Control Theoretic Smoothing Splines

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***Abstract*—In this paper some of the relationships between optimal control and statistics are examined. We produce generalized, smoothing splines by solving an optimal control problem for linear control systems, minimizing the L^2-norm of the control signal, while driving the scalar output of the control system close to given, prespecified interpolation points. We then prove a convergence result for the smoothing splines, using results from the theory of numerical quadrature. Finally, we show, in simulations, that our approach works in practice as well as in theory.**

***Index Terms*—Interpolation, linear systems, optimal control, smoothing splines.**

I. INTRODUCTION

IN this paper we continue the development begun in [8] of splines related to linear control systems. We refer to [8] for the rationale for the development. Classical polynomial splines and the splines developed in [8] are interpolating splines; that is, they are required to pass through specific points at specific times. In most applications, including trajectory planning, this is overly restrictive. We are usually content if the trajectory passes close to an assigned point at an assigned time. In this paper we will develop such generalized splines which are reflective of the dynamics of the underlying system.

What we will do is produce generalized splines by solving an optimal control problem for linear control systems. We will minimize the L^2-norm of the control signal, which is why we refer to the resulting curve as a spline, while driving the scalar output of the control system close to given, prespecified interpolation points.

This type of interpolation problem needs to be solved in a number of different control applications. When doing trajectory planning, for instance in air traffic control, we need to be able to generate curves that pass through predefined states at given times since we need to be able to specify the position that the system will be in at a sequence of times. However, in most situations, it is not really crucial that we pass through these points exactly, but rather that we go reasonably close to them, while minimizing the cost functional. This is a desired property for two apparent reasons. First of all, a small deviation from the prespecified point can result in a significant decrease in the cost, and secondly, when the data that we work with is noise contaminated, it is not even desirable to interpolate through these points exactly.

Grace Wahba [7] and her school have developed a theory of splines for statistics and they long ago recognized that interpolating splines are overly restrictive. In statistics a theory of smoothing splines, see [6], has been developed. The produced curves are indeed splines, but the nodal points are determined by the algorithm rather than being predetermined. All that is guaranteed is that the curves pass close to the desired points. We continue that development using the basic tool of the control of linear systems as was developed in [8].

The outline of the paper is as follows. In Section II we derive the basic spline functions. In Section III we show that by a change of basis we can develop numerically sound algorithms for the calculation of these splines. In Section IV we then prove a convergence result for smoothing splines using results from the theory of numerical quadrature. These results seem to be new even in the polynomial smoothing spline case. Detailed results about the rates of convergence are, however, beyond the scope of this paper.

II. DERIVATION

We consider the linear, single-input/single-output system

$$\begin{aligned}\dot{x} &= Ax + bu\\ y &= c^T x, \qquad x \in \mathbb{R}^n\end{aligned} \tag{1}$$

where A is a matrix and b, c are vectors of compatible dimensions. We further assume that the system is both controllable and observable. Our goal is to produce a control law $u(t)$ which drives the scalar output trajectory close to a sequence of set points at fixed times while maintaining control of the growth of the control function $u(t)$. We will label the times and points as

$$\{(t_i, \alpha_i): \quad i = 1, \cdots, m\} \tag{2}$$

where we assume that

$$0 < t_1 < t_2 < \cdots < t_m \le T. \tag{3}$$

We now formulate a cost function of the form

$$J(u) = \sum_{i=1}^{m} w_i (y(t_i) - \alpha_i)^2 + \rho \int_0^T u^2(t)\, dt \tag{4}$$

where the weights w_i and ρ are chosen to be strictly positive. Our goal is to minimize the quadratic functional J subject to the affine constraint

$$y(t) = c^T e^{At} x_0 + \int_0^t c^T e^{A(t-s)} bu(s)\, ds.$$

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Let

$$g_{t_i}(t) = \begin{cases} c^T e^{A(t_i - t)} b, & t \le t_i \\ 0, & \text{otherwise} \end{cases}$$

where the t_i's are the interpolation times. Let

$$\beta_i = c^T e^{At_i} x_0.$$

We also define a set of linear functionals as

$$L_{t_i}(u) = \int_0^T g_{t_i}(t) u(t)\, dt.$$

Now, using the notation that we have developed we have

$$y(t_i) = \beta_i + \int_0^T g_{t_i}(t) u(t)\, dt = \beta_i + L_{t_i}(u).$$

We substitute this into (4) to obtain

$$J(u) = \sum_{i=1}^m w_i (L_{t_i}(u) + \beta_i - \alpha_i)^2 + \rho \int_0^T u^2(t)\, dt$$

and letting $\beta_i - \alpha_i = \gamma_i$ gives us

$$J(u) = \sum_{i=1}^m w_i (L_{t_i}(u) + \gamma_i)^2 + \rho \int_0^T u^2(t)\, dt. \tag{5}$$

Our goal is to minimize this functional over the Hilbert space of square integrable functions on the interval $[0, T]$. We calculate the Fréchet derivative in the form

$$\begin{aligned}\lim_{\alpha \to 0} & \frac{1}{\alpha} (J(u + \alpha h) - J(u)) \\ &= \sum_{i=1}^m 2 w_i L_{t_i}(h)(L_{t_i}(u) + \gamma_i) + 2\rho \int_0^T h(t) u(t)\, dt \\ &= 2 \int_0^T \left[\sum_{i=1}^m w_i g_{t_i}(t)(L_{t_i}(u) + \gamma_i) + \rho u(t) \right] h(t)\, dt. \end{aligned} \tag{6}$$

Now, to ensure that u is a minimum we must have that the Fréchet derivative vanishes but this can only happen if

$$\sum_{i=1}^m w_i g_{t_i}(t)(L_{t_i}(u) + \gamma_i) + \rho u(t) = 0 \tag{7}$$

since h was chosen arbitrarily in $L^2[0, T]$ in (6).

We now consider the operator

$$T(u) = \sum_{i=1}^m w_i g_{t_i}(t) L_{t_i}(u) + \rho u(t).$$

We can rewrite this operator in the following form:

$$T(u) = \int_0^T \left(\sum_{i=1}^m w_i g_{t_i}(t) g_{t_i}(s) \right) u(s)\, ds + \rho u(t) \tag{8}$$

and we want to show that this operator T is one to one and onto.

Lemma 2.1: The set of functions $\{g_{t_i}(t)\colon i = 1, \cdots, m\}$ are linearly independent.

Proof: The proof is obvious and relies on the fact that the different g_{t_i}'s vanishes at different times. ■

We now establish that the operator T is one to one.

Lemma 2.2: The operator T is one to one for all choices of $w_i > 0$ and $\rho > 0$.

Proof: Suppose $T(u_0) = 0$, which would imply that

$$\sum_{i=1}^m w_i g_{t_i}(t) L_{t_i}(u_0) + \rho u_0(t) = 0$$

and hence that

$$\sum_{i=1}^m w_i g_{t_i}(t) a_i + \rho u_0(t) = 0$$

where a_i is the constant $L_{t_i}(u_0)$. This implies that any solution u_0 of $T(u_0) = 0$ is in the span of the set $\{g_{t_i}(t)\colon i = 1, \cdots, m\}$. Now consider a solution of the form

$$u_0(t) = \sum_{i=1}^m \tau_i g_{t_i}(t)$$

and evaluate $T(u_0)$ to obtain

$$\sum_{i=1}^m w_i g_{t_i}(t) L_{t_i} \left(\sum_{j=1}^m \tau_j g_j(t) \right) + \rho \sum_{i=1}^m \tau_i g_{t_i}(t) = 0.$$

Thus for each i

$$w_i \sum_{j=1}^m L_{t_i}(g_j) \tau_j + \rho \tau_i = 0.$$

The coefficient τ is then the solution of a set of linear equations of the form

$$(DG + \rho I)\tau = 0$$

where D is the diagonal matrix of the weights w_i and G is the Gramian with $g_{ij} = L_{t_i}(g_j)$. Now consider the matrix $DG + \rho I$ and multiply it on the left by D^{-1}, and consider the scalar

$$x^T \left(G + \rho D^{-1} \right) x = x^T G x + \rho x^T D^{-1} x > 0$$

since both terms are positive. Thus for positive weights and positive ρ the only solution is $\tau = 0$. ■

It remains to show that the operator T is onto.

Lemma 2.3: For $\rho > 0$ and $w_i > 0$ the operator T is onto.

Proof: Suppose T is not onto. Then there exists a nonzero function f such that $\int_0^T f(t) T(u)(t) dt = 0$ for all u. We have, after some manipulation, that

$$\begin{aligned} &\int_0^T f(t) T(u)(t)\, dt \\ &= \int_0^T \left[\int_0^T \sum_{i=0}^m w_i g_{t_i}(t) g_{t_i}(s) f(t)\, dt + \rho f(s) \right] u(s)\, ds \\ &= 0 \end{aligned}$$

and hence that

$$\int_0^T \sum_{i=0}^m w_i g_{t_i}(t) g_{t_i}(s) f(t)\,dt + \rho f(s) = 0.$$

By the previous lemma the only solution of this equation is $f = 0$ and hence T is onto. ■

We have thus proved the following proposition.

Proposition 2.1: The functional

$$J(u) = \sum_{i=1}^m w_i (L_{t_i}(u) + \gamma_i)^2 + \rho \int_0^T u^2(t)\,dt$$

has a unique minimum.

We now use (7) to find the optimal solution. As shown in the proof or Lemma 2.2, the optimal control is of the form

$$u_m(t) = \sum_{i=1}^m \tau_i g_{t_i}(t). \tag{9}$$

Substituting this into (7) we have upon equating coefficients of the $g_{t_i}(t)$ the system of linear equations

$$(DG + \rho I)\tau = -D\gamma \tag{10}$$

where γ is the vector of γ_i's from (5). As in the proof of the lemma the coefficient matrix is invertible and hence the solution exists and is unique. The resulting curve $y(t)$ is a spline. The major difference is that the nodal points are determined by the optimization instead of being predetermined. It is interesting to calculate the difference between the nodal points γ_k and the values of the spline function. Using (9) gives us

$$\begin{aligned} L_t(u_m) &= \sum_{i=1}^m \tau_i L_t(g_{t_i}) \\ &= \sum_{i=1}^m \tau_i \int_0^T g_t(s) g_{t_i}(s)\,ds. \end{aligned}$$

Evaluating this at $t = t_k$ gives

$$\begin{aligned} L_{t_k}(u_m) &= \sum_{i=1}^m \tau_i \int_0^T g_{t_k}(s) g_{t_i}(s)\,ds \\ &= e_k^T G\tau = -e_k^T \left[\gamma + \rho D^{-1}\tau\right] \end{aligned}$$

where e_k is the kth unit vector, Thus

$$L_{t_k}(u_m) + \gamma_k = -\rho w_k^{-1} \tau_k.$$

Now, τ is a linear function of the data γ and hence the spline depends linearly on the data.

However, inverting the matrix $DG + \rho I$ is not trivial. Since it is a Gramian we can expect it to be badly conditioned. In the next section we will show that by a change of basis we can produce a much better conditioned system.

III. Conditioning

We, once again, consider the basis of functions defined as

$$g_{t_i}(t) = \begin{cases} c^T e^{A(t_i - t)} b, & 0 \le t \le t_i \\ 0, & \text{otherwise.} \end{cases}$$

Our goal now is to define a new basis which has better numerical properties at the same time as the differentiability properties are preserved. We first assume that the values t_i are evenly spaced

$$t_i = t_0 + ih$$

where we, as before, set $t_0 = 0$. In this case the functions g_{t_i} assume a simpler form

$$g_{t_i}(t) = \begin{cases} c^T e^{Ahi} e^{-At} b, & 0 \le t \le ih \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$p(t) = t^n - \sum_{k=0}^{n-1} \xi_k t^k$$

be such that

$$p(e^{Ah}) = 0$$

i.e., p is the characteristic polynomial of e^{Ah}.

We now define a new basis

$$\begin{aligned} h_{n+k}(t) &= g_{t_{n+k}}(t) - \sum_{i=0}^{n-1} \xi_i g_{t_{i+k}}(t) \\ & \qquad k = 1, 2, \cdots, m-n \end{aligned} \tag{11}$$

where we have assumed that $m > n$.

For convenience of notation we define

$$h_i(t) = g_{t_i}(t) \qquad i = 0, \cdots, n.$$

We note the following facts.

Fact 1: $h_{n+k}(t) = 0$ for $t < kh,\ k = 1, \cdots, m-n$.

Fact 2: $h_{n+k}(t) = 0$ for $t > (k+n)h,\ k = 1, \cdots, m-n$.

Fact 3: span$\{g_{t_i}(t)\colon\ i = 0, \cdots, n\}$ = span$\{h_i(t)\colon\ i = 0, \cdots, n\}$.

Fact 4: $\int_0^T h_i(t) h_j(t)\,dt = 0$ for $|i-j| > n$.

We define a matrix, Ξ, that transforms the corresponding Gramians as shown in (12) at the bottom of the next page. We now state the following about the Gramian corresponding to our new basis.

Fact 5: $H = \Xi G \Xi^T$.

It follows from Fact 4 that H is zero except for a band of width $n+1$ around the main diagonal. In the case that the t_i's are not uniformly spaced the situation is a bit more complicated because a set of n e^{At_i} may or may not be linearly independent. The best that we can do is to define algorithmically the transformation. The matrix will be banded but the band may or may not be of constant width. We state the following result.

Proposition 3.1: If the t_i's are uniformly distributed in the interval $[0, T]$ then with probability 1 the set $\{c^T e^{At_i}\colon\ i = 1, \cdots, n\}$ is linearly independent.

Thus in this case the change of basis works and we have a banded matrix. Unfortunately, in this case the amount of computation required to reduce the matrix is prohibitive.

IV. Convergence

In this section we will prove that the smoothing splines defined above converge in a statistical sense. We make a few assumptions which we believe are necessary.

Assumption 4.1: For the system of (1) we assume $c^T b = \cdots = c^T A^{n-2} b = 0$.

Assumption 4.2: The matrix A has only real eigenvalues.

Assumption 4.3: The set of interpolation times, $\{t_i \colon i = 1, 2, \cdots\}$, ordered as in (3), are contained in the interval $[0, T]$.

Assumption 4.4: Let $f(t)$ be a C^∞ function on an interval that contains $[0, T]$.

Now, let u_N be the control that optimizes the functional

$$\begin{aligned} J_N(u) = \frac{1}{2N} \sum_{i=1}^{N} w_{iN} (L_{t_{iN}}(u) - f(t_{iN}))^2 \\ + \frac{\rho}{2} \int_0^T u^2(t)\, dt \end{aligned} \tag{13}$$

where the subscript N is used to denote the case where we interpolate close to N given points. The control, u_N, was shown to exist uniquely in Section II.

We now let u^* be the control that optimizes

$$J(u) = \frac{1}{2} \int_0^T (L_t(u) - f(t))^2\, dt + \frac{\rho}{2} \int_0^T u^2(t)\, dt \tag{14}$$

and make the following important assumption.

Assumption 4.5: The sequence of quadratures defined by the numbers w_{iN}, t_{iN} converge for all continuous functions defined on $[0, T]$, i.e.,

$$\lim_{N \to \infty} \frac{1}{2N} \sum_{i=1}^{N} w_{iN} h(t_{iN}) = \frac{1}{2} \int_0^T h(t)\, dt.$$

We will prove the following theorem.

Theorem 4.1: Under the Assumptions 4.1–4.5 the sequence of controls $\{u_N(t)\}_{N=1}^{\infty}$ converges to the function $u^*(t)$ in the L^2 norm and the sequence of smoothing splines $\{L_t(u_N)\}_{N=1}^{\infty}$ likewise converges to $L_t(u^*)$ in L^2 norm.

Proof of Theorem: We begin the proof by showing that u^* exists and is unique. We show this by explicit construction. We have shown previously that the functions u_N exist and are unique. Then we will argue that the minimizers of the functionals J_N converge to the minimizer of J. We divide the proof into a series of lemmas. □

Lemma 4.1: The function u^* exists and is unique.

Proof: We first observe that the cost functional J given by

$$J(u) = \int_0^T (L_t(u) - f(t))^2\, dt + \int_0^T u^2(t)\, dt \tag{15}$$

can be reduced to a standard linear-quadratic-optimization problem by a change of variable. Let

$$w(t) = L_t(u) - f(t).$$

By taking a sequence of derivatives we have

$$w^{(0)}(t) = \int_0^t c^T e^{A(t-s)} b u(s)\, ds - f(t) \tag{16}$$

$$w^{(1)}(t) = \int_0^t c^T A e^{A(t-s)} b u(s)\, ds - f^{(1)}(t) \tag{17}$$

$$\vdots$$

$$w^{(n-1)}(t) = \int_0^t c^T A^{n-1} e^{A(t-s)} b u(s)\, ds - f^{(n-1)}(t) \tag{18}$$

$$\begin{aligned} w^{(n)}(t) = c^T A^{n-1} b u(t) + \int_0^t c^T A^n e^{A(t-s)} b u(s)\, ds \\ - f^{(n)}(t). \end{aligned} \tag{19}$$

Here we have used our assumption that $c^T A^k b = 0$ for $k \le n-2$. Now let

$$p(t) = t^n - \zeta_{n-1} t^{n-1} - \cdots - \zeta_1 t - \zeta_0$$

be such that $p(A) = 0$. By using (16)–(19), and taking the appropriate weighted sum, we have

$$\begin{aligned} & w^{(n)} - \zeta_{n-1} w^{(n-1)} - \cdots - \zeta_0 w \\ & \quad = c^T A^{n-1} b u(t) - f^{(n)} + \zeta_{n-1} f^{(n-1)} + \cdots + \zeta_0 f. \end{aligned} \tag{20}$$

Writing this in state space form gives us

$$\begin{aligned} \frac{d}{dt} \hat{w}(t) = \hat{A} \hat{w}(t) + \left(c^T A^{n-1} b\right) e_n u(t) \\ + \left(-f^{(n)} + \zeta_{n-1} f^{(n-1)} + \cdots + \zeta_0 f\right) e_n \end{aligned} \tag{21}$$

$$\begin{pmatrix} 1 & 0 & & & & & & & & \\ 0 & 1 & 0 & & & & & & & \\ 0 & 0 & \ddots & 0 & & & & & & \\ & & & 1 & & & & & & \\ -\xi_0 & -\xi_1 & \cdots & -\xi_{n-1} & 1 & 0 & & & & \\ 0 & -\xi_0 & -\xi_1 & \cdots & -\xi_{n-1} & 1 & 0 & & & \\ & & & & & & \ddots & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & -\xi_0 & \cdots & -\xi_{n-1} & 1 \end{pmatrix} \tag{12}$$

where $\hat{w} = (w^{(0)}, \cdots, w^{(n-1)})^T$. In (21) e_n is the nth unit vector and $\hat{A}$ is in companion form and is similar to A. Let

$$F(s) = (f^n(s) - \zeta_{n-1} f^{n-1}(s) - \cdots - \zeta_0 f(s)).$$

Now, (21) defines a closed affine subspace in $L^2[0,T]$, and hence there is a unique function $u^* \in L^2[0,T]$ which gives a point of minimal norm in the affine subspace. ∎

We now characterize the function u^* of the previous lemma and show that it is at least C^∞.

Lemma 4.2: The optimal spline $L_t(u^*)$ is given by

$$\begin{aligned} &L_t(u^*) \\ &\quad= \begin{pmatrix} e_1^T & 0 \end{pmatrix} \exp\left(\begin{pmatrix} A & -e_n e_n^T \\ -e_1 e_1^T & -A^T \end{pmatrix} t\right) \begin{pmatrix} \hat{w}(0) \\ \lambda(0) \end{pmatrix} \\ &\qquad - \int_0^t \begin{pmatrix} e_1^T & 0 \end{pmatrix} \exp\left(\begin{pmatrix} A & -e_n e_n^T \\ -e_1 e_1^T & -A^T \end{pmatrix} (t-s)\right) \\ &\qquad \times \begin{pmatrix} e_n \\ 0 \end{pmatrix} F(s)\, ds \end{aligned}$$

and the optimal control is given by

$$u^*(t) = e_n^T \lambda(t)$$

where

$$\frac{d}{dt}\begin{pmatrix} \hat{w}(t) \\ \lambda(t) \end{pmatrix} = \begin{pmatrix} A & -e_n e_n^T \\ -e_1 e_1^T & -A^T \end{pmatrix} \begin{pmatrix} \hat{w}(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} e_n \\ 0 \end{pmatrix} F(s) \tag{22}$$

with data

$$\hat{w}(0) = \hat{w}_0, \qquad \lambda(T) = 0.$$

Proof: From the previous lemma we know there is a point of minimal norm in the affine subspace. We will explicitly construct that point using a construction similar to a calculation found in [3].

Define the linear affine subspace

$$\begin{aligned} &AF(\hat{w}_0, F(s)) \\ &\quad= \left\{(\hat{w}, u) \colon \hat{w} = \int_0^t e^{\hat{A}(t-s)} \left(c^T A^{n-1} b\right) e_n u(s)\, ds\right\} \\ &\qquad + \left(e^{\hat{A}t}\hat{w}_0 + \int_0^T e^{\hat{A}(t-s)} F(s) e_n ds, 0\right). \end{aligned} \tag{23}$$

The object is to construct the orthogonal complement to $AF(0,0)$. At this point the construction of the complement is found in [3]. We then construct the intersection of the orthogonal complement to $AF(0,0)$ and $AF(\hat{w}_0, F(s))$. The single point in this intersection is found by solving the two point boundary value problem

$$\frac{d}{dt}\begin{pmatrix} \hat{w}(t) \\ \lambda(t) \end{pmatrix} = \begin{pmatrix} A & -e_n e_n^T \\ -e_1 e_1^T & -A^T \end{pmatrix} \begin{pmatrix} \hat{w}(t) \\ \lambda(t) \end{pmatrix} - \begin{pmatrix} e_n \\ 0 \end{pmatrix} F(s) \tag{24}$$

with data

$$\hat{w}(0) = \hat{w}_0, \qquad \lambda(T) = 0$$

and where

$$u(t) = e_n^T \lambda(t).$$

It is necessary to determine if this two point boundary value problem has solutions. We solve the differential equation, assuming that it is an initial value problem, to obtain.

$$\begin{aligned} \begin{pmatrix} \hat{w}(t) \\ \lambda(t) \end{pmatrix} &= \exp\left(\begin{pmatrix} A & -e_n e_n^T \\ -e_1 e_1^T & -A^T \end{pmatrix} t\right) \begin{pmatrix} \hat{w}(0) \\ \lambda(0) \end{pmatrix} \\ &\quad - \int_0^t \exp\left(\begin{pmatrix} A & -e_n e_n^T \\ -e_1 e_1^T & -A^T \end{pmatrix} (t-s)\right) \\ &\quad \times \begin{pmatrix} e_n \\ 0 \end{pmatrix} F(s)\, ds. \end{aligned}$$

Setting $t = T$ we have a linear equation for $\lambda(0)$ and this equation has a unique solution if and only if it has a unique solution for $F = 0$. But for $F = 0$ this is the linear two point boundary value problem associated with the linear quadratic optimal control problem

$$J(u) = \int_0^T \hat{w}(t)^T e_1 e_1^T \hat{w}(t) + u^2(t)\, dt$$

with linear constraint

$$\frac{d}{dt}\hat{w} = \hat{A}\hat{w} + c^T A^{n-1} b e_n u(t).$$

This problem has a unique solution since the pair $(e_1, \hat{A})$ is observable and the pair $(\hat{A}, e_n)$ is controllable. It then follows from the linear quadratic optimal control theory that the two point boundary value problem has solutions, for all values of $\hat{w}(0)$, and these solutions exist on the interval $[0,T]$. (See for example [5].) We note that $\lambda(0) = P(0)\hat{w}(0)$ where $P(t)$ is the solution of the associated Riccati equation. Thus the lemma is proven. ∎

We now finish the proof of the theorem by proving convergence. From our assumption that f is C^∞, the control u^* is at least C^∞. We know that the minimizer of $J(u)$ is unique and we have shown that the minimizer of $J_N(u)$ is unique. We also know from the general theory of optimization [4] that the minimizer of a quadratic functional is given by the unique zero of the Fréchet derivative of the functional. Calculating the Fréchet derivatives we have the following two linear functionals:

$$\begin{aligned} DJ(u)(w) &= \int_0^T (L_t(u) - f(t)) L_t(w)\, dt \\ &\quad + \int_0^T u(t) w(t)\, dt \end{aligned} \tag{25}$$

$$\begin{aligned} DJ_N(u)(w) &= \sum_{i=1}^N w_{iN}(L_{t_{iN}}(u) - f(t_{iN}) L_{t_{iN}}(w) \\ &\quad + \int_0^T u(t) w(t)\, dt. \end{aligned} \tag{26}$$

It is clear that for each u and w that $DJ_N(u)(w)$ converges to $DJ(u)(w)$ provided the quadrature scheme converges for a sufficiently general class of functions. We now rewrite the Fréchet derivatives in terms of inner products by the simple expediency of interchanging the order of integration

$$\begin{aligned} DJ(u)(w) &= \int_0^T \left(\int_0^T g_t(s)(L_t(u) - f(t))\, dt + u(s) \right) \\ &\quad \times w(s)\, ds \end{aligned} \tag{27}$$

$$\begin{aligned} DJ_N(u)(w) &= \int_0^T \left(\sum_{i=1}^N w_{iN} g_{t_{iN}}(s)(L_{t_{iN}}(u) - f(t_{iN})) + u(s) \right) \\ &\quad \times w(s)\, ds. \end{aligned} \tag{28}$$

From the previous two equations we have that the convergence is independent of w since

$$\sum_{i=1}^N g_{t_{iN}}(s) w_{iN}(L_{t_{iN}}(u) - f(t_{iN})) + u(s) \tag{29}$$

converges to

$$\int_0^T g_t(s)(L_t(u) - f(t))\, dt + u(s) \tag{30}$$

for every $s \in [0, T]$. We are now concerned with the convergence of linear operators rather than linear functionals.

Let

$$B(s)(u) = \int_0^T g_t(s) L_t(u)\, dt + u(s) \tag{31}$$

and define $B_N(s)$ as

$$B_N(s)(u) = \sum_{i=1}^N g_{t_{iN}}(s) w_{iN} L_{t_{iN}}(u) + u(s). \tag{32}$$

Furthermore, let

$$b(s) = \int_0^T g_t(s) f(t)\, dt$$

and

$$b_N(s) = \sum_{i=1}^N g_{t_{iN}} w_{iN} f(t_{iN}).$$

Now it is clear that $b_N(s)$ converges to $b(s)$ pointwise and hence in L^2 norm. Thus given ϵ for N sufficiently large we have

$$|B_N(s)(u_N - u^*)| < \epsilon.$$

Now we know that $B_N(s)x = b_N(s)$ has a unique solution and hence that $B_N(s)$ is non singular. We can thus conclude that $u_N(s) - u^*(s)$ converges to zero pointwise and hence in L^2 norm. Thus the theorem is proven. ■

Comments: This is important because it shows exactly how the continuous spline is dependent on the data. We see from the theorem that the spline is the convolution of the function F with a kernel that is the semigroup of a Hamiltonian system. We also see that since the control is optimal with respect to the cost function, the resulting feedback controlled system is stable and hence perturbations in F are not blown up but die quite quickly. This, however, is really just straight forward results from the theory of linear quadratic optimal control.

We obtain as corollaries three important results.

Corollary 4.1: Let $w_{iN} = 1/N$ and let the sequence of t_i's be the observed values of a random variable uniformly distributed in the interval $[0, T]$ then the sequence of smoothing splines $\{L_t(u_N)\}_{N=1}^{\infty}$ converges to $L_t(u^*)$ in L^2 norm.

Corollary 4.2: Let $w_{iN} = 1/N$ and let $t_{iN} = iT/N$ (Riemann sum) then the sequence of smoothing splines $\{L_t(u_N)\}_{N=1}^{\infty}$ converges to $L_t(u^*)$ in L^2 norm.

Corollary 4.3: Let w_{iN}, t_{iN} be defined by a Gaussian quadrature scheme then the sequence of smoothing splines $\{L_t(u_N)\}_{N=1}^{\infty}$ converges to $L_t(u^*)$ in L^2 norm.

Proof of Corollaries: The corollaries are presented in the order of the rate of convergence. We begin with Corollary 4.1. This result is based on the "law of large numbers", and while the rate of convergence is painfully slow there are minimal assumptions about the location of points making it an extremely useful result. We state this standard result for ease of reference. □

Theorem 4.2 (Law of Large Numbers): Let $x_1, x_2, \cdots, x_n$ be independent and identically distributed random variables whose probability density function is denoted by $\mu(x)$. Let

$$I = \int_{-\infty}^{\infty} f(x)\mu(x)\, dx$$

exist, then

$$\frac{1}{n}\sum_{i=1}^n f(x_i)$$

converges to I in probability with n.

For Corollary 4.1 we have taken

$$\mu(x) = \begin{cases} \dfrac{1}{T}, & 0 \le x \le T \\ 0, & \text{otherwise} \end{cases}$$

in the theorem. It is worth noting that we could have modified the cost function $J(u)$ to obtain a more general result. If we take

$$J(u) = \int_0^{\infty} (L_t(u) - f(t))^2 \mu(t)\, dt + \int_0^{\infty} u^2(t)\, dt \tag{33}$$

most of the previous results still hold. However there are problems that need to be resolved such as what is the nature of the optimal solution and how does one interpret splines on an infinite interval. There are certain aspects of this that will be investigated in a future paper. For example, if $\mu(t) = \lambda e^{-\lambda t}$ this would produce a spline like smoothing function which would have some predictive value for future times which is one of the more serious drawbacks of spline approximation.

For Corollary 4.2 we appeal to the general theory of Riemann sums and use the following theorem which can be found in Davis's classical book [1].

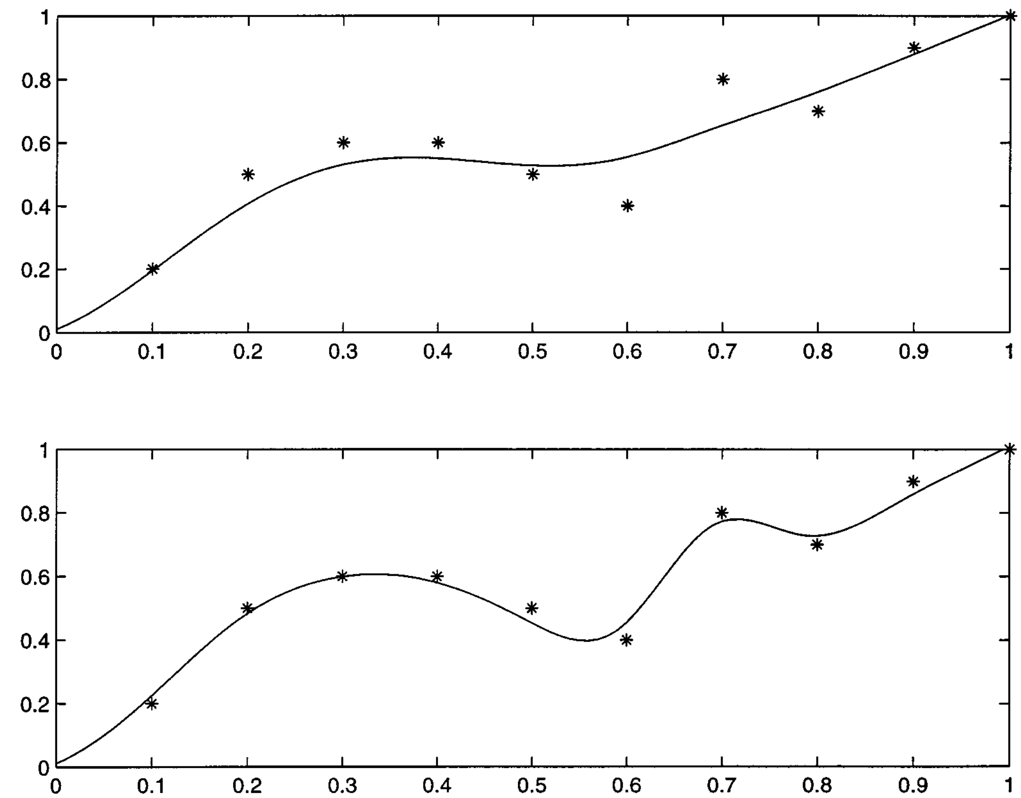

Fig. 1. Effect of decrease in the value of ρ.

Theorem 4.3 (Convergence of Riemann Sums): Let $f(x)$ be continuous in $[a, b]$. Then

$$\left| \int_a^b f(x)dx - h \sum_{i=1}^{n} f(a+kh) \right| \leq (b-a)w\left(\frac{b-a}{n}\right)$$

where

$$w(\delta) = \max_{x_1 - x_2 \leq \delta} |f(x_1) - f(x_2)|, \qquad a \leq x_1, x_2 \leq b.$$

There are various refinements of this theorem in the literature and we refer the reader to [2] for a survey and interpretation of the literature. For this result the rate of convergence is of the order of $1/n$ which is an improvement over the rate of convergence given by the law of large numbers, which is only $1/\sqrt{n}$. There are various improvements which can be made along this line. For example, we can use multiple point trapezoidal rules and multiple point Simpson's rules to obtain polynomial convergence of various orders. Again we refer the reader to [2] for many examples.

The results for Gaussian quadrature are quite diverse. Technically we have used the following theorem related to Legendre quadrature.

Theorem 4.4 (Legendre Quadrature): Let

$$E_n(f) = \int_0^T f(x)\,dx - \sum_{k=1}^{n} w_k f(x_k)$$

where the x_k are the zeros of the Legendre polynomial of degree n and the weights w_k are the weights of the associated quadrature scheme, then

$$E_n(f) = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\eta).$$

From this result we see that the order of convergence is non polynomial. We also see that it becomes harder and harder to give precise estimates because of the difficulty of estimating the higher derivatives of f. In [2] there are numerous results on the rates of convergence of some of the classical quadrature schemes but there does not seem to be a general procedure for finding rates for arbitrary weight functions.

V. Simulation Results

In this section we present a small example of the difference between interpolating splines and smoothing splines. We use only the simplest case but the difference are quite dramatic. Recall that the reason for investigating the smoothing splines was that the interpolating splines result in rather large accelerations even for well behaved trajectories, as mentioned already in Section I.

We have selected a set of 10 points between 0 and 1 on the interval [0, 1]. We have chosen to use the system

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

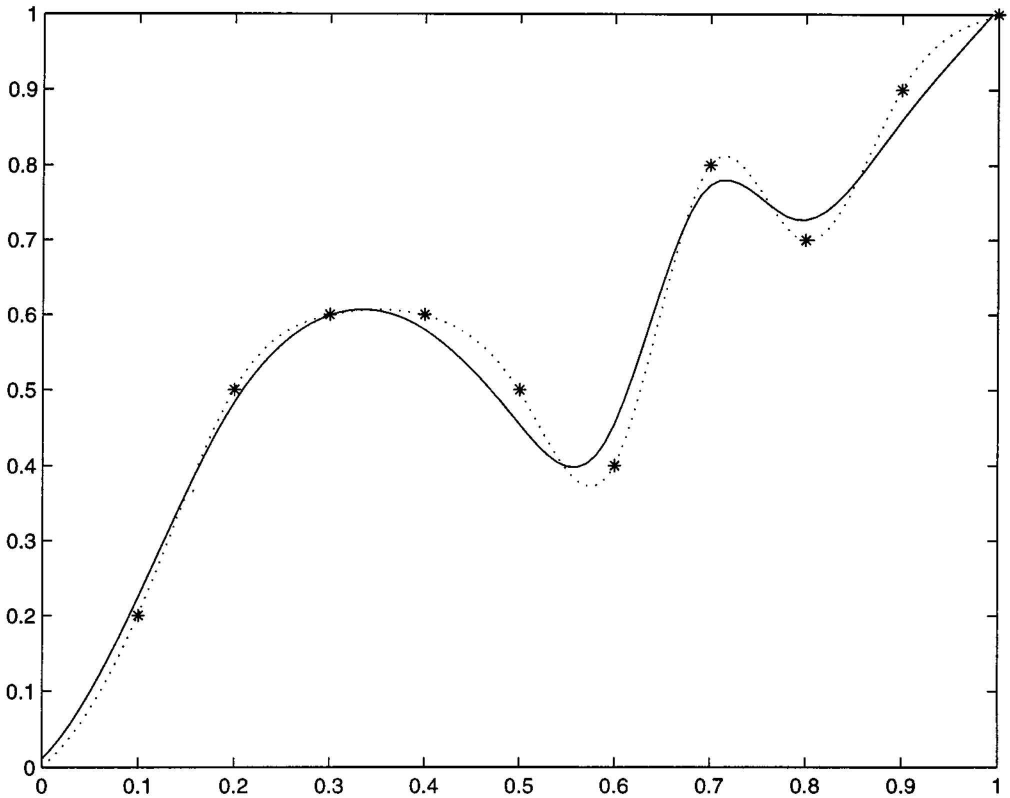

Fig. 2. Smoothing versus interpolating splines.

because of its simplicity and because it produces the familiar cubic spline. Other systems would have produced similar results. In the first graph in Fig. 1 we have chosen to weight the integral of the control by a weight of $\rho = 10^{-4}$ and in the second we have chosen a weight of $\rho = 10^{-5}$. We have a difference in control energy of an order of magnitude: or exactly, in the first

$$\frac{1}{2}\int_0^1 u^2(t)\,dt = 21.1$$

and in the second we have

$$\frac{1}{2}\int_0^1 u^2(t)\,dt = 307.$$

These two graphs illustrate nicely the tradeoff between control power and fit.

We also use the same system to compare interpolating and smoothing splines. In Fig. 2 we have used the same set of points as in Fig. 1, and for the smoothing splines we have used a weight of $\rho = 10^{-5}$. We calculate the control energy for both the smoothing and the interpolating splines. For the smoothing spline we have the cost of 307, but for the interpolating spline the cost more than doubles to

$$\frac{1}{2}\int_0^1 u^2(t)\,dt = 659.$$

Thus we see that there is a significant increase in control power and hence in accelerations between the two systems for a modest increase in fit.

VI. Conclusion

Polynomial smoothing splines have played an important role in nonparametric statistics. When data is corrupted by noise interpolating splines fail to represent the underlying process because of their inherent variation. Smoothing splines overcome this drawback at the expense of the introduction of more complexity and at the expense of obscuring the underlying process by the addition of the smoothing factor. In this paper we have shown that the smoothing splines converge to a very natural object that can be represented as the output of a forced Hamiltonian system associated with the continuous version of the cost function of the optimal control system.

This paper makes two contributions. The first is to show that the smoothing splines have a natural object to which they converge. This is parallel to the result for interpolating deterministic splines. The second contribution is to show that polynomial smoothing splines can naturally be considered as a special case of a more general class of splines associated with linear control theory. This is important in cases in which the underlying process is dynamic.

In this paper we have shown that the discrete spline converges to the continuous analog and we have shown that the rate of convergence of the cost function is dependent on the rate of convergence of the corresponding quadrature method. The quadrature method influences the spline through the choice of points and through the choice of weights. We have not studied how the choice of smoothing parameter enters into this calculation nor have we studied the rate of convergence of the spline itself. These topics are beyond the scope of this paper.

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