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A Direct Algebraic Approach to Observer Design Under Switching Measurement Equations

Mohamed Babaali, Magnus Egerstedt, and Edward W. Kamen

***Abstract*—Based on the algebraic transformation of a switched linear measurement equation into a nonlinear, yet deterministic, equation, an asymptotic state observer is constructed for discrete-time linear systems whose observations are generated according to randomly switching measurement modes. The observer, which combines the algebraic transformation with a Newton observer applied to the resulting nonlinear measurement equation, is shown to be locally exponentially convergent under arbitrary mode sequences.**

***Index Terms*—Observer design, sensor failures, switched systems.**

I. INTRODUCTION

The emergence of increasingly complex engineering systems has triggered an intense focus on novel control-theoretic areas of research, including sensor and actuator networks, decentralized control, and fault-tolerant control. In order for such complex systems to behave in a satisfactory manner, i.e., to be subjected to effective control strategies, it is vitally important that the measured sensory data be incorporated in the control loop under various forms of unreliability. In particular, in a number of applications, including manufacturing, telecommunications, and embedded systems, sensor failures occur intermittently and go undetected, while only a finite number of possible sensory modes of operation exist, and are known. In other words, even though it is unknown which mode of operation the sensors obey to at any given time instant, a characterization of all possible sensory modes is assumed available *a priori*. In this note, we consider the particular class of discrete-time linear dynamical systems with randomly switching measurement equations, and we propose a local exponential state observer for such systems.

In other words, we consider the single-output autonomous system

$$\begin{aligned} x_{k+1} &= Ax_k \\ y_k &= C(\theta_k)x_k \end{aligned} \tag{1}$$

where x_k and y_k are in $\mathbb{R}^n$ and $\mathbb{R}$, respectively, where the *mode* θ_k takes values in $\{1, \ldots, m\}$, and where $A, C(1), \ldots, C(m)$ are constant matrices of compatible dimensions. We assume that the mode sequence $\{\theta_k\}_{k=0}^{\infty}$ is arbitrary, indexing the measurement equation in such a way that $C(\theta_k)$ switches randomly among $C(1), \ldots, C(m)$, modeling the m different sensory modes. Throughout the note, we will further assume that A is invertible, which is a natural assumption for sampled linear systems. In fact, it is easily shown (e.g., [6]) that sampling a continuous-time linear system with *arbitrarily* switched measurement equations actually results in a switched linear system (1), which is not

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true if the system dynamics, i.e., the A matrix, undergoes switching as well.

It should be stressed that it is unknown which one of the m different measurement equations is in effect at any given time instant. In other words, we assume the mode sequence $\{\theta_k\}_{k=0}^{\infty}$ to be unknown, even though the set of possible measurement matrices $C(1), \ldots, C(m)$ is known. The goal of this work can now be stated as follows. Devise an asymptotic state observer for (1), assuming the switching sequence is arbitrary and unknown.

It appears that this problem has never been successfully addressed in a systematic manner. When the mode sequence is observed, it is well known (e.g., [10]) that a Kalman filter can, under some conditions, be used as an observer for (1), and recently, an linear matrix inequality-based approach has been proposed for designing Luenberger-like switching observers [2]. These results are not pertinent to this work because of the known-modes assumption. However, capitalizing on the latter approach and on failure detection techniques, an observer design methodology was proposed in [7]. Unfortunately, failure detection schemes require θ to be slowly varying, which is too restrictive for the problem at hand. Finally, note that the finite-time observability problem for switched linear systems has lately experienced a surge of interest [5], [14], though this line of work does not provide any direct guidance as to the construction of asymptotic observers.

The outline of this note is as follows. We introduce the direct algebraic approach (DAA) in Section II, and we construct the DAA-Newton observer in Section III. In Section IV, we analyze some geometric aspects of the observer, which enable us to prove its local exponential convergence in Section V.

II. DAA

In this section, we present the DAA, which was originally proposed in [9], and recently generalized to (1) in [3]. It is based on the following equation:

$$(y_k - C(1)x_k)\cdots(y_k - C(m)x_k) = 0 \tag{2}$$

which follows from $y_k - C(\theta_k)x_k = 0$ [which is the measurement equation in (1)] and from $\theta_k \in \{1, \ldots, m\}$. Now, defining the polynomial form g_k as

$$g_k(x) \triangleq (y_k - C(1)x)\cdots(y_k - C(m)x) \tag{3}$$

we have $g_k(x_k) = 0$, which can be viewed as an alternate measurement equation for (1). The DAA lies precisely in shifting our attention to designing an observer for the system thus obtained, that is

$$\begin{aligned} x_{k+1} &= Ax_k \\ g_k(x_k) &= 0 \end{aligned} \tag{4}$$

where $g_k(x_k) = 0$ is the new nonlinear, time-varying, yet deterministic measurement equation given in implicit form. Indeed, g_k is a deterministic polynomial form whose coefficients are determined by the available measurement y_k. Clearly, the uncertainty associated with the randomly switched measurement equation has been removed, and the need to determine θ_k circumvented. Note that a similar idea has been successfully applied to the data association problem in multiple target tracking, leading to the so-called "symmetric measurement equations" filter [8]. Similar ideas can also be found in fault detection and isolation [16], [1], and in system identification for multimodal systems [13], [15].

Unfortunately, the transformation of (1) into (4) does come with a price. The price one has to pay for the introduction of a nonlinear measurement equation is that local convergence is in general all one can hope for. In the next section, we complete the construction by combining the DAA with a nonlinear observer, thus obtaining an observer for our original system (1).

III. DAA-Newton Observer

In [12], a nonlinear observer design approach, which we refer to as the Newton observers approach, was proposed. As we will see, a Newton observer can successfully be combined with the DAA. The key idea is to fix an integer $N_B \geq n$, defined as the "block size," and to define a new measurement map G_k as $G_k(x) \triangleq (g_k(x)\ldots g_{k+N_B-1}(A^{N_B-1}x))^T$, or by recalling the expression of g_k in (3)

$$G_k(x) = \begin{pmatrix} \prod_{i=1}^{m}(y_k - C(i)x) \\ \vdots \\ \prod_{i=1}^{m}(y_{k+N_B-1} - C(i)A^{N_B-1}x) \end{pmatrix}. \tag{5}$$

We can now define the DAA-Newton observer for (1) as

$$\hat{x}_k^- = A\hat{x}_{k-1} \tag{6}$$

$$\hat{x}_k = \hat{x}_k^- - (G'_k(\hat{x}_k^-))^{\dagger}(G_k(\hat{x}_k^-)) \tag{7}$$

where $G'_k(x)$ is the Jacobian of $G_k(x)$, and where $J^{\dagger}$ is defined for any full-column-rank matrix J as

$$J^{\dagger} \triangleq (J^TJ)^{-1}J^T \tag{8}$$

and coincides with the pseudoinverse of J. This implies that $G'_k(\hat{x}_k^-)$ must have full column rank, and sufficient conditions for this to be satisfied are given in the next section.

Noting that

$$G_k(x_k) = 0 \tag{9}$$

which can be used as a new measurement equation replacing $g_k(x_k) = 0$ in (4), thus yielding the following *augmented* system:

$$\begin{aligned} x_{k+1} &= Ax_k \\ G_k(x_k) &= 0 \end{aligned} \tag{10}$$

the observer given by (6)–(7) appears as a direct interpretation of the Newton observers approach applied to (10): Equation (7), the "corrector" part of the observer, materializes a single iteration of Newton's method on (9) using $\hat{x}_k^-$ as the initial estimate of the root of G_k, exhibiting the "map inversion" viewpoint of [12]. The motivation behind the construction of G_k in (5) thus becomes obvious: Newton's method cannot be shown to converge to x_k if (9) is underdetermined, hence, the condition $N_B \geq n$.

Note that, by construction of G_k, future measurements must be available, i.e., the measurements $y_k, \ldots, y_{k+N_B-1}$ must be available for the computation of $\hat{x}_k$. This limitation can easily be overcome by adding a predictor after (7), i.e., by letting $z_k = A^{N_B-1}\hat{x}_{k-N_B+1}$ be the estimate of x_k, which results in a causal observer. For the sake of simplicity, we will study the convergence of $\hat{x}_k$ rather than that of z_k, and we observe that z_k will converge to x_k whenever $\hat{x}_k$ does, since

$$\|z_k - x_k\| \leq \|A\|^{N_B-1}\|\hat{x}_{k-N_B+1} - x_{k-N_B+1}\|. \tag{11}$$

IV. NONDEGENERACY

In this section, we address the following question: When does $G'_k(x)$ have full-column rank, so that its left inverse $G'_k(x)^\dagger$ exists, enambling the implementation of the DAA-Newton observer (6)–(7)? $G'_k(x)$, which can be expressed as

$$G'_k(x) = -\sum_{i=1}^{m} \begin{pmatrix} \left(\prod_{j\neq i} y_k - C(j)x\right) C(i) \\ \vdots \\ \left(\prod_{j\neq i} y_{k+N_B-1} - C(j)A^{N_B-1}x\right) C(i)A^{N_B-1} \end{pmatrix} \tag{12}$$

has been difficult to analyze, unless evaluated at $x = x_k$, in which case replacing y_k with $C(\theta_k)x_k$ in (12), its expression simplifies to

$$G'_k(x_k) = -\begin{pmatrix} \left(\prod_{j\neq\theta_k} (C(\theta_k) - C(j))x_k\right) C(\theta_k) \\ \vdots \\ \left(\prod_{j\neq\theta_{k+N_B-1}} (C(\theta_{k+N_B-1}) - C(j))A^{N_B-1}x_k\right) \\ \times C(\theta_{k+N_B-1})A^{N_B-1} \end{pmatrix}. \tag{13}$$

Actually, all we will study is $G'_k(x_k)$, and we will deduce that $G'_k(x)$ has full rank for x close enough to x_k, by taking advantage of the smoothness of G'_k. By analogy to scalar polynomials, we define *nondegeneracy* as follows.

Definition 1: x is a nondegenerate root of G_k if $G_k(x) = 0$ and $G'_k(x)$ has full-column rank. ♦

Note that nondegeneracy implies that x is the unique solution of $G_k(x) = 0$ in a neighborhood of x. The objective of this section thus becomes to establish necessary and sufficient conditions for x_k to be guaranteed to be a nondegenerate root of G_k for all $k \geq 0$, all initial states $x_0 \neq 0$, and all mode sequences $\{\theta_k\}_{k=0}^{\infty}$. Before that, we need to establish some notation and state some assumptions, which are the focus of the next paragraph.

Let a path θ of length N be a mode string $\theta_1\theta_2\ldots\theta_N$ with values in $\{1,\ldots,m\}$, let its length be denoted by $|\theta| = N$, and let Θ_N be the set of all m^N paths of length N. Two paths θ and θ' of the same length N are said to be disjoint if $\theta_k \neq \theta'_k\ \forall k \in \{1,\ldots,N\}$. Define the observability matrix $\mathcal{O}(\theta)$ of a path θ of length N as

$$\mathcal{O}(\theta) \triangleq \begin{pmatrix} C(\theta_1) \\ \vdots \\ C(\theta_N)A^{N-1} \end{pmatrix} \tag{14}$$

and the function $\mathcal{P}$ of a pair of paths θ^1 and θ^2 as

$$\mathcal{P}(\theta^1, \theta^2) \triangleq \mathcal{O}(\theta^1) - \mathcal{O}(\theta^2). \tag{15}$$

For a fixed integer N and $I \in 2^{\{1,\ldots,N\}}$ with $\mathrm{card}(I) = N'$, we further define the mapping

$$\phi_I\colon \mathbb{R}^{N\times n} \to \mathbb{R}^{N'\times n} \tag{16}$$

as the mapping extracting the submatrix of an $N \times n$ matrix corresponding to the lines indexed by I. If $I = \emptyset$, then ϕ_I outputs a matrix of rank zero, i.e., some zero matrix. We now state the following assumption for further analysis.

Assumption 1: Given an integer N, assume that for any pair of disjoint paths $\theta, \theta' \in \Theta_N$, and any $I \in 2^{\{1,\ldots,N\}}$

$$\mathrm{rank}(\phi_I(\mathcal{P}(\theta,\theta'))) < n \Rightarrow \mathrm{rank}(\phi_{\bar{I}}(\mathcal{O}(\theta))) = n \tag{17}$$

where $\bar{I}$ is the complement of I in $\{1,\ldots,N\}$. ♦

We can now state the main result of this section.

Proposition 1: x_k is a nondegenerate root of G_k for all $k \geq 0$, all initial states $x_0 \neq 0$, and all mode sequences $\{\theta_k\}_{k=0}^{\infty}$, if and only if Assumption 1 is satisfied at the block size (i.e., with $N = N_B$), and A is invertible. ♦

Note that if $x_k = 0$, then $G'_k(x_k) = 0$. Proposition 1, guaranteeing nondegeneracy everywhere except at the origin, is therefore the best that can be done. Note that Assumption 1 is decidable when N is fixed. All one has to do is check (17) for all pairs of disjoint paths $\theta, \theta' \in \Theta_N$ and all elements I of $2^{\{1,\ldots,N\}}$. However, it turns out that this is unnecessary. Instead, it is easily shown that it suffices to check (17) for the maximal[1] elements I of $2^{\{1,\ldots,N\}}$ satisfying $\mathrm{rank}(\phi_I(\mathcal{P}(\theta,\theta'))) < n$. Indeed, if $I' \subset I$, then $\mathrm{rank}(\phi_{I'}(\mathcal{P}(\theta,\theta'))) \leq \mathrm{rank}(\phi_I(\mathcal{P}(\theta,\theta')))$ and $\mathrm{rank}(\phi_{\bar{I}'}(\mathcal{O}(\theta))) \geq \mathrm{rank}(\phi_{\bar{I}}(\mathcal{O}(\theta)))$. Unfortunately, the decidability of determining whether a block size N exists satisfying Assumption 1, which is ultimately what will be required, is still an open question. The simpler problem of determining the existence of an integer N such that every path θ of length N satisfies $\mathrm{rank}(O(\theta)) = n$, i.e., determining whether (1) is *pathwise observable*, has been shown to be decidable in [4]. Furthermore, sufficient conditions for pathwise observability, including a nonpathological sampling criterion, have been established in [6]. What we do know, however, is that if Assumption 1 is satisfied at some N, then it will be satisfied at any $N' > N$. This can be deduced from Lemma 1 below, but can also easily be established directly.

Before proving Proposition 1, we first establish the following lemma, for which we need to define the parameterized function $\mathcal{G}_{\theta,x^*}$ as follows ($|\theta| = N$):

$$\mathcal{G}_{\theta,x^*}(x) \triangleq \begin{pmatrix} \prod_{i=1}^{m}(C(\theta_1)x^* - C(i)x) \\ \vdots \\ \prod_{i=1}^{m}(C(\theta_N)A^{N-1}x^* - C(i)A^{N-1}x) \end{pmatrix} \tag{18}$$

and we note that, letting $\theta^k \triangleq \theta_k\cdots\theta_{k+N_B-1}$, we get

$$G_k(x) = \mathcal{G}_{\theta^k,x_k}(x). \tag{19}$$

Lemma 1: x is a nondegenerate root of $\mathcal{G}_{\theta,x}$ for all $\theta \in \Theta_N$ and all $x \neq 0$ if and only if Assumption 1 is satisfied at N. ♦

Proof: We first give the expression of the Jacobian $\mathcal{G}'_{\theta,x}$ of $\mathcal{G}_{\theta,x}$ evaluated at x

$$\mathcal{G}'_{\theta,x}(x) = -\begin{pmatrix} \left(\prod_{j\neq\theta_1}(C(\theta_1) - C(j))x_k\right) C(\theta_1) \\ \vdots \\ \left(\prod_{j\neq\theta_N}(C(\theta_N) - C(j))A^{N-1}x_k\right) C(\theta_N)A^{N-1} \end{pmatrix}. \tag{20}$$

Now, suppose that there exist $x \neq 0$ and $\theta \in \Theta_N$ such that $\mathrm{rank}(\mathcal{G}'_{\theta,x}(x)) < n$. Let $I \in 2^{\{1,\ldots,N\}}$ be the set indexing the rows of $\mathcal{G}'_{\theta,x}(x)$ equal to zero. It follows from (20) that there exists a path $\theta' \in \Theta_N$ disjoint from θ such that $\phi_I(\mathcal{P}(\theta,\theta'))x = 0$, and since $x \neq 0$, such that $\mathrm{rank}(\phi_I(\mathcal{P}(\theta,\theta'))) < n$. Moreover, since $\bar{I}$ indexes the only nonzero rows of $\mathcal{G}'_{\theta,x}(x)$, it follows, once again from (20), that $\mathrm{rank}(\mathcal{G}'_{\theta,x}(x)) = \mathrm{rank}(\phi_{\bar{I}}(\mathcal{O}(\theta)))$, and therefore that $\mathrm{rank}(\phi_{\bar{I}}(\mathcal{O}(\theta))) < n$

Conversely, suppose the existence of a pair of disjoint paths $\theta, \theta' \in \Theta_N$, and that of a set $I \in 2^{\{1,\ldots,N\}}$ such that $\mathrm{rank}(\phi_I(\mathcal{P}(\theta,\theta'))) < n$ and $\mathrm{rank}(\phi_{\bar{I}}(\mathcal{O}(\theta))) < n$. Let x be any nontrivial vector in the nontrivial null space of $\phi_I(\mathcal{P}(\theta,\theta'))$. Then, as noted earlier, $\mathrm{rank}(\mathcal{G}'_{\theta,x}(x)) = \mathrm{rank}(\phi_{\bar{I}}(\mathcal{O}(\theta))) < n$. ■

[1]Maximality is understood here in the set inclusion sense.

We can now easily prove Proposition 1.

Proof: Suppose first that Assumption 1 is satisfied at N_B and that A is invertible. Then, for all $\{\theta_k\}_{k=0}^{\infty}$, if $x_0 \neq 0$, then $x_k \neq 0$ for all $k \geq 0$, and Lemma 1, with the help of (19), implies that x_k is a nondegenerate root of G_k for all $k \geq 0$.

Conversely, if A is singular, then there exists $x_0 \neq 0$ such that $x_1 = Ax_0 = 0$, which can only be degenerate. On the other hand, if instead Assumption 1 is not satisfied, then, again by Lemma 1 and (19), there exist $x_0 \neq 0$ and $\{\theta_k\}_{k=0}^{\infty}$ such that x_0 is a degenerate root of G_0. ■

V. Convergence

In this section, we show that the DAA-Newton observer (6)–(7) results in a local exponential observer for (1). We have the following.

Theorem 1: Assume that Assumption 1 is satisfied at the block size and that A is invertible. Then, whenever $x_0 \neq 0$, the DAA-Newton observer (6)–(7) results in a local exponential observer for (1), in the sense that there exists $\delta > 0$ and $\alpha < 1$ such that

$$\|\hat{x}_{k+1} - x_{k+1}\| \leq \alpha \|\hat{x}_k - x_k\| \tag{21}$$

for all $k \geq 0$ whenever $\|x_0 - \hat{x}_0^-\| \leq \delta$. ♦

Note that our definition of a local exponential observer does not imply that the rate α and radius δ of convergence are uniform over the entire state space: α and δ in Theorem 1 depend on x_0.

Before embarking on proving Theorem 1, a few comments are in order. First, the main role of the invertibility of A is to keep the states x_k away from the origin, so that $G_k'(x_k)$ can be guaranteed to be nonsingular for all $k \geq 0$ by Proposition 1, so that, in turn, $G_k'(\hat{x}_k^-)^\dagger$ can be guaranteed to exist for $\hat{x}_k^-$ close enough to x_k. In short, the need to keep the state away from the origin arises from the fact that we require $G_k'(x_k)$, rather than $G_k'(\hat{x}_k^-)$, to be of full-column rank. It deserves to be noted that if $x_0 = 0$, then $y_k = 0$ for all $k \geq 0$, regardless of the mode sequence. If the system is pathwise observable, which is actually implied by Assumption 1, then $x = 0$ will be the only possible solution to the observation problem, so this case can be dealt with separately.

Here, we first prove an essential lemma in Section V-A, before detailing the proof in Section V-B. For the remainder of the note, the Euclidean (or induced Euclidean) norm is denoted by $\|\cdot\|$, and the open ball of radius r centered on x by $B(x, r)$. The pth differential of a function G is written $G^{\{p\}}$, but we will sometimes write $G' = G^{\{1\}}$ and $G'' = G^{\{2\}}$. Finally, by $G \in \mathcal{C}^P(\mathbb{R}^n)$, we mean that G is p times continuously differentiable over $\mathbb{R}^n$.

A. Newton Observers

Consider the nonlinear system

$$\begin{aligned} x_{k+1} &= f(x_k) \\ G_k(x_k) &= 0 \end{aligned} \tag{22}$$

where the time-varying measurement map G_k is either square or overdetermined (i.e., $G_k : \mathbb{R}^n \to \mathbb{R}^N, N \geq n$), so that we can define a Newton observer for (22) as follows:

$$\hat{x}_k^- = f(\hat{x}_{k-1}) \tag{23}$$

$$\hat{x}_k = \mathcal{N}_k(\hat{x}_k^-) \tag{24}$$

where $\mathcal{N}_k$ is given by

$$\mathcal{N}_k(x) \triangleq x - (G_k'(x))^\dagger G_k(x). \tag{25}$$

In [12], the observer (23)–(24) was shown to be locally convergent for time-invariant systems that are controlled-invariant with respect to a compact set. Here, we present an extension of that result to a class of time-varying, possibly unstable autonomous systems described by (22). Furthermore, the following result does not require the strong observability assumptions of [12]. In particular, in [12], the augmented measurement equation is assumed to have a unique global solution (i.e., x such that $G_k(x) = 0$), which is, in general, not true for all $x_k \in \mathbb{R}^n$ if G_k is defined as in (5). This global observability condition therefore needs to be relaxed to a local observability condition. Moreover, the *observability rank condition* of [12] also has to be relaxed in a similar way. The following lemma incorporates these modifications. As can be anticipated, we will prove Theorem 1 by showing that the system in (10) satisfies the requirements of the following lemma, whose proof is given in the Appendix.

Lemma 2: Consider the system in (22). First, assume that f and $G_k, k \geq 0$, are in $\mathcal{C}^3(\mathbb{R}^n)$, and that f is globally L-Lipschitz (i.e., for all x, y in $\mathbb{R}^n$, $\|f(x) - f(y)\| \leq L\|x - y\|$). Furthermore, assume that given $x_0 \in \mathbb{R}^n$, there exists a sequence R_k of subsets of $\mathbb{R}^n$ such that

1) $x_k \in R_k, k \geq 0$;
2) defining d_k as $\mathrm{dist}(x_k, R_k^c)$, where R_k^c is the complement of R_k in $\mathbb{R}^n$, there exists $\beta > 0$ such that $d_{k+1} \geq \beta d_k > 0$;
3) and finally
a) $\exists g_p > 0, \gamma_p > 0$ such that $\sup_{x \in R_k} \|G_k^{\{p\}}(x)\| \leq g_p \gamma_p^k, p \in \{1, 2, 3\}$;
b) $\exists g_\dagger > 0, \gamma_\dagger > 0$ such that $\|(G_k'(x_k))^\dagger\| \leq g_\dagger \gamma_\dagger^k$.

Then there exist $c > 0$ and $\nu > 0$ such that $(1/2)\sup_{x \in X_k} \|\mathcal{N}_k''(x)\| \leq c\nu^k$, where $X_k = \{x \in R_k \,|\, \|(G_k'(x))^\dagger\| \leq 2g_\dagger \gamma_\dagger^k\}$, and moreover, the observer given by (23)–(24) results in a local exponential observer for (22), as if

$$\|\hat{x}_0^- - x_0\| \leq \delta, \qquad \text{then} \tag{26}$$

$$\|\hat{x}_{k+1} - x_{k+1}\| \leq \alpha \|\hat{x}_k - x_k\| \tag{27}$$

for all $k \geq 0$, whenever α and δ satisfy

- $0 < \alpha \leq \min\{\beta, (1)/(\gamma_\dagger \gamma_2), (1)/(\gamma_\dagger^2 \gamma_1 \gamma_2), (\beta)/(\gamma_\dagger \gamma_1), (1/\nu)\}$;
- $0 \leq \delta < \min\{(d_0)/(2), (1)/(4 g_\dagger g_2), (1)/(8 g_\dagger^2 g_1 g_2), (d_0)/(4 g_\dagger g_1), (\alpha / cL)\}$. ♦

B. Proof of Theorem 1

The proof lies in showing that the system in (10) satisfies the requirements for Lemma 2, under the assumptions of Theorem 1. Lemma 2 then allows one to select $\alpha < 1$, then $\delta > 0$, such that (27) holds for all $k \geq 0$ whenever (26) does.

First of all, the dynamics being linear and G_k being polynomial in the state, they are both in $\mathcal{C}^3(\mathbb{R}^n)$. Moreover, A being invertible, there exist $l > 0$ and $L > 0$ such that

$$l\|x\| \leq \|Ax\| \leq L\|x\| \tag{28}$$

$\forall x \in \mathbb{R}^n$. This implies that the dynamics is L-Lipschitz.

Next, since $x_0 \neq 0$, there exist $r_0 > 0, r_0' > 0$ such that $r_0 < \|x_0\| < r_0'$. Letting $r_k = r_0 l^k$ and $r_k' = r_0' L^k$, we get $x_k \in R_k, k \geq 0$, where $R_k \triangleq \{x \in \mathbb{R}^n \,|\, r_k < \|x\| < r_k'\}$. Clearly, $d_{k+1} \geq l d_k > 0$. It now remains to prove that conditions 3a) and 3b) in Lemma 2 are met.

At this point, we need to integrate the fact that the DAA-Newton observer converges for arbitrary mode sequences $\{\theta_k\}_{k=0}^{\infty}$. Recalling that $G_k(x) = \mathcal{G}_{\theta^k, x_k}(x)$, where $\theta^k \triangleq \theta_k \cdots \theta_{k+N_B-1}$, we get

$$\sup_{x \in R_k} \|G_k^{\{p\}}(x)\| \leq \max_{\theta \in \Theta_{N_B}} \sup_{x^* \in R_k} \sup_{x \in R_k} \|\mathcal{G}_{\theta, x^*}^{\{p\}}(x)\| \tag{29}$$

for $p \in \{1, 2, 3\}$, and we can therefore focus on bounding the right-hand side of (29) in proving that condition 3a) is met for any $\{\theta_k\}_{k=0}^{\infty}$. Since $r_k' = r_0' L^k$, and since $\mathcal{G}_{\theta, x^*}^{\{p\}}(x)$ is polynomial in x^* and x, it is straightforward to show that there exist $g_p > 0$ and $\gamma_p > 0, p \in \{1,2,3\}$, such that

$$\max_{\theta\in\Theta_{N_B}} \sup_{\|x^*\|\le r'_k} \sup_{\|x\|\le r'_k} \left\|\mathcal{G}^{\{p\}}_{\theta,x^*}(x)\right\| \le g_p\gamma_p^k \tag{30}$$

for $k \ge 0$, and since $R_k \subset \{x \,|\, \|x\| \le r'_k\}$, we get

$$\sup_{x\in R_k} \left\|G^{\{p\}}_k(x)\right\| \le g_p\gamma_p^k. \tag{31}$$

As for condition 3b), we have

$$\|(G'_k(x_k))^\dagger\| \le \max_{\theta\in\Theta_{N_B}} \sup_{x\in R_k} \|(\mathcal{G}'_{\theta,x}(x))^\dagger\|. \tag{32}$$

Lemma 1 tells us that $(\mathcal{G}'_{\theta,x}(x))^\dagger$ is defined, and therefore continuous (since it is rational in the entries of x) over the entire unit sphere. Therefore, since the unit sphere in $\mathbb{R}^n$ is compact, we get that

$$\mathcal{H} \triangleq \max_{\theta\in\Theta_{N_B}} \sup_{\|x\|=1} \|(\mathcal{G}'_{\theta,x}(x))^\dagger\| \tag{33}$$

is finite. Consequently, observing that $\|(\mathcal{G}'_{\theta,rx}(rx))^\dagger\| = (1)/(r_k^{m-1})\|(\mathcal{G}'_{\theta,x}(x))^\dagger\|$ for any $r \in \mathbb{R}^*$

$$\max_{\theta\in\Theta_{N_B}} \sup_{x\in R_k} \|(\mathcal{G}'_{\theta,x}(x))^\dagger\| = \frac{\mathcal{H}}{r_k^{m-1}} \tag{34}$$

hence, 3b) with $g_\dagger = (1)/(\mathcal{H}r_0^{m-1})$ and $\gamma_\dagger = (1)/(l^{m-1})$. ■

VI. Conclusion

An observer design approach has been presented for linear discrete-time systems with randomly switching measurement equations, and has been shown to produce local exponential observers under arbitrary switching.

Appendix

We first state the following standard result (adapted from [11, pp. 279–281, p. 309]), which establishes the convergence of Newton's method.

Theorem 2: Let G be a mapping from $\mathbb{R}^n$ to $\mathbb{R}^N$, where $N \ge n$, and assume that $G \in \mathcal{C}^3(\mathbb{R}^n)$. Assume further the following.

1) There is a point $x_1 \in X$ such that $(G'(x_1))^\dagger$ exists with $\|(G'(x_1))^\dagger\| \le \beta$ and $\|(G'(x_1))^\dagger G(x_1)\| \le \eta$.
2) There exists $r \ge 2\eta$ such that $\sup_{x\in T} \|G''(x)\| \le K$, where $T = B(x_1, r)$.
3) The constant $h = \beta\eta K$ satisfies $h < (1/2)$.

Then the sequence $x_{n+1} = \mathcal{N}(x_n) \triangleq x_n - (G'(x_n))^\dagger G(x_n)$ of successive approximations generated by Newton's method exists for all $n \ge 1$, remains in T, and converges to a solution of $G(x) = 0$. Moreover, the rate of convergence is given by

$$\|x_{n+1} - x^*\| \le \mu\|x_n - x^*\|^2 \tag{35}$$

where $\mu = (1/2)\sup_{x\in T}\|\mathcal{N}''(x)\|$. ♦

We next state the following straightforward lemma without proof.

Lemma 3: Let $G : R \to \mathbb{R}^N$, where R is an open subset of $\mathbb{R}^n$ and $N \ge n$. Assume that $G \in \mathcal{C}^2(R)$, and that there exists $x^* \in R$ such that $G(x^*) = 0$. Assume further that there exist two positive scalars $g_\dagger$ and g_2 such that $\|(G'(x^*))^\dagger\| \le g_\dagger$ and $\sup_{x\in R} \|(G''(x))\| \le g_2$, and let $r \le \min\{\mathrm{dist}(x^*, \bar{R}), (1)/(2g_\dagger g_2)\}$. then $\sup_{x\in B(x^*,r)} \|(G'(x))^\dagger\| \le 2g_\dagger$ and, moreover, x^* is the unique solution of $G(x) = 0$ in $B(x^*, r)$. ♦

The following lemma establishes the main step in the proof of Lemma 2.

Lemma 4: Under the assumptions of Lemma 2, if $\|\hat{x}_k^- - x_k\| \le \alpha^k\delta$, then $\|\hat{x}_k - x_k\| \le (\alpha/L)\|\hat{x}_k^- - x_k\|$. ♦

Proof: We first define, for $k \ge 0$

- $\beta_k = \|(G'_k(\hat{x}_k^-))^\dagger\|$;
- $\eta_k = \|(G'_k(\hat{x}_k^-))^\dagger G_k(\hat{x}_k^-)\|$;
- $h_k = \beta_k\eta_k g_2\gamma_2^k$;
- $\rho_k = \min\{d_k, (1)/(2g_\dagger\gamma_\dagger^k g_2\gamma_2^k)\}$, and $S_k = B(x_k, \rho_k)$;
- $\mu_k = (1/2)\sup_{T_k}\|\mathcal{N}''_k(x)\|$, where $T_k = B(\hat{x}_k^-, 2\eta_k)$, and we note that $\mu_k \le c\nu^k$ if $T_k \subset X_k$.

We next note that since $\delta < \min\{(d_0)/(2), (1)/(4g_\dagger g_2)\}$ and $\alpha \le \min\{\beta, (1)/(\gamma_\dagger\gamma_2)\}$, we have that $\alpha^k\delta < (\rho_k)/(2)$, and that $\hat{x}_k^- \in S_k$. Moreover, $S_k \subset R_k$ (with R_k given in Lemma 2) because $\rho_k \le d_k$. Therefore, by Lemma 3

$$\sup_{x\in S_k}\|(G'_k(x))^\dagger\| \le 2g_\dagger\gamma_\dagger^k. \tag{36}$$

Thus, since $\hat{x}_k^- \in S_k \subset R_k, \|(G'_k(\hat{x}_k^-))^\dagger\| \le g_\dagger\gamma_\dagger^k$, which implies that $\beta_k \le g_\dagger\gamma_\dagger^k$, and $\eta_k \le \|(G'_k(\hat{x}_k^-))^\dagger\| \cdot \|G_k(\hat{x}_k^-)\| \le g_\dagger\gamma_\dagger^k\|G_k(\hat{x}_k^-)\| \le g_\dagger\gamma_\dagger^k g_1\gamma_1^k\|\hat{x}_k^- - x_k\| \le g_\dagger\gamma_\dagger^k g_1\gamma_1^k\alpha^k\delta$. Therefore, $h_k = \beta_k\eta_k g_2\gamma_2^k \le \delta g_\dagger^2 g_1 g_2\alpha^k\gamma_\dagger^{2k}\gamma_1^k\gamma_2^k$, and since $\delta < (1)/(8g_\dagger^2 g_1 g_2) < (1)/(2g_1 g_\dagger^2 g_2)$ and $\alpha \le (1)/(\gamma_\dagger^2\gamma_1\gamma_2)$, we get

$$h_k < \frac{1}{2}. \tag{37}$$

We now consider the open ball T_k. From $\delta < \min\{(1)/(8g_\dagger^2 g_1 g_2), (d_0)/(4g_\dagger g_1)\}$ and $\alpha \le \min\{(1)/(\gamma_\dagger^2\gamma_1\gamma_2), (\beta)/(\gamma_\dagger\gamma_1)\}$, we get that $2\eta_k \le 2g_\dagger\gamma_\dagger^k g_1\gamma_1^k\alpha^k\delta < (\rho_k)/(2)$, which, given that $\alpha^k\delta < (\rho_k)/(2)$, implies that $T_k \subset S_k \subset R_k$. Therefore

$$\sup_{x\in T_k} \|G''_k(x)\| \le g_2\gamma_2^k. \tag{38}$$

Finally, by virtue of Theorem 2 and of (37) and (38), Newton's method would converge to a solution of $G_k(x) = 0$ inside T_k. By virtue of Lemma 3 and of the fact that $\rho_k = \min\{d_k, (1)/(2g_\dagger\gamma_\dagger^k g_2\gamma_2^k)\}, x_k$ is the unique solution of $G_k(x) = 0$ in S_k. Since $T_k \subset S_k$, Newton's method converges to x_k, and we get, from (35), that

$$\|\hat{x}_k - x_k\| \le \mu_k\|\hat{x}_k^- - x_k\|^2 \tag{39}$$

and since $T_k \subset X_k$ [thanks to the fact that $T_k \subset S_k \subset R_k$ and to (36)], we have that $\mu_k \le c\nu^k$, which, combined with $\delta < (\alpha/cL), \alpha \le (1/\nu)$, and (39), implies that

$$\|\hat{x}_k - x_k\| \le c\nu^k\alpha^k\delta\|\hat{x}_k^- - x_k\| \le \frac{\alpha}{L}\|\hat{x}_k^- - x_k\| \tag{40}$$

which completes the proof. ■

We finally establish the lemma.

Proof of Lemma 2: First, $\|\mathcal{N}''_k(x)\|$ needs to be adequately bounded. Schematically, note that for scalar $\mathcal{N}$, G, and x, we have

$$\mathcal{N}'' = \frac{(G')^3G'' - 2GG'(G'')^2 + G(G')^2G'''}{(G')^4}.$$

This shows that $\mathcal{N}''_k(x)$ is polynomial in $G_k^{\{p\}}(x), p \in \{1,2,3\}$, and $(G'_k(x))^\dagger$. Since these terms are bounded by exponentials over X_k, it is straightforward to bound the polynomial by an exponential, finding $c > 0$ and $\nu > 0$ such that $(1/2)\sup_{x\in X_k}\|\mathcal{N}''_k(x)\| \le c\nu^k, k \ge 0$.

We now show by induction on k that

$$\|\hat{x}_k - x_k\| \le \frac{\alpha}{L}\|\hat{x}_k^- - x_k\|, \qquad k \ge 0 \tag{41}$$

and note that (41), combined with the fact that f is globally L-Lipschitz, yields

$$\|\hat{x}_{k+1} - x_{k+1}\| \leq \alpha \|\hat{x}_k - x_k\|, \qquad k \geq 0 \tag{42}$$

which establishes (27). Note that we also get $\|\hat{x}_{k+1}^- - x_{k+1}\| \leq \alpha \|\hat{x}_k^- - x_k\|$.

Equation (41), for $k = 0$, is a direct consequence of Lemma 4 and of (26).

Now, assume that (41) is true up to time $k = K - 1$, or in other words that $\|\hat{x}_k - x_k\| \leq (\alpha/L)\|\hat{x}_k^- - x_k\|$ for $0 \leq k \leq K-1$. Since f is globally L-Lipschitz, we furthermore have that $\|\hat{x}_{k+1}^- - x_{k+1}\| \leq L\|\hat{x}_k - x_k\|$ for $0 \leq k \leq K-1$. Combining these last two facts, we get

$$\|\hat{x}_K^- - x_K\| \leq \alpha^K \|\hat{x}_0^- - x_0\| \leq \alpha^K \delta \tag{43}$$

which, again by Lemma 4, establishes (41) for $k = K$. ∎

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On the Structure of the Solutions of Discrete-Time Algebraic Riccati Equation With Singular Closed-Loop Matrix

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***Abstract*—The classical discrete-time algebraic Riccati equation (DARE) is considered in the case when the corresponding closed-loop matrix is singular. It is shown that in this case all the symmetric solutions of the DARE coincide along some directions. A parametrization of the set of solutions in terms of a reduced-order DARE is then obtained. This parametrization provides an algorithm (that appears to be computationally very attractive when the multiplicity of the eigenvalue $\lambda = 0$ of the closed-loop matrix is large) for the computation of the solutions of the DARE. The same issue for the generalized DARE is also addressed.**

***Index Terms*—Algebraic Riccati equation (ARE), closed-loop matrix, discrete-time linear quadratic (LQ) optimal control, order reduction, symplectic pencils.**

I. Introduction

The algebraic Riccati equation (ARE) arises naturally and plays a fundamental role in the solutions of many important problems in modern control theory ranging from optimal control and filtering [7], [1] to stochastic realization theory and identification [9].

In this note, we deal with the discrete-time ARE

$$X = F^\top X F - (F^\top X G + S)(R + G^\top X G)^{-1} \times (G^\top X F + S^\top) + Q \tag{1.1}$$

with $Q = Q^\top, F \in \mathbb{R}^{n\times n}, S, G \in \mathbb{R}^{n\times m}, R = R^\top \in \mathbb{R}^{m\times m}$ and we make the following standing assumption:

$$\Pi := \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \geq 0. \tag{1.2}$$

One of the key results of [2] was the possibility of reducing the order of the ARE in the case when R is singular. This possibility is essentially due to the fact that when R is singular the spectral density of the underlying spectral factorization problem has zeros in the origin and thus for any solution X of (1.1), the matrix $F_X := F - G(R+G^\top XG)^{-1}(G^\top XF+S^\top)$ is singular so that the corresponding closed-loop dynamics cannot be inverted. A simple way to see this fact is to consider the pencil

$$M - sL = \begin{bmatrix} F & 0 & G \\ Q & I & S \\ S^\top & 0 & R \end{bmatrix} - s \begin{bmatrix} I & 0 & 0 \\ 0 & F^\top & 0 \\ 0 & G^\top & 0 \end{bmatrix}. \tag{1.3}$$

Recall [6] that if X is a symmetric solution of (1.1) then there exist nonsingular $(2n+m)\times(2n+m)$ matrices P and Σ such that

$$P(M - sL)\Sigma = \begin{bmatrix} F_X - sI & sD & 0 \\ 0 & I - sF_X^\top & 0 \\ 0 & -sG^\top & R + G^\top X G \end{bmatrix} \tag{1.4}$$

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