B-splines and control theory

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Abstract

In this paper some of the relationships between B-splines and linear control theory is examined. In particular, the controls that produce the B-spline basis is constructed and compared to the basis elements for dynamic splines.

Key words: B-splines, control theory, dynamic splines, interpolation, approximation

1 Introduction

In this paper the connections between the theory of B-splines and control theoretic or dynamic splines are examined. The theory of B-splines is a well developed area of applied numerical analysis and interpolation theory, and the use of B-splines rivals that of Bezier curves in applicability to computer graphics and approximation theory. (See for example [1, 2].) On the other hand, the idea of dynamic splines was first used by Crouch and his colleagues in the determination of aircraft trajectories [3]. Quite independently Martin, Egerstedt, and their colleagues began exploiting the properties of controlled linear systems to solve interpolation and approximation problems.

Since the introduction of splines by Shoenberg, [4, 5], it has been recognized that they are extremely powerful tools both in application and theory. Many variants have been introduced over the years and this paper is an attempt to show how some of these variations are related.

In [6, 7, 8] it was recognized that the dynamic splines generalized the classical concepts of splines and that many applications were easy to formulate and solve using the control theoretic approach. The idea is to find a control that drives a linear, single-input, single-output control system of the form

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx \end{aligned}$$

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through, in the case of interpolation, a series of way points or close to a series of way points, in the case of smoothing. Here, $x \in \mathrm{R}^n$, $u, y \in \mathrm{R}$, and A, b, c are matrices and vectors of compatible dimensions. When adopting this control theoretic point of view, the goal becomes that of constructing the control directly rather than the actual spline.

Given a set of data of the form

$$\{(t_i, \alpha_i) : \; i = 1, \cdots, N\},$$

where $\alpha_i \in \mathrm{R}, \; i = 1, \ldots, N$, and $0 < t_1 < \ldots < t_N < T$ for some final time T, we generate two optimal control problems that produce the desired controls. The first problem, the problem of interpolating splines, is as follows.

Problem 1

$$\min_{u \in L_2[0,T]} \int_0^T u^2(t) dt$$

subject to the constraints

$$y(t_i) = \alpha_i, \; i = 1, \ldots, N.$$

As shown in [6], this problem can be solved by reducing it to the problem of finding the point of minimal norm on an affine linear variety in the Hilbert space, $L_2[0, T]$.

The second problem, the problem of smoothing or approximation, is formulated as follows.

Problem 2

$$\min_{u \in L_2[0,T]} \left(\int_0^T u^2(t) dt + \sum_{i=1}^{N} w_i (y(t_i) - \alpha_i)^2 \right),$$

where the weights satisfy $w_i \geq 0, \; i = 1, \ldots, N$.

The main goal of this paper is to understand the extent to which these two problems can be applied to the theory and application of B-splines.

2 A basis for B-splines

In this section we consider the standard basis for normalized, uniform B-splines. We take two approaches. The first is a modification of the general approach of de Boor [2] and is inspired by [9]. The second is a more geometric approach, where we determine the geometric properties of the basis in order to gain an understanding of its relationship to optimal control.

The following recursive algorithm for the computation of the basis elements of the B-splines is taken from [9]:

Algorithm 1 *Let* $N_{0,0}(s) = 1$ *and, for* $i = 1, 2, \cdots, k$,

$$\begin{cases} N_{0,i}(s) & = & \frac{1-s}{i} N_{0,i-1}(s) \\ N_{i,i}(s) & = & \frac{s}{i} N_{i-1,i-1}(s) \\ N_{j,i}(s) & = & \frac{i-j+s}{i} N_{j-1,i-1}(s) + \frac{1+j-s}{i} N_{j,i-1}(s), \; j = 1, \cdots, i-1. \end{cases}$$

We then define the basis element $B_k(s)$ as

$$B_k(s) = \begin{cases} N_{k-j,k}(s-j) & j \le s < j+1 \quad j = 0, \cdots, k \\ 0 & s < 0 \text{ or } k+1 \le s. \end{cases} \tag{1}$$

The spline function of degree k is given by a weighted sum of shifted B-splines for a fixed value of k, i.e. the spline function becomes

$$S_k(t) = \sum_{i=1}^{M} c_i B_k(t-i+1). \tag{2}$$

For now our primary objects of interest are the basis element $B_k(s)$. We first note that

$$B_k(0) = B_k^{(1)}(0) = \cdots = B_k^{(k-1)}(0) = 0,$$

as well as

$$B_k(k+1) = B_k^{(1)}(k+1) = \cdots = B_k^{(k-1)}(k+1) = 0,$$

where $B_k^{(l)}(\cdot)$ denotes the lth derivative.

We furthermore observe that $B_k(s)$ is a piecewise polynomial of degree k and that it is $k-1$ times continuously differentiable. These are of course just the properties that make it a polynomial spline. We are, however, particularly concerned with the characterization of the kth derivative. This function is piecewise constant and if we use a piecewise constant input to the controlled differential equation

$$\frac{d^k}{dt^k} y(t) = u(t)$$

we can generate the function $B_k(t)$. It is tedious to compute the derivatives of the general $B_k(t)$ so in the remainder of this paper we restrict ourselves to the cubic case.

In Table 1 we calculate the first few of the elements, using Algorithm 1. The derivatives of the cubic terms are calculated and presented in Table 2. To define the generating basis function we furthermore make the appropriate shifts, presented in Table 3. We are not interested in the quadratic case so we calculate the derivatives of the cubic terms in Table 2

3 A geometric approach

We know that the B-spline, B_3, should have the property that the $B_3(0) = B_3^{(1)}(0) = B_3^{(2)}(0) = 0$ and $B_3(4) = B_3^{(1)}(4) = B_3^{(2)}(4) = 0$, where the spline is defined on the interval $[0, 4]$, in order to ensure that it has two continuous derivatives over the entire real line.

We first observe that

$$B_3(t) = \frac{a}{3} t^3, \quad 0 \le t \le 1$$

and that

$$B_3(t) = \frac{d}{3}(t-4)^3, \quad 3 \le t \le 4$$

for some choice of a and d. Now, on the interval $[1, 2]$

$$B_3^{(1)}(t) = b(t-1)^2 + \alpha(t-1) + \gamma$$

and in order for $B_3^{(1)}(t)$ to be continuous we must have $\gamma = a$. For $B_3^{(2)}(t)$ to be continuous we must furthermore have $\alpha = 2a$. Thus we have that

$$B_3^{(1)}(t) = b(t-1)^2 + 2a(t-1) + a.$$

We can show in a similar fashion that on the interval $[2, 3]$ we must have

$$B_3^{(1)}(t) = c(t-3)^2 - 2d(t-3) + d.$$

We now have four free parameters a, b, c, d that need to be determined.

In order to achieve continuity of the first derivatives at $t = 2$ we must have

$$B_3^{(1)}(2) = b + 2a + a = c + 2d + d,$$

and in order to have continuity of the second derivatives we must have

$$B_3^{(2)}(2) = 2b + 2a = -2c - 2d.$$

Thus we have used two of the degrees of freedom. We now integrate $B_3^{(1)}(t)$ to obtain $B_3(t)$. Since $B_3^{(1)}(t)$ is continuous on the interval $[0, 4)$ the function

$$y(t) = \int_0^t B_3^{(1)}(s)ds$$

will likewise be continuous and will have the proper degrees of zeros at $t = 0$. We must insure that

$$y(4) = 0$$

but this can happen if and only if

$$\int_0^4 B_3^{(1)}(s)ds = 0.$$

We begin by writing $B_3^{(1)}(t)$ as

$$B_3^{(1)}(t) = \begin{cases} at^2 & 0 \le t \le 1 \\ b(t-1)^2 + 2a(t-1) + a & 1 \le t \le 2 \\ c(t-3)^2 - 2d(t-3) + d & 2 \le t \le 3 \\ d(t-4)^2 & 3 \le t \le 4 \end{cases} \tag{3}$$

and $B_3^{(2)}(t)$ as

$$B_3^{(2)}(t) = \begin{cases} 2at & 0 \le t \le 1 \\ 2b(t-1) + 2a & 1 \le t \le 2 \\ 2c(t-3) - 2d & 2 \le t \le 3 \\ 2d(t-4) & 3 \le t \le 4. \end{cases} \tag{4}$$

Integrating $B_3^{(1)}(t)$ gives

$$B_3(t) = \int_0^t B_3^{(1)}(s)ds,$$

which in turn gives us the following

$$B_3(t) = \begin{cases} \frac{a}{3}t^3 & 0 \leq t \leq 1 \\ \frac{a}{3} + \frac{b}{3}(t-1)^3 + a(t-1)^2 + a(t-1) & 1 \leq t \leq 2 \\ \frac{7a+b+c+6d}{3} + \frac{c}{3}(t-3)^3 - d(t-3)^2 + d(t-3) & 2 \leq t \leq 3 \\ \frac{7a+b+c+7d}{3} + \frac{d}{3}(t-4)^3 & 3 \leq t \leq 4. \end{cases} \tag{5}$$

Now, in order for $B_3(4) = 0$ we must have

$$7a + b + c + 7d = 0.$$

Solving this system of three equations in four unknowns we have

$$\begin{aligned} 3a + b - c - 3d &= 0 \\ a + b + c + d &= 0 \\ 7a + b + c + 7d &= 0, \end{aligned}$$

or in other words

$$\begin{aligned} a &= -d \\ b &= -c \\ b &= -3a. \end{aligned}$$

Thus we may use a as the free parameter to obtain

$$B_3(t) = \begin{cases} a\frac{1}{3}t^3 & 0 \leq t \leq 1 \\ a(\frac{1}{3} - (t-1)^3 + (t-1)^2 + (t-1)) & 1 \leq t \leq 2 \\ a(\frac{1}{3} + (t-3)^3 + (t-3)^2 - (t-3)) & 2 \leq t \leq 3 \\ a\frac{-1}{3}(t-4)^3 & 3 \leq t \leq 4. \end{cases} \tag{6}$$

From this we see that a is just a scaling parameter and the continuity of the derivatives can easily be checked.

Now that the parameters have been chosen we can evaluate the second and third derivatives to obtain

$$B_3^{(2)}(t) = \begin{cases} 2at & 0 \leq t \leq 1 \\ -6a(t-1) + 2a & 1 \leq t \leq 2 \\ 6a(t-3) + 2a & 2 \leq t \leq 3 \\ -2a(t-4) & 3 \leq t \leq 4. \end{cases} \tag{7}$$

$$B_3^{(3)}(t) = \begin{cases} 2a & 0 \leq t < 1 \\ -6a & 1 \leq t < 2 \\ 6a & 2 \leq t < 3 \\ -2a & 3 \leq t < 4. \end{cases} \tag{8}$$

Letting

$$u = B_3^{(3)}(t)$$

gives us that the spline function can be uniquely generated by the control system

$$\frac{d^3}{dt^3}x = u, \;\; x(0) = \dot{x}(0) = \ddot{x}(0) = 0,$$

for a given choice of a.

4 An optimal control approach

A natural question to ask is if the basis element for the B-splines $B_k(t)$ is optimal with respect to some standard optimal control law in the same sense that interpolating and smoothing splines are optimal. Because of the initial and terminal conditions care must be taken in the formulation of the optimization problem. In this section we continue to restrict our attention to cubic case for ease of computation. In principle we could work with any dimension but in practice the calculations would become very cumbersome. However, it is natural to use the system

$$\frac{d^3}{dt^3}x = u$$

because in this case we can prescribe the correct boundary values in a very natural manner as

$$x(0) = \dot{x}(0) = \ddot{x}(0) = 0$$

and

$$x(4) = \dot{x}(4) = \ddot{x}(4) = 0.$$

We are thus asking for a control that is piecewise constant to generate the B-spline. From the proceeding work we have that the control

$$B_3^{(3)}(t) = \begin{cases} 2a & 0 \leq t < 1 \\ -6a & 1 \leq t < 2 \\ 6a & 2 \leq t < 3 \\ -2a & 3 \leq t < 4 \end{cases} \tag{9}$$

is the desired control. Now in the space of B-splines with nodes at the integers, the B-spline from the previous section is certainly optimal with respect to some optimal control law due to its uniqueness. However, a reasonable question to ask is if it is the solution to the following problem.

Problem 3

$$\min_{u \in L_\infty[0,4]} \|u\|_{L_\infty}$$

subject to the constraints

$$\frac{d^3}{dt^3}x = u,$$

$$x(0) = \dot{x}(0) = \ddot{x}(0) = 0,$$

$$x(4) = \dot{x}(4) = \ddot{x}(4) = 0,$$

and $x(2) = \xi$ *for given* $\xi (\neq 0)$.

In other words, are the nodes forced by some choice of the optimal control law? Surprisingly the answer is no. There is a bang-bang control law that does better than the uniform B-spline, as we will see in what follows.

4.1 Dual Optimization

Since the B-spline passes through the point ξ at time $t = 2$, the augmented optimization constraints become

$$\begin{aligned}
\ddot{x}(4) &= \int_0^4 u(t)dt = 0 \\
\dot{x}(4) &= \int_0^4 \int_0^t u(s)dsdt = \int_0^4 (4-t)u(t)dt = 0 \\
x(4) &= \int_0^4 \int_0^t \int_0^s u(r)drdsdt = \tfrac{1}{2}\int_0^4 (4-t)^2 u(t)dt = 0 \\
x(2) &= \tfrac{1}{2}\int_0^4 (2-t)_2^2 u(t)dt = \xi,
\end{aligned}$$

where $(2-t)_2^2 = (2-t)^2$ if $t \leq 2$ and 0 otherwise. These constraints can in turn be rewritten, adopting an inner product notation, as

$$\begin{aligned}
&\langle 1, u\rangle = 0 \\
&\langle 4-t, u\rangle = 0 \\
&\langle (4-t)^2, u\rangle = 0 \\
&\langle (2-t)_2^2, u\rangle = 2\xi,
\end{aligned}$$

where the inner product is taken between elements in $L_\infty[0,4]$ and $L_1[0,4]$, which is the dual space of $L_\infty[0,4]$.

Now, in [10] the following standard theorem in dual optimization can be found:

Theorem 1 *Let X be a Banach space and let $X^\star$ be the dual of X. Given $y_i \in X$, $i = 1, \ldots, p$, suppose that*

$$D = \{x^\star \in X^\star \mid \langle y_i, x^\star\rangle = c_i,\ i = 1, \ldots, p\}$$

is nonempty. Then

$$\min_{x^\star \in D} \|x^\star\| = \max_{\|Ya\| \leq 1} c^T a,$$

where $c = (c_1, \ldots, c_p)^T$ and $Ya = y_1 a_1 + \ldots + y_p a_p$. Furthermore, the optimal $\hat{a}$ and $\hat{x}^\star$ satisfy

$$\langle Y\hat{a}, \hat{x}^\star\rangle = \|Y\hat{a}\| \cdot \|\hat{x}^\star\|.$$

By applying Theorem 1 to our problem, the dual maximization problem becomes

$$\max_{\|Ya\|_{L_1} \leq 1} \xi a_4,$$

where $a = (a_1, a_2, a_3, a_4)^T$ and

$$Ya = a_1 + a_2(4-t) + a_3(4-t)^2 + a_4(2-t)_2^2.$$

If $a^\star \in L_1[0,4]$ solves the dual problem then the optimal $u^\star$ has to satisfy

$$\langle Ya^\star, u^\star\rangle = \|Ya^\star\|_{L_1}\|u^\star\|_{L_\infty}.$$

This directly gives that $|u^\star|$ has to be constant on the entire interval and that it only changes sign when Ya changes sign. It is thus a bang-bang controller that solves the problem.

Since Ya is a quadratic function for both $t \leq 2$ and $t > 2$ and is continuous, together with its first derivative, at $t = 2$, we see that Ya and hence $u^\star$ changes sign three times at most. Moreover if the number of sign changes of $u^\star$ in the interval $(0,4)$ is less than three, we can easily show that the constraints $\ddot{x}(4) = \dot{x}(4) = x(4) = 0$ cannot be satisfied. We thus assume that $u^\star$ changes sign three times in $(0,4)$ and

$$u^\star(t) = \begin{cases} U & 0 \leq t < t_1 \\ -U & t_1 \leq t < t_2 \\ U & t_2 \leq t < t_3 \\ -U & t_3 \leq t < 4 \end{cases} \tag{10}$$

with $0 < t_1 < t_2 < t_3 < 4$. Then the constraints $\ddot{x}(4) = \dot{x}(4) = x(4) = 0$ are computed respectively as

$$\begin{aligned}
&\textstyle\int_0^4 u^\star(t)dt = U(2t_1 - 2t_2 + 2t_3 - 4) = 0, \\
&\textstyle\int_0^4 (4-t)u^\star(t)dt = U\left[-(4-t_1)^2 + (4-t_2)^2 - (4-t_3)^2 + 8\right] = 0, \\
&\textstyle\int_0^4 (4-t)^2 u^\star(t)dt = U\left[-\frac{2}{3}(4-t_1)^3 + \frac{2}{3}(4-t_2)^3 - \frac{2}{3}(4-t_3)^3 + \frac{64}{3}\right] = 0.
\end{aligned}$$

Thus with

$$r_1 = 4 - t_1, \ \ r_2 = 4 - t_2, \ \ r_3 = 4 - t_3,$$

we get

$$\begin{cases} r_1 - r_2 + r_3 = 2 \\ r_1^2 - r_2^2 + r_3^2 = 8 \\ r_1^3 - r_2^3 + r_3^3 = 32. \end{cases} \tag{11}$$

Solving this system of algebraic equations with $4 > r_1 > r_2 > r_3 > 0$ yields the unique solution

$$r_1 = 2 + \sqrt{2}, \ \ r_2 = 2, \ \ r_3 = 2 - \sqrt{2},$$

and we obtain optimal switching times as

$$t_1 = 2 - \sqrt{2}, \ \ t_2 = 2, \ \ t_3 = 2 + \sqrt{2}. \tag{12}$$

On the other hand, the value of U is obtained from the constraint $x(2) = \xi$ with $\xi = 2/3$, namely

$$\frac{1}{2}\int_0^2 (2-t)^2 u^\star(t)dt = \frac{2}{3}U(2-\sqrt{2}) = \xi = \frac{2}{3}$$

yielding

$$U = \frac{2+\sqrt{2}}{2}. \tag{13}$$

Finally, the optimal solution $x(t)$ is obtained as

$$\frac{1}{U}x(t) = \begin{cases} \frac{1}{6}t^3 & 0 \le t < t_1 \\ \frac{1}{6}t^3 - \frac{1}{3}(t-t_1)^3 & t_1 \le t < 2 \\ \frac{1}{6}(4-t)^3 - \frac{1}{3}(t_3-t)^3 & 2 \le t < t_3 \\ \frac{1}{6}(4-t)^3 & t_3 \le t < 4. \end{cases} \tag{14}$$

In Figure 1, the solution $x(t)$ is depicted together with the B-spline $B_3(t)$ (dotted line).

4.2 Bang-Bang Control

There is another way in which to approach this problem. We can assume that there is a bang-bang control law and simply ask if the nodes are forced.

Let

$$u(t) = \begin{cases} 1 & 0 \le t < t_1 \\ -1 & t_1 \le t < t_2 \\ 1 & t_2 \le t < t_3 \\ -1 & t_3 \le t < 4. \end{cases} \tag{15}$$

We only assume that $0 \le t_1 \le t_2 \le t_3 \le 4$. We directly calculate the integrals of u as

$$\int_0^t u(s)ds = \begin{cases} t & 0 \le t < t_1 \\ t_1 - (t-t_1) & t_1 \le t < t_2 \\ -(t_3-4) + (t-t_3) & t_2 \le t < t_3 \\ -(t-4) & t_3 \le t < 4 \end{cases} \tag{16}$$

Now, in order to have continuity at $t = t_2$ we must have

$$t_1 - (t_2 - t_1) = -(t_3 - 4) + (t_2 - t_3) \tag{17}$$

and

$$\int_0^t (t-s)u(s)ds = \begin{cases} \frac{1}{2}t^2 & 0 \le t < t_1 \\ \frac{1}{2}t_1^2 + t_1(t-t_1) - \frac{1}{2}(t-t_1)^2 & t_1 \le t < t_2 \\ -\frac{1}{2}(t_3-4)^2 - (t_3-4)(t-t_3) + \frac{1}{2}(t-t_3)^2 & t_2 \le t < t_3 \\ -\frac{1}{2}(t-4)^2 & t_3 \le t < 4. \end{cases} \tag{18}$$

Again in order to have continuity at $t = t_2$ it must hold that

$$\frac{1}{2}t_1^2 + t_1(t_2-t_1) - \frac{1}{2}(t_2-t_1)^2 = -\frac{1}{2}(t_3-4)^2 - (t_3-4)(t_2-t_3) + \frac{1}{2}(t_2-t_3)^2, \tag{19}$$

as well as

$$\begin{aligned} &\tfrac{1}{2}\textstyle\int_0^t (t-s)^2 u(s)ds = \\ &= \begin{cases} \frac{1}{3!}t^3 & 0 \le t < t_1 \\ \frac{1}{3!}t_1^3 + \frac{1}{2}t_1^2(t-t_1) + \frac{1}{2}t_1(t-t_1)^2 - \frac{1}{3!}(t-t_1)^3 & t_1 \le t < t_2 \\ -\frac{1}{3!}(t_3-4)^3 - \frac{1}{2}(t_3-4)^2(t-t_3) - \frac{1}{2}(t_3-4)(t-t_3)^2 + \frac{1}{3!}(t-t_3)^3 & t_2 \le t < t_3 \\ -\frac{1}{3!}(t-4)^3 & t_3 \le t < 4. \end{cases} \end{aligned} \tag{20}$$

Again in order to have continuity at $t = t_2$ we must have

$$\begin{aligned}&\tfrac{1}{3!}t_1^3 + \tfrac{1}{2}t_1^2(t_2 - t_1) + \tfrac{1}{2}t_1(t_2 - t_1)^2 - \tfrac{1}{3!}(t_2 - t_1)^3 = \\ &= -\tfrac{1}{3!}(t_3 - 4)^3 - \tfrac{1}{2}(t_3 - 4)^2(t_2 - t_3) - \tfrac{1}{2}(t_3 - 4)(t_2 - t_3)^2 + \tfrac{1}{3!}(t_2 - t_3)^3.\end{aligned} \tag{21}$$

Now, let

$$a = t_1 - 0 \tag{22}$$

$$b = t_2 - t_1 \tag{23}$$

$$c = t_3 - t_2 \tag{24}$$

$$d = 4 - t_3. \tag{25}$$

Substituting these into Equations (17), (19) and (21) gives that

$$a - b = d - c \tag{26}$$

$$\frac{1}{2}a^2 + ab - \frac{1}{2}b^2 = -\frac{1}{2}d^2 - cd + \frac{1}{2}c^2 \tag{27}$$

$$\frac{1}{3!}a^3 + \frac{1}{2}a^2b + \frac{1}{2}ab^2 - \frac{1}{3!}b^3 = \frac{1}{3!}d^3 + \frac{1}{2}d^2c + \frac{1}{2}dc^2 - \frac{1}{3!}c^3 \tag{28}$$

and we get a fourth equation

$$a + b + c + d = 4. \tag{29}$$

Adding the two linear equations (26) and (29) together gives

$$a + c = 2$$

and when subtracting them we get

$$b + d = 2.$$

Substituting for a and b in the quadratic equation we have

$$(2 - c)^2 + 2(2 - c)(2 - d) - (2 - d)^2 = c^2 - 2cd - d^2.$$

Expanding the left hand side we have

$$4 - 4c + c^2 + 8 - 4d - 4c + 2cd - 4 + 4d - d^2 = c^2 - 2cd - d^2,$$

which upon collecting terms gives

$$cd - 2c + 2 = 0.$$

Manipulating the cubic equation gives

$$(a + b)^3 - 2b^3 = (d + c)^3 - 2c^3.$$

Substituting the linear terms for a and b gives

$$(4 - (c + d))^3 - 2(2 - d)^3 = (d + c)^3 - 2c^3$$

and expanding the left hand side gives

$$-(c+d)^3 + 12(c+d)^2 - 48(c+d) + 64 + 2(d^3 - 6d^2 + 12d - 8) = (d+c)^3 - 2c^3$$

Further simplification gives

$$2(c^3 + 3c^2d + 3cd^2 + d^3) - 12(c^2 + 2cd + d^2) + 48(c+d) - 64 - 2d^3 + 12d^2 - 24d + 16 - 2c^3 = 0$$

$$(c^2d + cd^2) - 2c^2 - 4cd + 8c + 4d - 8 = 0$$

Using

$$cd = 2c - 2$$

we have

$$(2c-2)c + (2c-2)d - 2c^2 - 4(2c-2) + 8c + 4d - 8 = 0$$

$$2c^2 - 2c + 2cd - 2d - 2c^2 - 8c + 8 + 8c + 4d - 8 = 0$$

$$+2c - 4 + 2d = 0$$

We now have reduced the system to the following simple set of equations.

$$\begin{aligned} c + d &= 2 \\ a + c &= 2 \\ b + d &= 2 \\ cd - 2c + 2 &= 0 \end{aligned}$$

Solving this system of equations gives a positive solution

$$c = b = \sqrt{2}, \quad a = d = 2 - \sqrt{2}$$

and finally

$$t_1 = 2 - \sqrt{2}, \quad t_2 = 2, \quad t_3 = 2 + \sqrt{2},$$

which is consistent with the solution to the optimal control problem in the previous subsection.

5 The Construction of Optimal Splines Using B-splines

As we saw in the preceding section the basis elements for uniform B-splines are not optimal in any usual sense. However we can find in the class of all B-splines optimal choices. We will construct two types of splines in analogy with dynamic splines, interpolating and approximating.

Let a data set D be given in R as

$$D = \{\alpha_i \in \mathrm{R} : \quad i = 1, \cdots, N\}.$$

Consider the system

$$\frac{d^k}{dt^k} x(t) = u(t)$$

and a restricted set of controls

$$C = \{u(t): \quad u(t) = \sum_{i=1}^{M} \tau_i \frac{d^k}{dt^k} B_k(t-i+1), \quad \tau_i \in \mathrm{R}\}.$$

We choose the two cost functions from the theory of dynamic splines

$$J(u) = \int_{-\infty}^{\infty} u^2(t)dt \tag{30}$$

and

$$J(u) = \int_{-\infty}^{\infty} u^2(t)dt + \sum_{i=1}^{N} w_i(x(t_i) - \alpha_i)^2 \tag{31}$$

and we now pose two related problems:

Problem 4 (Interpolation)

$$\min_{u \in C} \int_{-\infty}^{\infty} u^2(t)dt$$

subject to the constraints

$$x(t_i) = \alpha_i, \quad i = 1, \cdots, N$$

and

Problem 5 (Approximation)

$$\min_{u \in C} \int_{-\infty}^{\infty} u^2(t)dt + \sum_{i=1}^{N} w_i(x(t_i) - \alpha_i)^2$$

We can integrate over the entire real line since the B-splines and all of their derivatives vanish outside of a compact set and since the control is allowed to only be a finite sum. Now these problems are both finite dimensional because the space of controls is finite dimensional. It should be noted that they differ from Problem 1 and Problem 2 only in the space of controls and that the number of basis elements is not necessarily the same as the number of data points. This is different that Problem 1 and Problem 2. There the number and the form of the basis elements is determined by the number of data points. Since in this case we have chosen a basis that constraint is lifted. We now proceed to reduce the form of the problems. We substitute the control function into $J(u)$. For the cost function of equation (30) we have

$$J(\tau) = \int_{-\infty}^{\infty} (\sum_{i=1}^{M} \tau_i \frac{d^k}{dt^k} B_k(t-i+1))^2 dt$$

and we see that we must integrate for every i and j

$$\int_{-\infty}^{\infty} \frac{d^k}{dt^k} B_k(t-i+1) \frac{d^k}{dt^k} B_k(t-j+1)dt$$

We only note that if $i-j>k$ then this integral is zero. We will explicitly calculate the integrals in the case that $k=3$. We recall the definition of $B_3(t)$ to get the third derivative as

$$\frac{d^3}{dt^3}B_3(t)=\begin{cases} 1 & 0\le t<1\\ -3 & 1\le t<2\\ 3 & 2\le t<3\\ -1 & 3\le t<4 \end{cases} \tag{32}$$

We now explicitly calculate the products.

$$\frac{d^3}{dt^3}B_3(t)\frac{d^3}{dt^3}B_3(t)=\begin{cases} 1 & 0\le t<1\\ 9 & 1\le t<2\\ 9 & 2\le t<3\\ 1 & 3\le t<4 \end{cases} \tag{33}$$

$$\frac{d^3}{dt^3}B_3(t)\frac{d^3}{dt^3}B_3(t-1)=\begin{cases} 0 & 0\le t<1\\ -3 & 1\le t<2\\ -9 & 2\le t<3\\ -3 & 3\le t<4\\ 0 & 4\le t<5 \end{cases} \tag{34}$$

$$\frac{d^3}{dt^3}B_3(t)\frac{d^3}{dt^3}B_3(t-2)=\begin{cases} 0 & 0\le t<1\\ 0 & 1\le t<2\\ 3 & 2\le t<3\\ 3 & 3\le t<4\\ 0 & 4\le t<5\\ 0 & 5\le t<6 \end{cases} \tag{35}$$

$$\frac{d^3}{dt^3}B_3(t)\frac{d^3}{dt^3}B_3(t-3)=\begin{cases} 0 & 0\le t<1\\ 0 & 1\le t<2\\ 0 & 2\le t<3\\ -1 & 3\le t<4\\ 0 & 4\le t<5\\ 0 & 5\le t<6\\ 0 & 6\le t<7 \end{cases} \tag{36}$$

and

$$\frac{d^3}{dt^3}B_3(t)\frac{d^3}{dt^3}B_3(t-i)=0 \quad i=4,5,\cdots \tag{37}$$

We then form the Table 4.

With $\tau=(\tau_1,\tau_2,\cdots,\tau_M)^T$, we can then write

$$J(\tau)=\tau^T G\tau$$

where G is the grammian whose mth row is given by

$$(0,\cdots,0,-1,6,-15,20,-15,6,-1,0,\cdots,0)$$

and the number 20 is in the $m\times m$ spot. The first row is given by

$$(20,-15,6,-1,0,\cdots,0)$$

and the last row is given by

$$(0, \cdots, 0, -1, 6, -15, 20).$$

The matrix G is positive definite since the basis elements are independent functions.

We now calculate the constraints as functions of τ. We have after integrating that

$$x(t) = \sum_{i=1}^{M} \tau_i B_3(t - i + 1)$$

and hence the constraint is given by

$$x(t_j) = \sum_{i=1}^{M} \tau_i B_3(t_j - i + 1) = \alpha_j, \quad j = 1, 2, \cdots, N.$$

From the structure of B_3 we see that

$$B_3(t_j - i + 1) \neq 0$$

if and only if

$$i - 1 < t_j < i + 3.$$

Let B denote the matrix such that

$$B\tau = \alpha$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)^T$. Problem 4 then reduces to the following

$$\min_{\tau \in \mathrm{R}^M} \tau^T G \tau \tag{38}$$

subject to

$$B\tau = \alpha. \tag{39}$$

Now the equation $B\tau = \alpha$ may have no solutions, i.e. the conditions are inconsistent, or it may have a unique solution in which case the minimization problem is irrelevant or it may have infinitely many solutions in which case the minimization problem is non empty. In the latter two cases the problem is solved via the pseudo inverse or if one prefers the equivalent problem of finding the point of minimum norm in the affine variety defined by

$$V_\alpha = \{\tau : \quad B\tau = \alpha\}$$

which in fact defines the pseudo inverse. Note that if the number of data points N is larger than the number of basis elements, M, then there is a high probability that the system $B\tau = \alpha$ will be inconsistent. It is also noted that if $B\tau = \alpha$ is consistent and the matrix B is of row full rank, then the optimal solution $\tau^\star$ is given explicitly as

$$\tau^\star = G^{-1} B^T (B G^{-1} B^T)^{-1} \alpha. \tag{40}$$

Problem 5, the problem of approximation, can be rewritten in a similar manner as

$$\begin{aligned} J(\tau) &= \tau^T G\tau + \sum_{i=1}^{N} w_i(\sum_{j=1}^{M} \tau_j B_3(t_i - j + 1) - \alpha_i)^2 \\ &= \tau^T G\tau + \sum_{i=1}^{N} w_i(e_i^T B\tau - \alpha_i)^2 \\ &= \tau^T G\tau + \tau^T B^T W B\tau - 2\alpha^T W B\tau + \alpha^T W\alpha \end{aligned}$$

where W is the diagonal matrix with the weights w_i on the diagonal. Problem 5 reduces to the following

$$\min_{\tau \in \mathrm{R}^M} \tau^T G\tau + \tau^T B^T W B\tau - 2\alpha^T W B\tau. \tag{41}$$

The solution to this minimization problem is given by the solution to the linear equation

$$(G + B^T W B)\tau = B^T W\alpha. \tag{42}$$

The matrix G is positive definite and the matrix $B^T W B$ is at least positive semi-definite so the system has a unique solution for τ. The optimal solution $\hat{\tau}^\star$ is thus given as

$$\hat{\tau}^\star = (G + B^T W B)^{-1} B^T W\alpha. \tag{43}$$

Moreover if W is positive-definite, then the matrix inversion lemma gives

$$(G + B^T W B)^{-1} B^T W = G^{-1} B^T (W^{-1} + BG^{-1} B^T)^{-1}$$

and we have an alternative expression for $\hat{\tau}^\star$ as

$$\hat{\tau}^\star = G^{-1} B^T (W^{-1} + BG^{-1} B^T)^{-1}\alpha. \tag{44}$$

Here if the matrix B is of row full rank, (44) is further written as

$$\hat{\tau}^\star = G^{-1} B^T (BG^{-1} B^T)^{-1} \left[I + W^{-1}(BG^{-1} B^T)^{-1}\right]^{-1} \alpha.$$

Thus, as $W = \mathrm{diag}\{w_1, w_2, \cdots, w_N\} \to \mathrm{diag}\{+\infty, +\infty, \cdots, +\infty\}$, W^{-1} reduces to zero matrix, and we see that the solution $\hat{\tau}^\star$ converges to the solution $\tau^\star$ for interpolation problem given in (40) .

6 Control Points and Control Polygons

The concepts of control points and control polygons are essential to the application of B-splines and for that matter Bezier curves. A spline, $s(t)$, of degree k in R^n is constructed using the basis of B-splines $B_k(t)$ as

$$s(t) = \sum_{i=1}^{M} \tau_i B_k(t - i + 1)$$

and the set of points

$$\{\tau_i \in \mathrm{R}^n : \quad i = 1, \cdots, M\}$$

is the set of control points. The control polygon is the polygonal line connecting the control points. The control points determine the shape of the spline function.

In the preceding section we constructed optimal weights in the scalar case. By repeating this procedure or by using a more complicated set of dynamics we can produce optimal vector valued weights. Thus given a set of data points

$$\{\alpha_i \in \mathrm{R}^n : \quad i = 1, \cdots, N\}$$

we can produce a set of control points optimal for this set of data, either as interpolation or as approximation. To see how this might be done consider a real curve $p(t) \in \mathrm{R}^n$. Our goal is to reproduce this curve using optimal B-splines. If we are precise in our description of the curve choose N points that lie on the curve,

$$D = \{p(t_i) : i = 1, \cdots, N\}.$$

We will use these points as data to construct the control points. The designer must choose these points and he must decide on the degree of the spline that he wants to construct. We assume that we are constructing a spline of degree k. Then as in the preceding section let

$$C = \{u(t) : \ u(t) = \sum_{i=1}^{M} \tau_i \frac{d^k}{dt^k} B_k(t - i + 1), \ \tau_i \in \mathrm{R}^n\}.$$

The set C consists of all allowable controls that we use in the construction of the optimal spline.

7 An example of optimal construction

From a given set of data points on some given curve $p(t)$ in R^3

$$D = \{\alpha_i \in \mathrm{R}^3 : \ \alpha_i = p(t_i), \ i = 1, 2, \cdots, N\}, \tag{45}$$

we reconstruct the curve by the optimal interpolation and approximation by B-splines. Here, as $p(t)$, we take a cubic spline in R^3 given by

$$p(t) = \sum_{i=1}^{M} p_i B_3(t - i + 1) \tag{46}$$

where $\{p_i \in \mathrm{R}^3 : \ i = 1, 2, \cdots, M\}$ is a given set of control points.

One of the important applications of splines is in the design of characters, and in the sequel we present the results as character patterns generated from the curves in R^3 as well as the curves themselves. Figure 2 shows a Japanese character pronounced 'ru', which is generated from $p(t)$ as follows. We consider a virtual writing device modeled by a cone with the tip moving along

$p(t)$ in a three dimensional space $o - xyz$ and a virtual writing plane $o - xy$. The axis of the cone is taken to be parallel to z-axis, and the cross sectional area between the cone and the writing plane results in a circle. As the tip moves, the circle moves in the plane and result in the character pattern as illustrated in the figure on the left. In this example, a set of 20 control points $\{p_i \in \mathrm{R}^3 : \ i = 1, 2, \cdots, 20\}$ counting multiplicities is used and is shown by 'squares' together with the control polygon in xy-plane. The figure on the right is the pattern obtained in this fashion, and it may be considered as a good model of an actual brush-written character.

Such a curve or character pattern is then reconstucted using optimal interpolation and approximation by cubic B-splines as

$$\begin{aligned} s(t) &= \sum_{i=1}^{M} \tau_i B_3(t - i + 1) \\ s(t) &= (x(t)\ y(t)\ z(t))^T, \ \ \tau_i = (\tau_{x,i}\ \tau_{y,i}\ \tau_{z,i})^T \end{aligned} \tag{47}$$

for various cases of data points and parameters. In order to deal with curves in R^3, we apply the method developed in Section 5 for scalar case to each of the three elements independently.

7.1 Uniform data points

The number of data points used is $N = 10$ and $p(t)$ is sampled at

$$\{t_i : \ i = 1, 2, \cdots, 10\} = \{3, 5, 7, 9, 11, 13, 15, 17, 19, 20\}.$$

Thus they are located uniformly in the time interval $(3, 20)$. The result of optimal interpolation by $s(t) \in \mathrm{R}^3$ is shown in Figure 3 as the constructed character and curve $x(t)$ on the left and right respectively. The 'cross' denotes the data points α_i, and 'square' and 'circle' denote the projection of the computed control points τ_i onto the $o - xy$ plane. We may observe that the original character is quite well recovered by interpolation.

On the other hand, Figures 4 - 6 show the results by optimal approximation where the weighting matrix W is taken respectively as $W = I, 10I, 100I$ with the same weights for each component of $s(t)$. It can be seen that, starting with the greately deformed pattern in Figure 4, the original one is recovered more accurately as the weights increase. We also verified that by increasing the weights further the pattern approaches the interpolation result in Figure 3 as was analyzed in Section 5. It might be worth noting that characters are written in a variety of deformed styles in Japanese calligraphy, which may be modeled by a suitable weight adjustment in the optimal approximation as described here.

7.2 Non-uniform data points

The number of data points used is $N = 18$ and $p(t)$ is sampled at

$$\{t_i : \ i = 1, 2, \cdots, 18\} = \{3, 4, 5, 6, 7, 8, 8.1, 8.2, 11, 12, 13, 14, 14.1, 14.2, 17, 18, 19, 20\}.$$

We see that some of the points are located dense locally. As Figure 7 shows, interpolating construction presents oscillatory pattern which is far from the original one. We believe this is a phenomena which is often pointed out in the theory of interpolation. Notice here that y component of $s(t) \in \mathrm{R}^3$ is plotted in the figure on the right.

Figures 8 - 11 are the results by optimal approximation for the weight $W = I, 100I, 10^{\,8}I, 10^{10}I$ respectively. In contrast to the interpolation, we observe the efficacy of approximating construction: For wide range of weights, say up untill $10^{\,8}I$, an oscillation seems to be suppressed. In particular, the original pattern is well reconstructed when $W = 100I$ in spite of such unbalanced data points. Again, the case of large W, $W = 10^{\,10}I$, the result almost coincides with the case of interpolation. This together with the analysis in Section 5, namely the approximation coincides with the interpolation in the limit of incereasing W, indicates that the approximating construction is stable numerically for wide range of weight values.

8 Conclusions

In this paper we investigate the connections between B-splines and linear control theory.We show how the B-spline basis functions can be obtained by driving a third order control system with a piecewise constant input. However, we also show that the B-splines are in fact suboptimal with respect to an infinity-norm minimization, and that the solution to this problem is of the bang-bang type. We show that an optimal set control points can be constructed within the space of B-splines. Finally an example is developed to demonstrate the efficacy of this construction.

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Table 1: $N_{j,k}$ for $k = 0, 1, 2, 3$.

$k : j$	0	1	2	3
0	1			
1	$(1-s)$	s		
2	$\frac{1}{2}(1-s)^2$	$\frac{1}{2}(1+2s-2s^2)$	$\frac{1}{2}s^2$	
3	$\frac{1}{3!}(1-s)^3$	$\frac{1}{3!}(4-6s^2+3s^3)$	$\frac{1}{3!}(1+3s+3s^2-3s^3)$	$\frac{1}{3!}s^3$

Table 2: The derivatives of $N_{j,3}$

j	0	1	2	3
$N_{j,3}$	$\frac{1}{3!}(1-s)^3$	$\frac{1}{3!}(4-6s^2+3s^3)$	$\frac{1}{3!}(1+3s+3s^2-3s^3)$	$\frac{1}{3!}s^3$
$N_{j,3}^{(1)}$	$\frac{-1}{3!}3(1-s)^2$	$\frac{1}{3!}(-12s+9s^2)$	$\frac{1}{3!}(3+6s-9s^2)$	$\frac{1}{3!}3s^2$
$N_{j,3}^{(2)}$	$(1-s)$	$\frac{1}{3!}(-12+18s)$	$\frac{1}{3!}(6-18s)$	s
$N_{j,3}^{(3)}$	-1	3	-3	1

Table 3: The derivatives of $N_{j,3}$ shifted appropriately.

j	0	1	2	3
$N_{j,3}$	$\frac{1}{3!}(1-(s-3))^3$	$\frac{1}{3!}(4-6(s-2)^2+3(s-2)^3)$	$\frac{1}{3!}(1+3(s-1)+3(s-1)^2-3(s-1)^3)$	$\frac{1}{3!}s^3$
$N_{j,3}^{(1)}$	$\frac{-1}{2!}(1-(s-3))^2$	$\frac{1}{2!}(-4(s-2)+3(s-2)^2)$	$\frac{1}{2!}(1+2(s-1)-3(s-1)^2)$	$\frac{1}{2!}s^2$
$N_{j,3}^{(2)}$	$(1-(s-3))$	$(-2+3(s-2))$	$(1-3(s-1))$	s
$N_{j,3}^{(3)}$	-1	3	-3	1

Table 4: The inner-products of shifted B-splines

i	$\int_{-\infty}^{\infty} \frac{d^3}{dt^3} B_3(t) \frac{d^3}{dt^3} B_3(t-i) dt$
0	20
1	-15
2	6
3	-1
$i > 3$	0

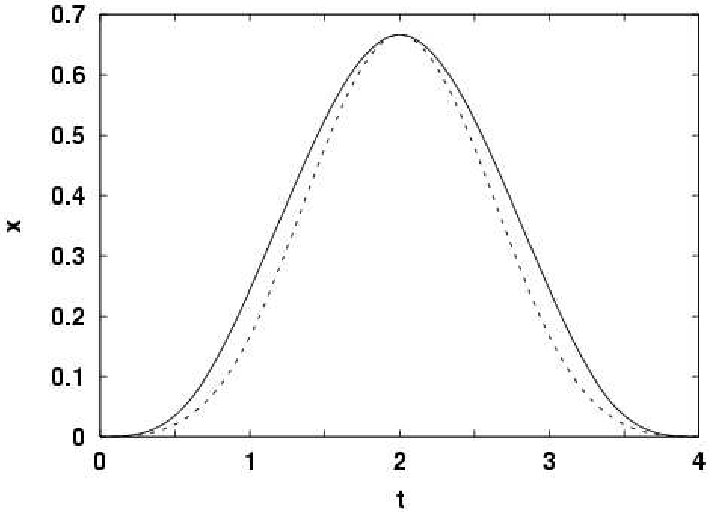

Figure 1: The optimal bang-bang solution.

Figure 2: Japanese character 'Ru' generated by cubic spline.

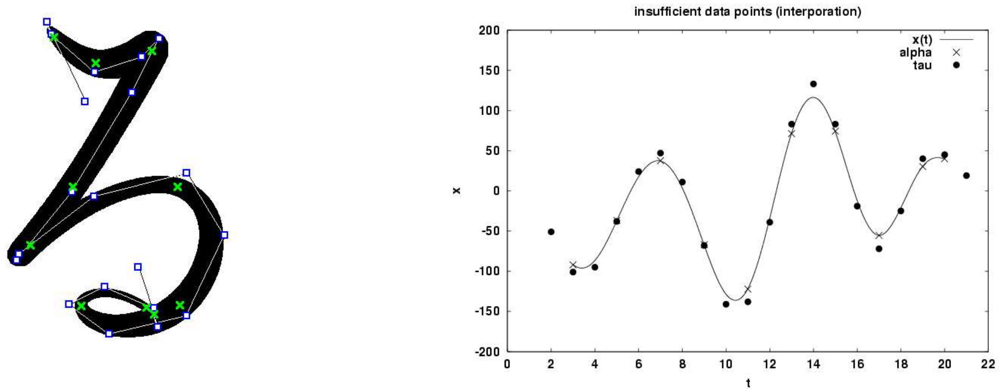

Figure 3: Interporating construction for uniform data points.

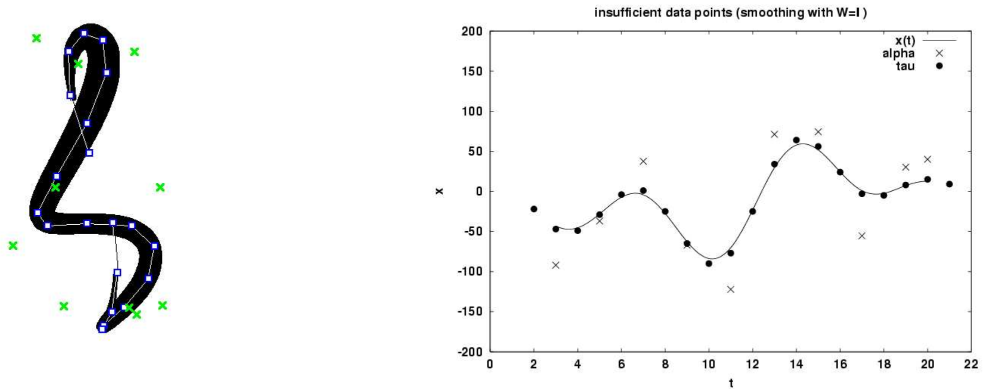

Figure 4: Approximating construction for uniform data points with $W = I$.

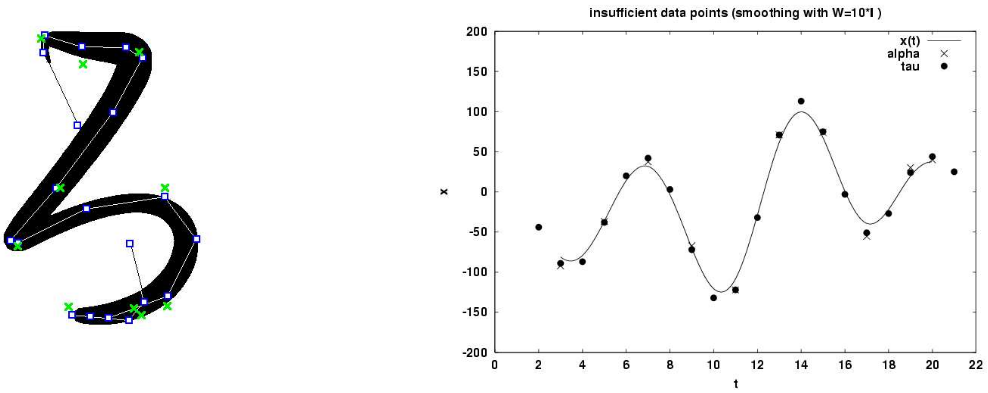

Figure 5: Approximating construction for uniform data points with $W = 10I$.

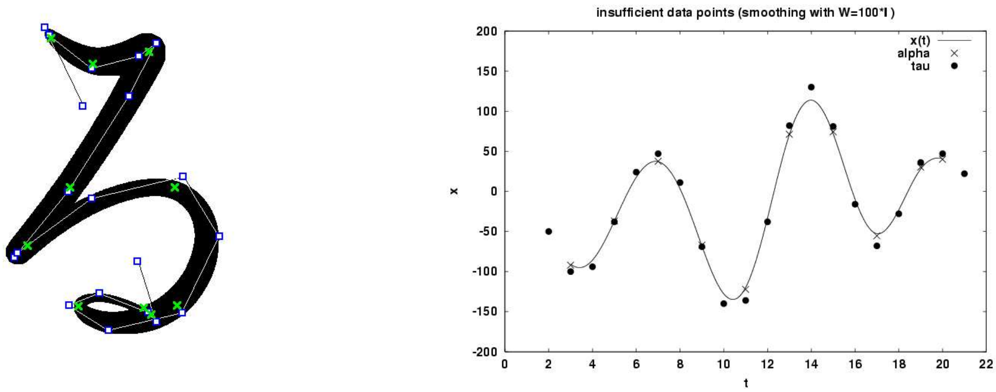

Figure 6: Approximating construction for uniform data points with $W = 100I$.

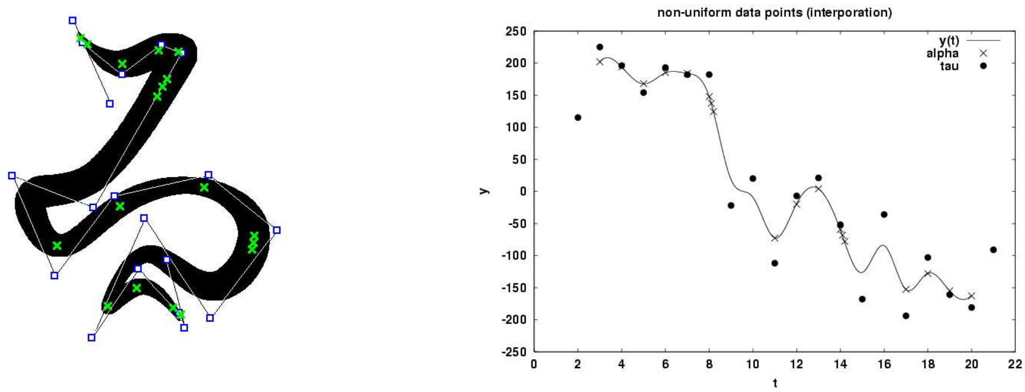

Figure 7: Interporating construction for non-uniform data points.

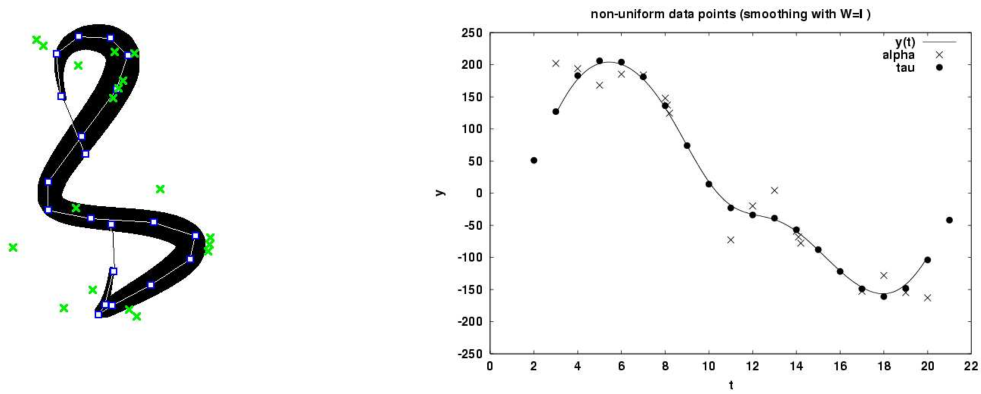

Figure 8: Approximating construction for non-uniform data points with $W = I$.

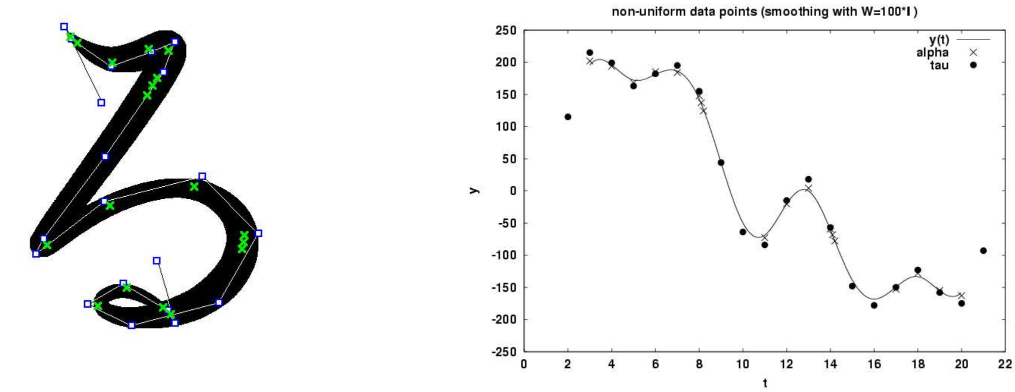

Figure 9: Approximating construction for non-uniform data points with $W = 100I$.

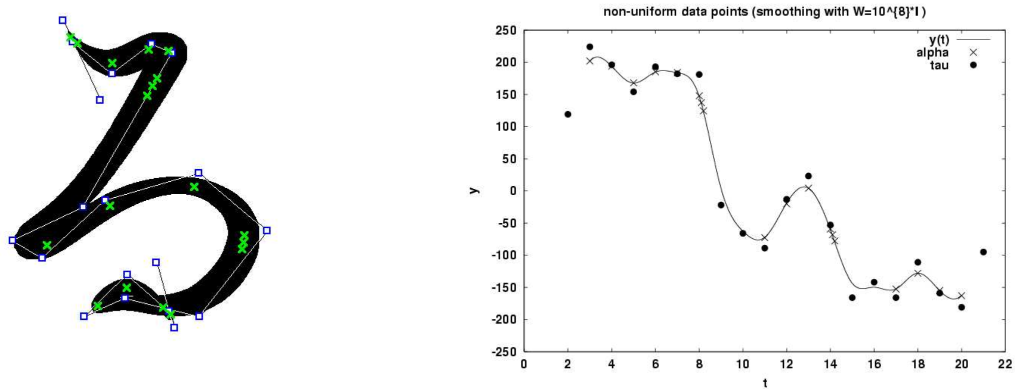

Figure 10: Approximating construction for non-uniform data points with $W = 10^{8}I$.

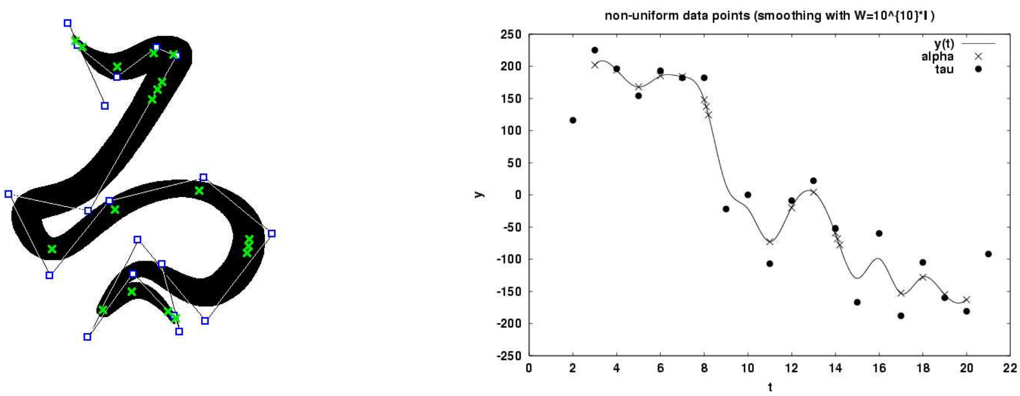

Figure 11: Approximating construction for non-uniform data points with $W = 10^{10}I$.