Connectedness Preserving Distributed Coordination Control over Dynamic Graphs

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***Abstract*— This paper presents a solution to the limited information rendezvous problem over dynamic interaction graphs. In particular, we show how we, by adding appropriate weights to the edges in the graphs, can guarantee that the graph stays connected. In previous work on graph-based coordination, connectedness have been assumed, and this paper thus shows how to overcome this limitation even when the graphs are subject to dynamic changes.**

I. Introduction

During the last decade, analysis and control of group behaviors of teams of autonomous agents has attracted significant interest due to the emergence of multi-agent robotics applications, sensor and actuator networks, and distributed embedded systems. What makes this problem challenging is that the agents are subjected to limitations on the available information, which has made graph-based models useful and natual tools for encoding these limitations [3], [6], [7], [8], [15].

The history behind this work can be traced back to Reynolds' "boids" model [1], where three *adhoc* protocols for autonomous agents, namely separation, alignment and cohesion, were defined. A special case of the "boids" model was proposed by Vicsek et al. [2], and an elegant example of graph-based control was provided in [3], where directional cohesion was achieved for Vicsek's model. Other notable contributions in this area were given in [4] with an analysis of swarm stability with fixed network topology. Moreover, in [5] social potentials were used to instrument cohesion in swarms, while flocking under switched topologies was studied in [6], where a theoretical frame-work was proposed based on graph theory. However, connectedness of the underlying graphs had to be assumed, which was also the case in [7], where stability of the flocking was proved by analyzing the algebraic connectivity of the induced graph. Moreover, in [8], state-dependent dynamic graphs were studied from a combinatoric point-of-view.

Alternative approaches to the coordination control problem were presented in [9], while a leader-follower assignment paradigm was studied in [10]. A model-independent coordination strategy was moreover proposed in [11], where a virtual leader was used to represent the desired trajectory. Hybrid control frameworks were proposed in [13] for multi-agent coordination control, while the complexity of multi-agent coordinations was studied in [12] from an information theoretic point-of-view. *These previous results all relied on the explicit assumption of connectedness of the interaction graphs.* To the best of our knowledge, the connectedness problem is left open so far in the literature. However, as it will be shown later, this assumption is vulnerable under some particular but not uncommon circumstances.

In this paper we will mainly focus on providing a solution to the rendezvous problem, i.e. the problem of driving the agents to a common point. It should be noted that this problem is solved if either connectedness is assumed [7], or connectedness is only required at distinct times [14]. In this paper we show how to make the graph stay connected for all times, and the outline of the paper is as follows: In Section 2 we review some previous results and recall some basic notions in algebraic graph theory. In Section 3 we show how to add weights in the static graph case, followed by the dynamic case in Section 4. The paper concludes with a collection of simulation results in Section 5.

II. Background

We first establish some notation and review some previous results. Given N agents $x_1, \ldots, x_N$ evolving in $\mathbb{R}^n$, we assume that the dynamics is simply given by

$$\dot{x}_i = u_i. \tag{1}$$

A fixed set of communication links is established between certain agents. By a Static Interaction Graph (SIG) $\mathcal{G} = (V, E)$ we understand the graph where the nodes $V = \{v_1, \ldots, v_N\}$ correspond to the different agents. Moreover, the edge set $E \subset V \times V$ is a set of unordered pairs of agents, with $(v_i, v_j) = (v_j, v_i) \in E$ if and only if a communication link exists between agents i and j, and we will use $V(\mathcal{G})$ and $E(\mathcal{G})$ to denote the node and edge sets respectively. Such graph-based encodings of the coordination have proved to be useful,

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and in particular algebraic graph-theory has provided tools for controlling and analyzing the coordinations. For example, an intuitive control law for handling the rendezvous problem is given by:

$$\dot{x}_i = \sum_{j \in N(i)} (x_j - x_i), \tag{2}$$

where $N(i)$ denotes the set of agents adjacent to agent i, $i = 1, 2, \ldots, N$.

Under this control law, it can be shown that all agents approach the same point asymptotically, provided that the SIG is connected. Even though this is not a new result, see for example [15], we will here sketch a proof based on algebraic graph theory.

First, we need to introduce an *orientation* associated with the SIG $\mathcal{G}$, i.e. a declaration of direction to each edge $\sigma : E(\mathcal{G}) \to \{-1, 1\}$ such that if $(v_i, v_j) \in E(\mathcal{G})$ then $\sigma(v_i, v_j) = -\sigma(v_j, v_i)$. Using this orientation, we can form the oriented graph $\mathcal{G}^\sigma$ by associating the orientation σ with $\mathcal{G}$. If now $E(\mathcal{G}) = \{e_1, \ldots, e_M\}$ then the $N \times M$ *incidence matrix* of $\mathcal{G}^\sigma$ is $\mathcal{I}(\mathcal{G}^\sigma) = [b_{ij}]$, where

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the head of } e_j \\ -1 & \text{if } v_i \text{ is the tail of } e_j \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Through the incidence matrix, we can now define the orientation-independent Laplacian as follows:

Definition 2.1: Given $\mathcal{G} = (V, E)$ together with an arbitrary orientation σ of $\mathcal{G}$, the Laplacian $\mathcal{L}(\mathcal{G}) \in \mathbb{N}^{N \times N}$ is given by

$$\mathcal{L}(\mathcal{G}) = \mathcal{I}(\mathcal{G}^\sigma)\mathcal{I}(\mathcal{G}^\sigma)^T. \tag{4}$$

The Laplacian has a number of well-studied properties, found for example in [16], including

1) $\mathcal{I}(\mathcal{G}^\sigma)\mathcal{I}(\mathcal{G}^\sigma)^T = \mathcal{I}(\mathcal{G}^{\sigma'})\mathcal{I}(\mathcal{G}^{\sigma'})^T$ for all orientations σ, σ', i.e. the Laplacian is orientation-independent.
2) $\mathcal{L}(\mathcal{G})$ is symmetric and non-negative definite.
3) Let $\{\lambda_i\}_{i=1}^N$ be the sorted eigenvalues of $\mathcal{L}(\mathcal{G})$, then $0 \leq \lambda_1 \leq \lambda_2 \cdots \leq \lambda_N$. Moreover, $\lambda_1 = 0$ and $\lambda_2 > 0$ if $\mathcal{G}$ is connected.
4) If $\mathcal{G}$ is connected then the set of eigenvectors $\nu_1, \ldots, \nu_N$ form an orthogonal basis in $\mathbb{R}^N$, and $\nu_1 = 1/\sqrt{N}\mathbf{1}$, where $\mathbf{1}$ denotes the vector with every entry equal to one. In other words, if $\mathcal{G}$ is connected then the null-space $\mathcal{N}(\mathcal{L}(\mathcal{G})) = span\{\mathbf{1}\}$.

If we denote component x_i as $x_i = (x_{i,1}, \ldots, x_{i,n})^T$ and let $x^T = (x_1^T, \ldots, x_N^T)$, we can define the component-wise operator $c(x, j) = (x_{1,j}, \ldots, x_{N,j})^T \in \mathbb{R}^N$, $j = 1, \ldots, n$, and note that along each component, the control law given in Equation 2, becomes

$$\frac{d}{dt}c(x, j) = -\mathcal{L}c(x, j). \tag{5}$$

Here we have dropped $\mathcal{L}$'s explicit dependence on $\mathcal{G}$, which we will continue to do whenever this dependence is clear from the context.

Now, as pointed out in [15], [16], if $\mathcal{G}$ is connected then the eigenvector corresponding to the semi-simple eigenvalue 0 is $\mathbf{1}$. This, together with the non-negativity of $\mathcal{L}$ and the fact that $span\{\mathbf{1}\}$ is $\mathcal{L}$-invariant, is enough to show that $c(x, j)$ approaches $span\{\mathbf{1}\}$ asymptotically.

However the main reason why graph-based abstractions are useful is that they can encode the dynamic aspects of the communication exchange in a very natural manner. Since all real sensors and transmitters have finite range, information exchange links may appear or be lost as the agents move around. If we let the maximal distance at which two agents can be separated and still exchange information be given by Δ, then we can form the Dynamic Interaction Graph (DIG) $\mathcal{G}(t) = (V, E(t))$, where $(v_i, v_j) = (v_j, v_i) \in E(t)$ if and only if $\|x_i(t) - x_j(t)\| \leq \Delta$. Note here that the edge set might be time-varying. However, the previously mentioned stability result is still useful in that it holds for *all connected graphs*. Moreover, since $c(x, j)^T c(x, j)$ is a Lyapunov function to the system in Equation 5, for any connected graph $\mathcal{G}$, the control law

$$\frac{d}{dt}c(x(t), j) = -\mathcal{L}(\mathcal{G}(t))c(x(t), j) \tag{6}$$

drives the system to span$\{\mathbf{1}\}$ asymptotically *as long as* $\mathcal{G}(t)$ *is connected for all* $t \geq 0$.

This is a very intriguing result and it shares the common feature with other graph-based results, e.g. [3], [7], in that it hinges on the connectedness assumption. Unfortunately, this property has to be assumed rather than proved, and in Figure 1 an example is shown when connectedness is lost when using Equation 6 to control the system.

What we will do for the remainder of this paper is to show how this assumption can be overcome by modifying Equation 6 in such a way that connectedness can be proved to hold for all times. This will close the gap encountered in the literature on graph-based multi-agent control.

III. Weighted Graph-Based Feedback

In this section we will draw inspiration from the previous section and modify the control law in Equation 6 in order to ensure that the graph stays connected. However, this modification must be structured in such a way that the control laws stay distributed. One obvious choice is to let

$$\dot{x}_i = d_i \sum_{j \in N(i)} w_{ij}(x_j - x_i), \tag{7}$$

where $w_{ij} = w_{ji}$. In this case we get

$$\frac{d}{dt}c(x, j) = -D\mathcal{I}^\sigma W \mathcal{I}^{\sigma T} c(x, j), \ j = 1, \ldots, n, \tag{8}$$

where $D = \text{diag}(d_1, d_2, \ldots, d_n) \in \mathbb{R}^{N \times N}$ and $W = \text{diag}(w_1, \ldots, w_M) \in \mathbb{R}^{M \times M}$ ($M = |E(\mathcal{G}^\sigma)|$) are positive definite (as long as $d_i, w_{ij} > 0$) weight matrices.

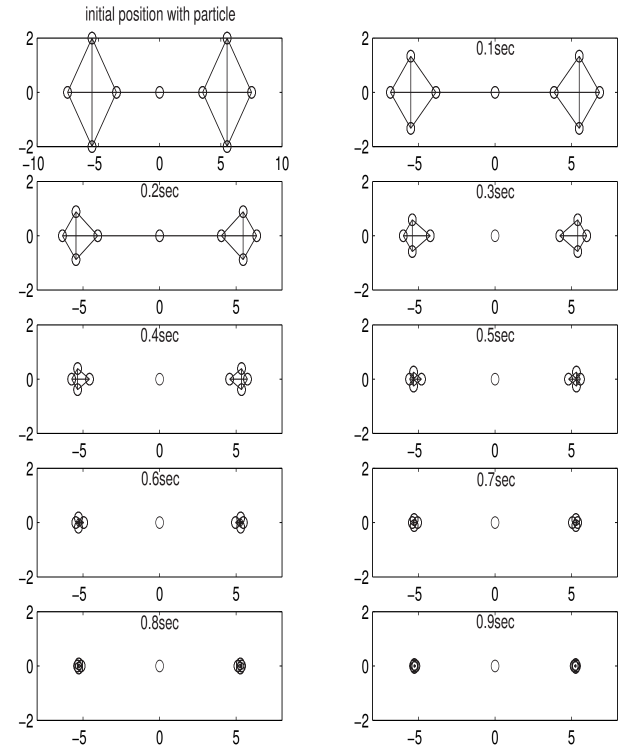


Fig. 1. A progression is shown where connectedness is lost even though the initial graph is connected ($\Delta = 4$).

The interpretation is that D associates a weight with each node while W associates a weight with each edge.

We will see that we, in fact, can let $D = I$ and still guarantee connectedness, and we define the weighted Graph-Laplacian as

$$\mathcal{L}_w \triangleq \mathcal{I}^\sigma W \mathcal{I}^{\sigma T},$$

where as before $W \in \mathbb{R}^{M\times M}$ is a diagonal matrix with each element corresponding to an edge. These weights can either be time dependent or time independent.

Since the main problem that must be dealt with is to ensure that no connections are lost we focus on the individual inter-agent distances. Given an ordering σ of $\mathcal{G}$ (connected SIG), and an edge $(v_i, v_j) \in E(\mathcal{G})$ such that $\sigma(v_i, v_j) = 1$, we let l_{ij}^σ denote the edge vector between the agents i and j, i.e. $l_{ij}^\sigma = x_i - x_j$. Hence, if we let Δ be the critical cut-off distance, we propose to use the following control strategy:

$$\dot{x}_i(t) = \sum_{j\in N(i)} \frac{(x_j(t) - x_i(t))}{(\|l_{ij}^\sigma(t)\| - \Delta)^2 \|l_{ij}^\sigma(t)\|}. \tag{9}$$

Along individual dimensions, the dynamics of the group then becomes

$$\frac{d}{dt} c(x(t), j) = -\mathcal{L}_w(t) c(x(t), j), \quad j = 1, 2, \dots n \tag{10}$$

where

$$\begin{aligned} W(t) &= diag(w_i(t)), \quad i = 1, 2, \dots |E(\mathcal{G})|, \\ w_i(t) &= \frac{1}{(\|l_i^\sigma(t)\| - \Delta)^2 \|l_i^\sigma(t)\|}. \end{aligned} \tag{11}$$

Here we have arranged the edges such that w_i and l_i correspond to edges weight i and edge vector i respectively. We will use this notation interchangeably with w_{ij}, l_{ij} whenever it is clear from the context.

For the purpose of analysis, we first define the interior of the valid edge set as $\mathcal{D} := \{x \in \mathbb{R}^{n\times N} \mid (v_i, v_j) \in E(\mathcal{G}) \Leftrightarrow \|l_{ij}\| < \Delta\}$, then define the edge tension function $V_{ij} : \mathcal{D} \to \mathbb{R}$:

$$V_{ij}(x) = \begin{cases} \frac{1}{\Delta - \|l_{ij}^\sigma(t)\|} & \text{if } \|l_{ij}^\sigma(t)\| < \Delta \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

and the total tension energy of the graph $V : \mathcal{D} \to \mathbb{R}$:

$$V(x) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} V_{ij}(x). \tag{13}$$

Lemma 3.1: Given an initial position $x(0) \in \mathcal{D}$ corresponding to a connected SIG and $0 < V_0 < \infty$, the set $\Omega := \{x(t) \mid V(x) \le V_0, t \ge 0\}$ is an invariant set to the system under the control law in Equation 9.

Proof:
With the definition of the tension energy, Equation 9 can be rewritten as:

$$\dot{x}_i = -\sum_{j\in N(i)} \frac{\partial V_{ij}}{\partial x_i} = -\frac{\partial V}{\partial x_i} = -\nabla_{x_i} V(x).$$

Therefore the time derivative of V is

$$\begin{aligned} \dot{V}(x) &= \nabla_x V(x) \dot{x}(t)^T \\ &= -\sum_{i=1}^{N} \dot{x}_i^T \dot{x}_i \\ &= -\sum_{j=1}^{n} c(x(t), j)^T \mathcal{L}_w^2 c(x(t), j). \end{aligned} \tag{14}$$

From the definitions, $V_{ij}(x), V(x) \in \mathcal{C}^1(\mathcal{D})$. Also, due to the finite initial value and to the fact that $\dot{V}(x)$ is nonpositive, the invariancy and the proof follow. ∎

Being positive, V_{ij} is also bounded from above by V_0, i.e. $V_{ij} \le V_0$, since $\left\|l_{ij}^\sigma(t)\right\| \to \Delta \Rightarrow V_{ij}(x(t)) \to \infty$. Therefore no edge will be cut off during the maneuver, and we can now state the main SIG theorem.

Theorem 3.2: Given a connected SIG $\mathcal{G}$, the multi-agent system (1) with the control law (9) asymptotically converges to the centroid $\bar{x}$, which is static.

Proof:
The proof of convergence is based on LaSalle's invariance theorem. Let $\mathcal{D}$, Ω be defined as before and let $\mathcal{E}$:=span$\{\mathbf{1}\}$. From Lemma 3.1, we know that Ω is positive invariant with respect to (9). Moreover $\dot{V}(x) \le 0$,

with equality only when $c(x(t), j) \in \mathcal{E}, \forall j$. Furthermore, $\mathcal{E}$ itself is $\mathcal{L}_w$ invariant from which convergence follows. (It is worth noticing that even though Ω is not compact in this case, the set $\mathcal{E}$ is totally enclosed in Ω, i.e. $\mathcal{E} \subset \otimes$ and $\partial\bar{\Omega} \cap \partial\bar{\mathcal{E}} = \Phi$, where ∂S denotes the boundary of S and $\bar{S}$ denotes its closure, so LaSalle's theorem is still applicable.)

Next we need to show that the agents converge to the centroid. The centroid is denoted as

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i,$$

and the component-wise dynamics of the centroid is

$$\frac{d}{dt}\overline{c(x,j)} = \frac{1}{N}\mathbf{1}^T\frac{d}{dt}c(x,j) = -\frac{1}{N}\mathbf{1}^T\mathcal{L}_w c(x,j).$$

Now since, $\mathbf{1}^T\mathcal{L}_w = (\mathcal{L}_w\mathbf{1})^T = 0$, we directly have that $\dot{\bar{x}} = 0$. Since $c(x,j) \to \text{span}\{1\}$ we can denote this point as ξ_j we get

$$\overline{c(x,j)} = \frac{1}{N}\sum_{j=1}^{N}\xi_j = \xi j$$

and the proof follows. ∎

Note that we still have extra freedom available to us by using the additional weight matrix D as

$$\dot{x} = -DL_w x.$$

As long as D is diagonal and positive definite $\mathcal{N}(DL_w) = \text{span}\{\mathbf{1}\}$ and the controller still drives the system to $\text{span}\{\mathbf{1}\}$ without loosing connectedness. However, in this case $x_i \to \bar{x}_w$ where $\bar{x}_w$ is given by

$$\bar{x}_w = \frac{1}{\text{tr}(D^{-1})}\sum_{i=1}^{N}(d_i^{-1})x_i(0) \tag{15}$$

where d_i is the ith diagonal element of D, and $\text{tr}(D)$ denotes the trace of matrix D. The proof is similar to the above one except that this time the relation is

$$\overline{c(x,j)} = \frac{1}{\text{tr}(D^{-1})}\sum_{j=1}^{N}d_i^{-1}\xi_j = \xi_j\frac{\sum_{j=1}^{N}d_i^{-1}}{\text{tr}(D^{-1})} = \xi_j$$

That concludes this section, where a SIG was always assumed. In what follows we will show that the same strategy remains valid even if the graph is allowed to change as the agents move around in the environment.

IV. Dynamic Graphs

As already pointed out, as the agents move, the interaction graph $\mathcal{G}$ may change. Whether or not the previous stability result still holds in this case will be the focus of this section. Since

$$\lim_{\|l_{ij}\|\uparrow\Delta} V_{ij}(\|l_{ij}\|) = \infty$$

we can not simply add new edges as soon as they are encountered. Instead we need a protocol for adding edges. As the dynamic interaction graph, DIG, evolves, an edge is added to E if (v_i, v_j) was previously not an edge and $\|l_{ij}\| \leq (\Delta - \delta)$, where $\delta > 0$ is the *switching threshold*. In this way, we have built in some hysteresis into the system, which allows us to state the following theorem.

Theorem 4.1: Following the control law (9), a group of agents starting from a connected graph will stay connected and a common Lyapunov function can be found under the above protocol for adding new edges.

Proof:
We claim that $\mathcal{W} = \frac{1}{2}x^T x$ is a Lyapunov function for the controlled system (9) since

$$\begin{aligned}\dot{\mathcal{W}} &= \nabla\mathcal{W}\dot{x} \\ &= -\sum_{j=1}^{n} c(x,j)^T\mathcal{L}_w c(x,j) \leq 0,\end{aligned} \tag{16}$$

given a connected SIG $\mathcal{G}$.

The equality is valid only when $c(x,j) \in \mathcal{N}(\mathcal{L}) = \text{span}\{\mathbf{1}\}$ for all j. Consider the result from the previous section, in which the centroid is proved to be static and to be the rendezvous point. We can thus conclude that $\mathcal{W}$ is a valid Lyapunov function, for static graph.

Since $\mathcal{W}$ does not depend on the structure of the graph, it is in fact a common Lyapunov function for arbitrary connected graphs, which means that stability is guaranteed as far as the graphs stay connected. (Note that a similar argument was presented in [15], based on the connectedness assumptions.)

Since no edges will be lost, as already proved in the previous section, and by the protocol for adding new edges, there are no infinite jumps in the total tension function $V(x)$. Hence connectedness will not be lost during switching either. Hence (5) also preserve connectedness for in the dynamic case. ∎

V. Examples

Here we will show simulations describing rendezvous behavior under different control laws. In the simulations, $\Delta = 4$, $\delta = 0.05$. Figure 2 shows the movement with $D = I$, i.e. Equation 9, while Figure 3 shows the movement of the rendezvous with a weight matrix $D = \text{diag}([1\ 1\ 1\ 1\ 1\ 0.5\ 0.5\ 0.5\ 0.5])$. The trajectories are shown in Figure 4. From the simulation, we find that the connectedness is maintained even under very pathological setups. Moreover, the weighted rendezvous is converging to a weighted centroid, as per Equation 15. Because the weights are symmetrically distributed about y axis, so the rendezvous point is still on x-axis, but shift toward right side where the agents are weighted heavily.

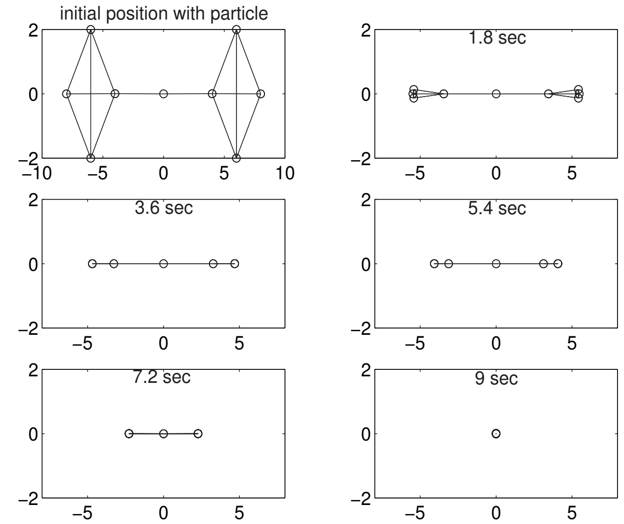


Fig. 2. A progression is shown where connectedness is maintained during the rendezvous maneuver, with $D = I$. Depicted are the positions of the agents and the edges in the DIG as a function of time.

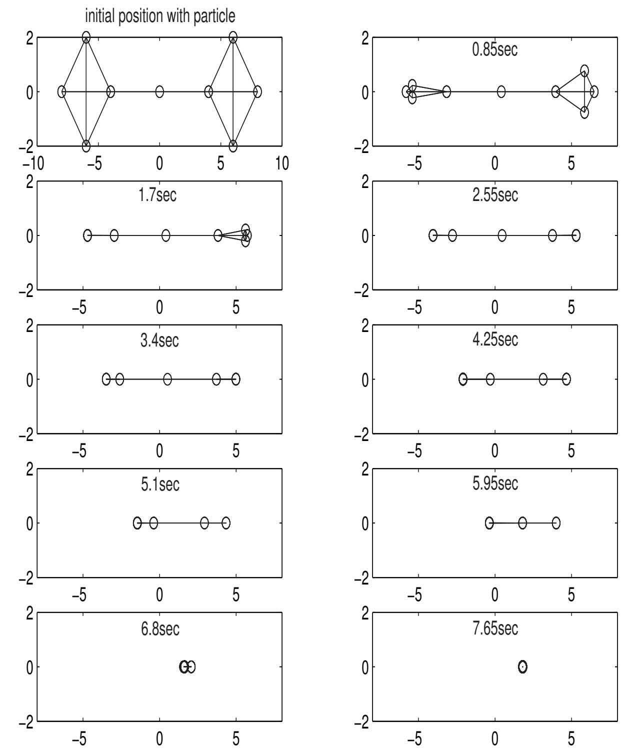
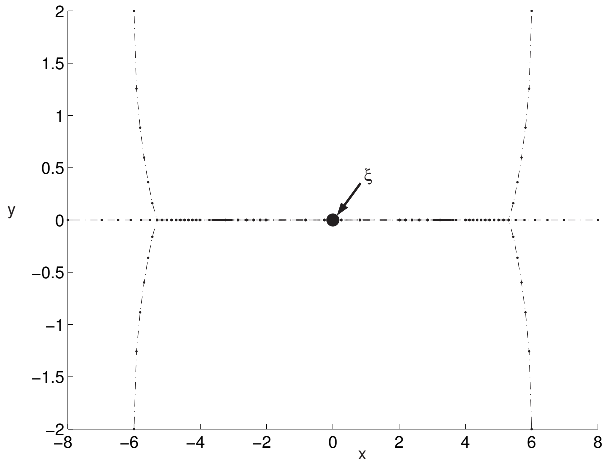
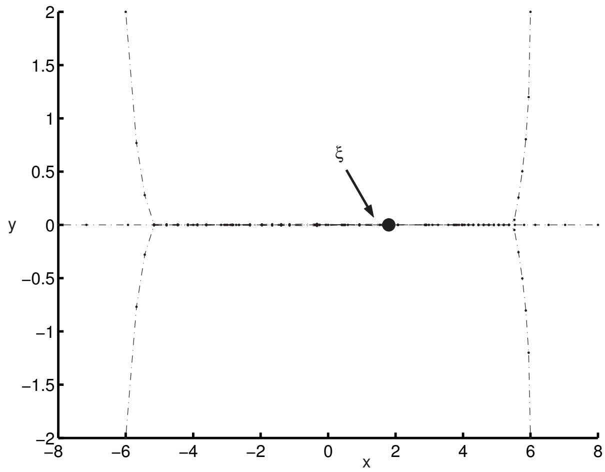


Fig. 3. A progression is shown where connectedness is maintained during the rendezvous maneuver, with $D \neq I$.



(a)



(b)

Fig. 4. Trajectory for (a) nonweighted and (b) weighted rendezvous.

VI. Conclusion

A graph-based nonlinear feedback control law is studied for distributed coordination control of multi agent system. The nonlinear feedback law is based on weighted graph Laplacians and it is proved to solve the rendezvous problem. Furthermore, the proposed control law is also proved to be able to guarantee that the connectedness is not lost during maneuvers. As to our knowledge, this is the first time that a general result is given on dynamic graphs.

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