\mathcal{G}_N$, for $N < 5$, is realizable in $\mathcal{G}_{N,\delta}$, we enumerate all possible graphs for $N < 5$ and give realizations for each graph. Since we are dealing with a small number ($N < 5$), the enumeration strategy works well. The number of possible graphs to check can be further reduced by noting that we need to consider only connected graphs. The justification for this comes from the previous two lemmas. From [4], we know that the number of all possible connected graphs are:

N	# connected graphs
1	1
2	1
3	2
4	6

The graphs and examples of their realizations are given in Figure 4, which completes the proof. ■

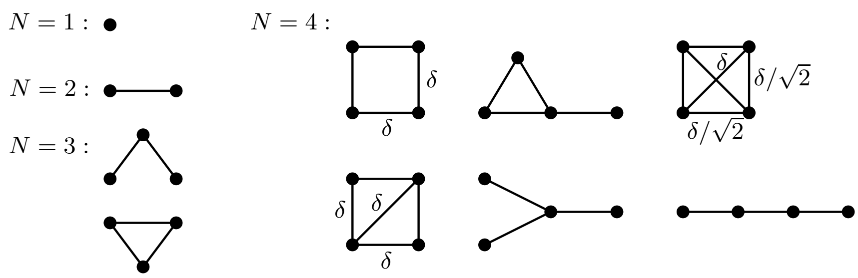

Fig. 4. Possible realizations for all connected $G \in \mathcal{G}_{N,\delta}$ for $N \leq 4$.

Corollary 1. *If each connected component $\mathcal{G}_i$ of a graph $\mathcal{G} \in \mathcal{G}_N$ belongs to $\mathcal{G}_{M_i}$, $M_i < 5$, then the graph has a realization in $\mathcal{G}_{N,\delta}$.*

It is thus clear that formations can produce a wide variety of connectivity graphs for N vertices. This includes graphs that have disconnected subgraphs or totally disconnected graphs with no edges. However the problem of coordinated control only becomes well-defined if no "sub-formations" of robots are totally isolated from the rest of the formation. This follows from the fact that there are no deterministic ways in which a "lost" robot (or group of robots) can be brought back within sensor range of the others in a completely decentralized system [19, 22]. This means that for all practical purposes, it reasonable to assume that the connectivity graph $\mathcal{G}(t)$ of the formation $\mathcal{F}(t)$ should always remain *connected* (in the sense of connected graphs) for all time t. For notational convenience, we use $\mathcal{G}^c_{N,\delta} \subseteq \mathcal{G}_{N,\delta} \subseteq \mathcal{G}_N$ to denote the set of all connected connectivity graphs.

3.2 Structure of Connectivity Graphs

It is interesting to see if the connectivity graphs possess any special structures that can be of use when coordinating multi-agent formations. In fact, we will show that connectivity graphs are made up of atomic graphs that can be combined to produce more complex graphs. This decomposition will prove to be helpful in the study of the topological properties of robot formations.

Definition 4 (Image of a Formation in $\mathbb{R}^2$). *If a given formation $\mathcal{F} = (x_1, x_2, \ldots x_N) \in \mathcal{C}^N(\mathbb{R}^2)$ has the connectivity graph $G = (\mathcal{V}, \mathcal{E}) = \Phi_N(\mathcal{F}(t))$, then each edge $e_k = \{v_{k_1}, v_{k_2}\} \in \mathcal{E}$ can be mapped to $\mathbb{R}^2$ by a map $f_k : \mathbb{R} \to \mathbb{R}^2$ given by $f_k(s) = sx_{k_1} + (1-s)x_{k_2}$, for $s \in [0,1]$. We call the image of the mapping f_k, the image of the edge in $\mathbb{R}^2$. The image of a formation, $I_\mathcal{F} \subset \mathbb{R}^2$, is defined as the union of the images of all edges in the connectivity graph of the formation:*

$$I_\mathcal{F} = \bigcup_{e_k \in \mathcal{E}} f_k([0,1]) \subset \mathbb{R}^2. \tag{8}$$

Note that this set is constructed by mapping each vertex v_i of the graph to its position x_i and each edge $e_k = \{v_{k_1}, v_{k_2}\}$ to the line segment between x_1 and x_2. If it is clear from the context what formation is under consideration, we will, with a slight abuse of notation, sometimes use I_G instead of $I_\mathcal{F}$.

Sometimes it will furthermore be convenient to describe the image of a subgraph $H = (\mathcal{E}_H, \mathcal{V}_H)$ of the connectivity graph G of formation $\mathcal{F}$. Here $\mathcal{E}_H \subset \mathcal{E}$ and $\mathcal{V}_H \subset \mathcal{V}$. In this case, we refer to the image of the subgraph H as

$$I_H = \bigcup_{e_k \in \mathcal{E}_H} f_k([0,1]) \subset \mathbb{R}^2, \qquad H \subseteq G = \Phi_N(\mathcal{F}) \in \mathcal{G}_{N,\delta}. \tag{9}$$

The image can thus be thought of as what a graph would "look like" if it was drawn in the plane. Note that this is different from the concept of planar graphs [15] or imbedding graphs in $\mathbb{R}^2$, where edges are not necessarily mapped to straight lines.

Definition 5 (Crossing Edges). *Two edges $e_i, e_j \in \mathcal{E}$ of the graph are said to be crossing if $f_i(s) = f_j(t)$ for some $s, t \in (0,1)$ and the set $f_i([0,1]) \cap f_j([0,1])$ has dimension* 0.

According to this definition, edge intersection at some vertex of the two edges does not count as a crossing. Moreover, the condition that the intersection set is of dimension 0 rules out edge intersections of collinear points.

It is interesting to note that the points in the image I_G can be categorized as smooth or non-smooth, where smoothness is defined in the setting of smooth manifolds. Any point x in the image that is not one of the robot positions $\{x_i\}_{i=1}^N$ or the crossing points is smooth, i.e. there always exists a small neighborhood $\mathcal{B}_\epsilon(x)$, and a homeomorphism $h : \mathcal{B}_\epsilon(x) \cap I_\mathcal{G} \to \mathbb{R}$.

Proposition 2. *An image of a formation of 4 vertices has a pair of crossing edges only if is isomorphic to either $\mathcal{U}_1, \mathcal{U}_2$, or $\mathcal{U}_3$, given in Figure 5.*

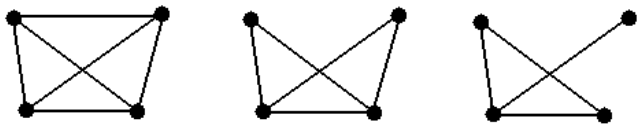

Fig. 5. The crossing generators $\mathcal{U}_1$,$\mathcal{U}_2$ and $\mathcal{U}_3$.

Proof: From the enumeration table for $N = 4$, and accompanying figures, it directly follows that only the graphs in Figure 5 can be realized with crossing edges. ■

The three graphs $\mathcal{U}_1$,$\mathcal{U}_2$,$\mathcal{U}_3$ are called the *crossing generators* of all connectivity graphs. Furthermore, if two edges, $e_i = (v_i, v_i')$ and $e_j = (v_j, v_j')$ of a connectivity graph are crossing in the image, then we say that their *crossing neighborhood vertices* is the set $\{v_i, v_i', v_j, v_j'\}$. Now, if G is the connectivity graph of a formation $\mathcal{F}$, and has the image $I_{\mathcal{F}}$, then the maximal subgraph spanned by each of its crossing neighborhood vertices is isomorphic to one of the crossing generators $\mathcal{U}_1$,$\mathcal{U}_2$ or $\mathcal{U}_3$. Hence it seems like connectivity graphs could be built by somehow combining crossing generators, which leads us to the standard definition of graph amalgamations [5]:

Definition 6 (Amalgamation of Graphs). *Let G and G' be two graphs and let $f : H \to H'$ be an isomorphism from a subgraph H of G to a subgraph H' of G'. The* amalgamation *of the two graphs, denoted as $G *_f G'$ is obtained from the union of G and G' and by identifying the subgraphs H and H' according to the isomorphism.*

Since the crossing generators have a special status as subgraphs of connected connectivity graphs, and they all contain K_3 as a subgraph, it seems natural to introduce the following amalgamation:

Definition 7 (Δ-Amalgamation of Crossing Generators). *If $\mathcal{U}_i$,$\mathcal{U}_j \in \mathcal{G}_4$ are two crossing generators, $H \subset \mathcal{U}_i$ and $H' \subset \mathcal{U}_j$ are subgraphs s.t. $H, H' \simeq \mathcal{K}_3$, and there is an isomorphism $\Delta : H \to H'$ between the respective subgraphs, then their amalgamation according to the isomorphism Δ is called a Δ-amalgamation, denoted by $\mathcal{U}_i *_\Delta \mathcal{U}_j$.*

We call this a "Δ-amalgamation" to signify that the amalgamation is taken over a triangular subgraph (shaped like a Δ), and it should be mentioned that there are several ways in which the Δ-amalgamation can be taken between any two crossing generators, depending on the choice of H and H'. In the context of connectivity graphs of formations, Δ - amalgamations are used to generate unions of the type, $\bigcup_i I_{G_i}$, where each $G_i \subseteq G$ is a valid connectivity graph in $\mathcal{G}_{4,\delta}$ and each $G_i \simeq \mathcal{U}_j$ for some $1 \leq j \leq 3$.

We see that the Δ-amalgamation $G_{i_1} *_\Delta G_{i_2}$ is well defined for connectivity graphs as long as $G_{i_1} *_\Delta G_{i_2}$ is a valid connectivity graph in $\mathcal{G}^c_{5,\delta}$. By generalizing this for an arbitrary number of amalgamations, the operation is well defined if

$$\underbrace{G_{i_1} *_\Delta G_{i_2} *_\Delta \dots G_{i_k}}_{(k-1)-amalgamations} \in \mathcal{G}^c_{4+k,\delta}. \tag{10}$$

The Δ-amalgamation operation for arbitrary graphs can be repeated to generate a whole family of graphs from the crossing generators. If we let $\Sigma = \sigma_1.\sigma_2.\dots.\sigma_K$ be a finite string defined over $\{1, 2, 3\}$, then we denote a member of this family as:

$$\mathcal{G}_\Sigma \simeq \mathcal{U}_{\sigma_1} *_\Delta \mathcal{U}_{\sigma_2} *_\Delta \dots *_\Delta \mathcal{U}_{\sigma_K}. \tag{11}$$

If we have repeated Δ-amalgamations of subgraphs of a connectivity graph, as is the case in Equation (10), there always exists a finite string Σ such that

$$\mathcal{G}_\Sigma \simeq G_{i_1} *_\Delta G_{i_2} *_\Delta \dots G_{i_k} \in \mathcal{G}^c_{4+k,\delta}. \tag{12}$$

An interesting question here would be to ask what connectivity graphs exist for an arbitrary string Σ. However, we leave this as a subject of future investigations.

Definition 8 (Atomic Crossing Graph). *Each well defined, repeated Δ-amalgamation, as defined in Equation (12), is called an Atomic Crossing Graph.*

We will denote the image of an atomic crossing graph as $I_{\mathcal{G}_\Sigma}$, by referring to its isomorphic graph $\mathcal{G}_\Sigma$, when the detail of the amalgamation is clear from the context.

Now, let $\mathcal{E}_\times \subseteq \mathcal{E}$ be the set of all crossing edges, and $\mathcal{V} \supseteq \mathcal{V}_\times = \{v \in \mathcal{V} \mid v \in e \text{ for some } e \in \mathcal{E}_\times\}$, then $H_\times = (\mathcal{E}_\times, \mathcal{V}_\times)$ is the subgraph of $G \in \mathcal{G}_{N,\delta}$ made up of all crossing edges of the connectivity graph of the formation $\mathcal{F} = (x_1, x_2, \dots x_N)$. We denote by $\overline{H}_\times$, the complement of $H_\times$, consisting of all non-crossing edges of the connectivity graph. I.e.

$$G = H_\times \cup \overline{H}_\times, \tag{13}$$

and we are now ready to state the following key proposition:

Proposition 3.

$$I_{H_\times} \subset \bigcup_{j\in J} I_{\mathcal{G}_{\Sigma_j}} \subseteq I_G \subset \mathbb{R}^2, \tag{14}$$

where J is some finite indexing set, and each $x \in I_{\mathcal{G}_{\Sigma_i}} \cap I_{\mathcal{G}_{\Sigma_j}} \setminus \{x_k\}_{k=1}^N$, for $i, j \in J$, is smooth.

Proof: See [29] for details.

This proposition reveals some important structural properties of connectivity graphs. It turns out that these properties are useful for obtaining a simplicial representation of connectivity graphs, which will subsequently help in understanding the topological shape of the formations. It is a well known fact from algebraic topology [16] that the study of topological shapes of compact closed manifolds is synonymous to the study of triangulations of those manifolds. These triangulations are called cell-complexes or simplicial complexes. A simplicial complex of dimension 1 can be thought of as an image of a connectivity graph with no crossing edges. The only non-smooth points in the complex are the images of the vertices. Therefore, if the crossing edges are removed from a connectivity graph, its image is a well-defined simplicial complex.

Now, Proposition 3 and Equation (13) lead to some interesting observations. All points in the image $I_{\overline{H}_\times}$ are smooth except for the vertex points. This makes $I_{\overline{H}_\times}$ a well defined simplicial complex of dimension 1. Therefore, the problem of obtaining a simplicial representation for the entire connectivity graph is reduced to finding such a representation for $I_{H_\times}$. If the image of each atomic crossing graph can be converted into a simplicial complex, by removal of images of crossing edges, then the union of the individual simplicial complexes would be a well-defined simplicial complex, as guaranteed by Proposition 3. In fact, in [29] it is shown that each G_Σ contains a "maximal simplicial structure", denoted by G^*_Σ (which is a subgraph of G_Σ), so that we can obtain a maximal simplicial subgraph G^* in G as

$$G^* = \left(\bigcup_i G^*_{\Sigma_i} \right) \cup \overline{H}_\times \subseteq \left(H_\times \cup \overline{H}_\times \right) = G. \tag{15}$$

We can further add simplicial structure to this, by gluing 2-simplexes to each triangular cycle, in which case we obtain a *maximal simplicial complex*, spanned by the connectivity graph. An example of such a construction is shown in Figures 6-7. Here, the connectivity graph of a randomly generated formation is shown together with its maximal simplicial complex.

This maximal simplicial complex is the object associated with the topological shape of the formation. This object can thus be studied using tools from standard algebraic topology to obtain its genus, fundamental groups, homological groups, etc. Hence we have the tools for associating all of these topological properties with a certain formation, which circumvents the difficulty associated with studying the topologies of semi-algebraic varieties directly.

4 Future Work: Global Behaviors From Local Rules

The introduction of connectivity graphs for characterizing the local interactions between robots in multi-agent formations serves two purposes. First, since these interactions imply constraints on the movements of the individual robots, it is vitally important that the set of feasible formations can be characterized in a precise manner. Secondly, and perhaps more importantly,

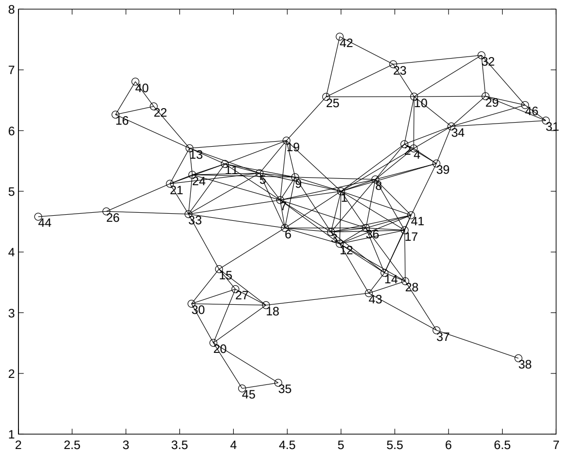

Fig. 6. Connectivity graph.

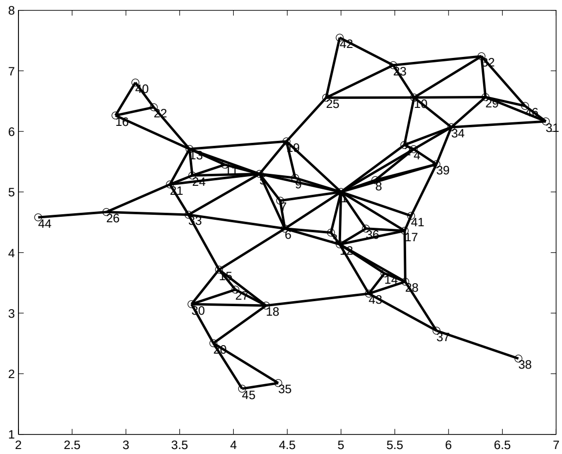

Fig. 7. The maximal simplicial subgraph of the graph in Figure 6.

these graphs provide guidance as to how the information should flow between different agents in order for the team of robots to come up with plans for achieving global objectives in a decentralized manner. These plans may include descriptions of what robot should take on what role in the formation, how it should move to achieve this, as well as what formation to use in the first place.

In [1], the situation was investigated where the robots were moving in such a way as to minimize some given formation error at the same time as a short-range obstacle-avoidance "behavior" was employed for negotiating obstacles. Since the particular formation currently in force can be thought of as the formation best suited for a particular environment, one can easily picture a scenario in which different formations should be used at different times. As an example, consider the way in which schooling fish switch from a spread formation to a much tighter one when traversing narrow passages. In [1], the measure of how well as certain formation was suited to the environment was a function of how much the obstacle-avoidance behavior deformed that particular formation. If one defines a formation error E_i as a positive definite measure of how much the robots deviate from formation i, a simple switch-law would be to switch to formation i if and only if $E_i < E_j, \ \forall j \neq i$. An example of this is shown in Figure 8, where three robots start out in a triangular formation. But, as the obstacles force the robots together, a line formation becomes better suited to the environment. (The technical details of this can be found in [1].)

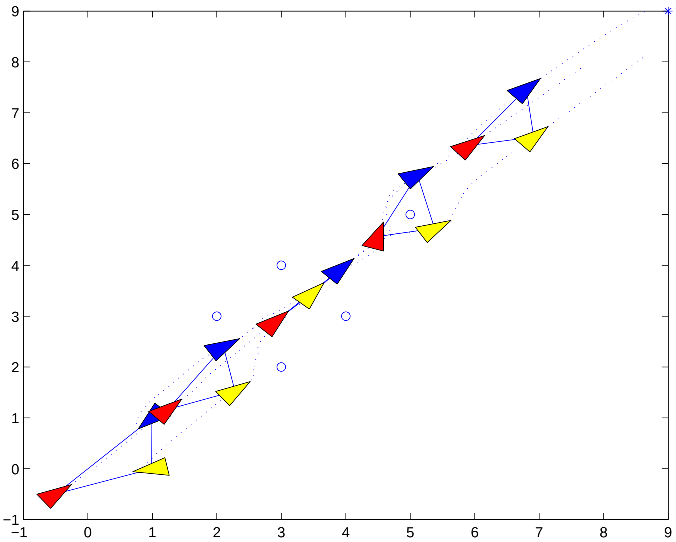

Fig. 8. Three robots switching between triangular and line formations.

However, these results were obtained under the assumption of perfect and complete information sharing, i.e. the individual robots knew the positions of all other robots as well as what formation was currently in force. But, as we have argued for repeatedly in this paper, this is an unrealistic and non-scalable assumption in a number of multi-agent applications. The key question is thus how such a result can be transferred to the decentralized setting. It is clear that some information must be shared between neighboring robots in order for them to achieve the desired global behavior. And, as the connectivity graphs provide us with a description of what robots an individual robot can interact with, it would be useful to complement this modeling formalism with a formalism in which it can be understood exactly what information should be shared.

In [23], Klavins investigates the *communication complexity* associated with different communication schemes, which provides one measure of how successful a particular communication-based coordination scheme is. Similarly, Olifati-Saber *et. al.* propose graph-based agreement protocols in [33] for maintaining formations. Similar results can be obtained for the connectivity graphs. An example of this is a recently developed decentralized algorithm for finding Hamilton paths through connectivity graphs of genus zero. Such paths are paths through the graph that visits every vertex exactly once, which provides a useful starting point when moving the robots into a given formation, as shown in [29]. However, much work remains to be done in this area, and in this paper we have merely presented a formalism in which local interactions can be modeled, and communication strategies can be analyzed, in a straight forward manner.

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