# On the Structural Complexity of Multi-Agent Robot Formations

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***Abstract*— We present a complexity measure for studying the structural complexity of multi-agent robot formations. We base this measure on the total information flow in the system, which is due to sensory perception and communication among agents. We show that from an information theoretic point of view, perception and communication are fundamentally the same. We describe the information flows for different protocols and show that the broadcast protocol corresponds to the worst case complexity for a given formation. This upper bound is found to be remarkably similar to a complexity measure of graphs defined in the context of molecular chemistry.**

## I. Introduction

When designing control strategies for distributed, multi-agent robot systems, it is vitally important that the number of prescribed local interactions is managed in a scalable manner. In other words, it should be possible to add new robots to the system without causing a significant increase in the communication and computational burdens of the individual robots. But, one strategy that achieves this is a strategy where no interactions are present, which is clearly unsatisfactory from a number of vantage points. Therefore, an additional requirement when designing multi-agent coordination strategies should be that enough local interactions are present in order to ensure the proper execution of the task at hand.

Hence, a fundamental question that arises when studying such multi-agent systems is how to properly define the notion of "complexity". The traditional, algorithmic notion of the complexity of a system is related to how difficult it is to *describe* it. Therefore, most of the measures of complexity are closely related to the *Algorithmic Information Content* (AIC) in a system [1]. However, as noted in the molecular chemistry literature [5], [6], [30], there is an inherent difference between *descriptive complexity* and *structural complexity*, where the latter measures the interactions, size, and asymmetry in the physical structure. As of yet, no universal concept of structural complexity has emerged [30], and in molecular chemistry, the solution has been to define the complexity measures relative to the particular problem under investigation. A similar program can be carried out within the context of formation control. It is clear that when talking about robot formations, any measure of the complexity of the formations should take into account the size of the formation, the number of communication links or interactions in the formation, and possibly also the degree of symmetry in the formation.

Molecular chemists have mainly described the structural complexity of molecules by defining measures on their corresponding graphs [5]. Fortunately, there is a corresponding notion of formation graphs, induced by robot formations, [24], [25], [29], where the structural information in the formation is captured. Therefore, it seems appropriate to study the structural complexity of multi-agent robot formations using their graphs. We will frequently refer to our work on *connectivity graphs* [24] of robot formations in order to make this notion concrete.

It should be emphasized that if we were to use descriptive measures of complexity, like the AIC, a formation that has a complete connectivity graph, e.g. $K_5$, as shown in Fig. 1, would be easier to describe than the "ring-like formation", with corresponding connectivity graph $C_5$. However, this is counter intuitive to our notion of complexity in robot formations since there are more *local interactions* in $K_5$ than in $C_5$, and therefore $K_5$ is expected to have greater structural complexity. A less straight forward situation is illustrated in Fig. 2. Should less interactions necessarily mean less complexity in this case? One may argue that $C_5$ should be less complex because it has more symmetry than the star graph $S_4$, despite having one more link.[1] It should be noted, however, that when formulating a measure of complexity for robot formations, it need not produce an absolute order on all connectivity graphs (although the order has to be observed in its own class e.g. among all rings, all stars, all complete graphs). This means that we are more interested in relative complexity. We should thus be able to at least differentiate between *very complex* formations and *very simple* ones. Hence, the complexity measure need not be unique for every graph.

### *A. Symmetry*

There is a distinction made in molecular chemistry between *intrinsic* and *extrinsic* complexities [5]. When studying the complexity of a molecule according to the number of interactions in the structure (bonds), symmetry does not play a role since symmetric structures do not necessarily imply fewer interactions. This is referred to as *intrinsic* complexity. But, when studying problems of synthesis or fabrication, in which a process is iterated to build a complex structure, symmetry clearly implies lesser complexity [30]. The same

[1] It can moreover be noted that star graphs cease to exist as connectivity graphs above $N = 6$, as shown in [24].

Fig. 1. $C_5$ vs $K_5$

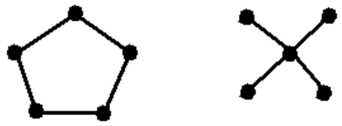

Fig. 2. $C_5$ vs $S_4$

approach should be taken when studying robot formations. The complexity of a robot formation that is already in place should not be affected by the potential symmetries of its corresponding connectivity graph. However, when studying the problem of *synthesizing* formations, it can be argued that it is easier to produce symmetric structures [19]. This is the case since in a completely symmetric formation, the same program can be executed by the different robots, which is not the case for asymmetric formations. However, in this paper we limit the scope to the intrinsic complexities, and leave extrinsic complexities to a future endeavor.

### *B. Perception, Communication, and Information Flow*

Individual agents in a formation can interact with each other in two distinct ways, via perception or communication. However, both of these two types of interactions result in a total information flow in the system, which means that they should both be taken into account when the complexity measure is defined. The sensors are used to gain information about other agents and the environment, while the communication channels are used to directly relay information to other agents. The concept of information flow among agents should thus be tied to the complexity of the formation in order to capture interactions in terms of perception and communications respectively. However, how does information flow due to perception differ from information flow due to communication? Are these two sides of the same coin or should they be treated as fundamentally different?

### *C. Criteria*

Given the above mentioned considerations, we will define a complexity measure of robot formations, related to the complexity of its connectivity graphs, which has a remarkable similarity with the complexity measures defined on graphs by molecular chemists. In [30], the complexity of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has been suggested as

$$C(G) = \sum_{v_i \in \mathcal{V}} \left( deg(v_i) + \sum_{v_j \in V, v_i \neq v_j} \frac{deg(v_j)}{d(v_i, v_j)} \right), \quad (1)$$

where $d : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^+$ is some distance function defined between vertices. As we will see, this definition will help us to characterize intrinsic complexity of robot formations quite nicely.

This paper is organized as follows: We will first introduce connectivity graphs of formations in Section II. Following this, we will discuss the equivalence between perception and communication from an information theoretic point of view, in Section III. Then, we will propose a definition of the intrinsic complexity of robot formations, in Section IV and explain its relation to the complexity of graphs.

## II. Formations and Connectivity Graphs

In order to see how a graph-based complexity measure is appropriate when studying multi-agent formation, we, in this section, recall some previous results and definitions of connectivity graphs. The technical details can be found in [24], [25] but we include this treatment for the sake of clarity. Throughout this paper it will be assumed that the robots are planar, and that they can interact with neighboring robots (through perception or communication) that are no further than $\delta$ away.

*Definition 2.1 (Formations and Their Configuration Spaces):* The configuration space $\mathcal{C}^N(\mathbb{R}^2)$ of the robot formation is made up of all ordered $N$-tuples in $\mathbb{R}^2$, with the property that no two points coincide.

$$\mathcal{C}^N(\mathbb{R}^2) = (\mathbb{R}^2 \times \mathbb{R}^2 \times \ldots \mathbb{R}^2) - \Delta,$$

where $\Delta = \{(x_1, x_2, \ldots, x_N) : x_i = x_j \text{ for some } i \neq j\}$.

The evolution of the formation can be represented as a trajectory $\mathcal{F} : \mathbb{R}_+ \to \mathcal{C}^N(\mathbb{R}^2)$, usually written as $\mathcal{F}(t) = (x_1(t), x_2(t), \ldots x_N(t))$ to signify time evolution [24]. The spatial relationship between robots can be represented as a graph in a very straightforward manner, in which the vertices of the graph represent the robots, and the pair of vertices on each edge tells us that the corresponding robots are within sensor range $\delta$ of each other.[2]

*Definition 2.2 (Connectivity Graph of a Formation):* Let $\mathcal{G}_N$ denote the space of all possible graphs that can be formed on $N$ vertices $V = \{v_1, v_2, \ldots, v_N\}$. Then we can define a function $\Phi_N : \mathcal{C}^N(\mathbb{R}^2) \to \mathcal{G}_N$, with $\Phi_N(\mathcal{F}(t)) = \mathcal{G}(t)$, where $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t)) \in \mathcal{G}_N$ is the *connectivity graph* of the formation $\mathcal{F}(t)$. Furthermore, $v_i \in \mathcal{V}$ represents robot $i$ at position $x_i$, and $\mathcal{E}(t)$ denotes the edges of the graph, with $e_{ij}(t) = e_{ji}(t) \in \mathcal{E}(t)$ if and

[2]Here, $\delta$ is used to signify the limited effective range of the sensors as well as the range within which a communication channel is available.

only if $\|x_i(t) - x_j(t)\| \leq \delta$, $i \neq j$. In other words $\Phi_N(\mathcal{F}(t))$ is given by the graph

$$(\{v_i\}_{i=1}^N, \{(v_i, v_j) \text{ s.t. } i \neq j, \|x_i(t) - x_j(t)\| \leq \delta\}).$$

The movements of the individual robots in the formation may result in the removal or addition of edges in the graph. Therefore, $\mathcal{G}(t)$ is a dynamic structure. If no constraints are put on the robot dynamics (except of course that the motion is continuous), the resulting formations can produce a wide variety of graphs on $N$ vertices. However, due to the special way in which the *connectivity graphs* are defined, not all graphs correspond to formations in $\mathcal{C}^N(\mathbb{R}^2)$. The consequence of this observation is that $\Phi_N$ is not onto for every $N$. We denote by $\mathcal{G}_{N,\delta} \subseteq \mathcal{G}_N$ the space of all possible connectivity graphs of $N$ robots, with sensor range $\delta$. It is thus an interesting exercise to try to characterize what graphs are connectivity graphs in the sense that they can be realized as a formation in the configurations space:

*Definition 2.3 (Realization of a Graph in $\mathcal{C}^N(\mathbb{R}^2)$):* A connectivity graph $\mathcal{G} \in \mathcal{G}_N$ can be *realized* in $\mathcal{C}^N(\mathbb{R}^2)$ if $\Phi_N^{-1}(\mathcal{G})$ is nonempty. In other words, a realization of $\mathcal{G}$ is any $\mathcal{F} \in \mathcal{C}^N(\mathbb{R}^2)$ such that $\Phi_N(\mathcal{F}) = \mathcal{G}$.

In [25], the following theorem was proved:

*Theorem 2.1:* $\mathcal{G}_{N,\delta} \subsetneq \mathcal{G}_N$ if and only if $N \geq 5$.

It is clear that different formations can produce a wide variety of graphs with $N$ vertices. This includes graphs that have disconnected subgraphs, or totally disconnected graphs with no edges. However, the problem of switching between different formations or of finding interesting structures within a formations can only be tackled if no "sub-formations" of robots are completely isolated from the rest of the formation. This is because there is no deterministic way of bringing a "lost" robot (or group of robots) back within sensor range of the others in an autonomous, decentralized system (as noted in [19]). This means that the connectivity graph $\mathcal{G}(t)$ of the formation $\mathcal{F}(t)$ should always remain *connected* (in the sense of connected graphs) for all time. For notational convenience we will use $\mathcal{G}^c_{N,\delta} \subseteq \mathcal{G}_{N,\delta} \subseteq \mathcal{G}_N$ to denote the set of all connected graphs of $N$ vertices that satisfy the connectivity condition in Definition 2.2, for a given sensor range $\delta$.

## III. Perception Vs. Communication

Any measure of how complex a certain formation is has to capture the amount of information that flows between the different agents in a meaningful manner. This exchange of information between agents is due to the local interactions among agents. There are two kinds of local interactions in multi-agent robotic systems. One is due to sensory perception of neighboring robots and the other is due to the communication channels. When defining complexity measures, one thus either have to unify these two types of local interactions, or define two different complexity costs associated with them. Hence, it is natural to ask whether these interactions differ fundamentally from each other. If we can show that there is no fundamental difference, it will simplify our task of characterizing complexity in terms of local interactions by not explicitly mentioning the cause of the interactions. We explore this issue in this section.

Since we are interested in this issue from an information theoretic point of view, we pose the following problem in an information theoretic setting. If $X, Y$ are two random variables, we will denote by $I(X;Y)$, the amount of information gained about $X$ by knowing $Y$. The entropy of each random variable will be denoted by $H(X)$ and $H(Y)$ respectively, and $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$ (see for example [7], [32]), where $X|Y$ and $Y|X$ are conditional random variables. If a variable $Z$ of $M$ components is defined over a finite field, we will refer to its space as the lattice $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \cdots \times \mathbb{Z}_{k_M} \subset \mathbb{R}^M$ to emphasize quantization, where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

*Problem 3.1:* Suppose the state of a system $X = [x_1, x_2, \ldots x_M]^T \in \mathbb{R}^M$ is measured by sensor $\mathcal{S}$, providing the measurements $Z = [z_1, z_2 \ldots z_M]^T \in \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \cdots \times \mathbb{Z}_{k_M} \subset \mathbb{R}^M$, where $k_i \in \mathbb{N}$ for $1 \leq k \leq M$. Knowledge about $X$ is also transmitted by a remote agent over a communication channel $\mathcal{C}$ as a vector $Y = [y_1, y_2, \ldots y_M]^T \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \cdots \times \mathbb{Z}_{N_M} \subset \mathbb{R}^N$, where $N_i \in \mathbb{N}$ for $1 \leq i \leq N$. Here, the state $x_i$ is assumed to be described by $y_i$. Each component $y_i$ of $Y$ is encoded independently of other components, and each symbol in each component is equally likely. i.e. $p_i(y_i) = \frac{1}{N_i}$. Then, we ask the following questions:

1) Does there always exist a virtual sensor $S'$ which provides the same information as the communication channel $\mathcal{C}$?
2) Does there always exist a virtual communication channel $\mathcal{C}'$ which provides the same information as the sensor $\mathcal{S}$?

These two questions seem to be the same in the sense that they both ask for sufficient conditions for the following to hold:

$$\begin{aligned} I(X;Z) &= I(X;Y) \\ H(X|Z) &= H(X|Y), \end{aligned}$$

but we will see, in what follows, that Question 1 is easier to answer.

### A. Example

We start our investigation by studying a simplified, yet illustrative example: Suppose that robot $i$ is transmitting its position to robot $j$ over a lossless communication channel. It is moreover assumed that robot $i$ measures its position with respect to $(0,0)$, i.e. with respect to robot $j$. It encodes its radial position $r$ as $\tilde{r}$ using $N_r$ symbols $\{r_1, r_2 \ldots r_{N_r}\}$, with resolution $\frac{\delta}{N_r}$, using the probability distribution $p(\tilde{r})$. Similarly, if there are $N_\theta$ symbols $\{\theta_1, \theta_2 \ldots \theta_{N_r}\}$ for quantizing the bearing $\tilde{\theta}$, with resolution $\frac{2\pi}{N_\theta}$, using the distribution

$p(\tilde{\theta})$, then the joint probability distribution of the symbol set is $p(\tilde{r},\tilde{\theta})$. The entropy of the coding scheme is given by

$$\begin{aligned} H(\tilde{r},\tilde{\theta}) &= -\sum_{i=1}^{N_r}\sum_{j=1}^{N_\theta} p(r_i,\theta_j)\log_2 p(r_i,\theta_j) \\ &= H(\tilde{r}) + H(\theta|\tilde{r}). \end{aligned}$$

We now assume that $\tilde{r}$ is independent of $\tilde{\theta}$, so that $p(\tilde{r},\tilde{\theta}) = p(\tilde{r})p(\tilde{\theta})$ and $p(\tilde{\theta}|\tilde{r})p(\tilde{\theta})$. Furthermore, assume that all symbols are equally likely i.e. $p(\tilde{r}) = \frac{1}{N_r}$ and $p(\tilde{\theta}) = \frac{1}{N_\theta}$. We then get

$$\begin{aligned} H(\tilde{r},\tilde{\theta}) &= H(\tilde{r}) + H(\tilde{\theta}|\tilde{r}) = H(\tilde{r}) + H(\tilde{\theta}) \\ H(\tilde{r},\tilde{\theta}) &= \log_2(N_\theta + N_r) \end{aligned}$$

The assumptions made here about uniform resolutions and independence are reasonable as they are usually true for real sensors. It is the processing (e.g. polar to Cartesian conversion in this case), that produces non-uniform resolutions or correlations [3].

*1) Communication:* Suppose that position $(\tilde{r},\tilde{\theta})$ is transmitted to robot $j$ from robot $i$. Then the amount of information that is gained about robot $i$'s true position, $(r,\theta)$, is given by the mutual information

$$I(r,\theta;\tilde{r},\tilde{\theta}) = H(r,\theta) - H(r,\theta|\tilde{r},\tilde{\theta}).$$

Since robot $i$ is equally likely to be in any position in $\mathcal{B}_\delta(0)$, we may assume that

$$f(r,\theta) = f(r)f(\theta) = \frac{1}{\delta}\frac{1}{2\pi} = \frac{1}{2\pi\delta}.$$

Now, the uncertainty associated with robot $i$'s true position can thus be calculated as

$$\begin{aligned} H(r,\theta) &= H(r) + H(\theta|r) = H(r) + H(\theta) \\ H(r,\theta) &= -\int_0^\delta f(r)\log_2 f(r)dr - \int_0^{2\pi} f(\theta)\log_2 f(\theta)d\theta \\ &= \log_2\delta + \log_2(2\pi) = \log_2(2\pi\delta). \end{aligned}$$

Next we compute $H(r,\theta|\tilde{r},\tilde{\theta})$. Observe that:

$$\begin{aligned} H(r,\theta|\tilde{r},\tilde{\theta}) &= H(r|\tilde{r},\tilde{\theta}) + H(\theta|r,\tilde{r},\tilde{\theta})) \\ &= H(r|\tilde{r}) + H(\theta|\tilde{\theta}), \end{aligned} \tag{2}$$

where

$$\begin{aligned} H(r|\tilde{r}) &= \sum_{i=1}^{N_r} p(r_i)\left(\int f(r|r_i)\log_2 f(r|r_i)dr\right) \\ H(\theta|\tilde{\theta}) &= \sum_{i=1}^{N_\theta} p(\theta_i)\left(\int f(\theta|\theta_i)\log_2 f(\theta|\theta_i)d\theta\right). \end{aligned}$$

If we now assume a uniform quantizer, we get the uniform distributions:

$$\begin{aligned} f(r|r_k) &= \frac{N_r}{\delta},\ r\in[r_k - \frac{\delta}{2N_r}, r_k + \frac{\delta}{2N_r}],\ 1\le k\le N_r \\ f(\theta|\theta_k) &= \frac{N_\theta}{2\pi},\ \theta\in[\theta_k - \frac{\pi}{N_\theta}, \theta_k + \frac{\pi}{2N_\theta}],\ 1\le k\le N_\theta. \end{aligned}$$

Therefore,

$$\begin{aligned} H(r|\tilde{r}) &= -\sum_{i=1}^{N_r}\frac{1}{N_r}\left(\int_{r_k-\frac{\delta}{2N_r}}^{r_k+\frac{\delta}{2N_r}} \frac{N_r}{\delta}\log_2\frac{N_r}{\delta}dr\right) \\ H(r|\tilde{r}) &= \log_2\frac{\delta}{N_r}. \end{aligned}$$

Similarly,

$$\begin{aligned} H(\theta|\tilde{\theta}) &= \log_2\frac{2\pi}{N_\theta} \\ \text{So } H(r,\theta|\tilde{r},\tilde{\theta}) &= \log_2\left(\frac{2\pi\delta}{N_r N_\theta}\right). \end{aligned}$$

The amount of information gained by communication is therefore

$$\begin{aligned} I_c(r,\theta;\tilde{r},\tilde{\theta}) &= H(r,\theta) - H(r,\theta|\tilde{r},\tilde{\theta}) \\ &= \log_2 N_r N_\theta. \end{aligned}$$

*2) Perception:* Now, consider the situation where robot $j$ is sensing robot $i$'s true position by an on-board sensor with radial resolution $\Delta r$ and angular resolution $\Delta\theta$.[3] Moreover, assume that the range of the sensor is limited to $R$ i.e. $\hat{r}\le R$. The sensor then gives the position $(\hat{r},\hat{\theta})$. By repeating this argument in the angular case, we get

$$H(r,\theta|\hat{r},\hat{\theta}) = H(r|\hat{r},\hat{\theta}) + H(\theta|r,\hat{r},\hat{\theta})).$$

If we further assume that the sensor measurements $\hat{r}$ and $\hat{\theta}$ are mutually independent, then

$$H(r,\theta|\hat{r},\hat{\theta}) = H(r|\hat{r}) + H(\theta|\hat{\theta}),$$

where

$$\begin{aligned} H(r|\hat{r}) &= -\sum_{i=0}^{R/\Delta r-1} p(i\Delta r)\int_0^R f(r|i\Delta r)\log_2 f(r|i\Delta r)dr \\ H(\theta|\hat{\theta}) &= -\sum_{i=0}^{2\pi/\Delta\theta-1} p(i\Delta\theta)\int_0^{2\pi} f(\theta|i\Delta\theta)\log_2 f(\theta|i\Delta\theta)d\theta. \end{aligned}$$

[3]This is a reasonable assumption, since most range sensors such as ultrasonic or infra-red sensors, are typically mounted in a ring of individual sensors with a certain, fixed range resolution [22].

*3) Equivalent Virtual Sensors:* Let us now try to build an equivalent virtual sensor to the communication channel described above. Suppose there exists a sensor with independent measurements $\hat{r}, \hat{\theta}$, whose range is limited to $R = \delta$. Let the resolutions of the sensor measurements be $\Delta r = R/N_r$, $\Delta\theta = 2\pi/N_\theta$ respectively. If the errors $e_r = r - \hat{r}$, $e_\theta = \theta - \hat{\theta}$ in the sensor's measurements are modeled by uniform distributions, given by:

$$\begin{aligned} f(e_r) &= \frac{1}{\Delta r}, \qquad e_r \in [-\frac{\Delta r}{2}, \frac{\Delta r}{2}], \\ f(e_\theta) &= \frac{1}{\Delta\theta}, \qquad e_\theta \in [-\frac{\Delta\theta}{2}, \frac{\Delta\theta}{2}], \end{aligned}$$

then it can be seen directly that $I_c(r, \theta; \hat{r}, \hat{\theta}) = I_p(r, \theta; \tilde{r}, \tilde{\theta}) = \log_2(N_r N_\theta)$.

This thus provides us with sufficient conditions for the equality to hold between perception and communications.

### B. The General Case

*Theorem 3.1:* For any communication link $\mathcal{C}$ that satisfies the assumptions in Problem 3.1, there always exist a virtual sensor $S'$ that provides the same information as the communication channel.

*Proof:*

By the setup in Problem 3.1, we have

$$I(X;Y) = \log_2(\prod_{i=1}^{N} N_i).$$

If we follow steps, similar to the previous example, we can construct a "virtual sensor" $\mathcal{S}'$ which is equivalent to the communication channel $\mathcal{C}$ as follows. Let the virtual sensor give measurements $Z' \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \cdots \times \mathbb{Z}_{N_M} \subset \mathbb{R}^M$, with resolutions

$$\begin{aligned} \Delta z'_i &= \frac{\max(y_i) - \min(y_i)}{N_i} \\ f(z'_i - x_i) &= f(x_i | i\Delta z') = \frac{1}{\Delta z_i} \end{aligned}$$

Then it can be directly verified that

$$I(X;Z') = H(X;Y) = \log_2(\prod_{i=1}^{N} N_i).$$

∎

Now, we come to our second question, namely the problem of creating a "virtual" communication channel $\mathcal{C}'$ equivalent to a given sensor. If $I(X;Z)$ is the amount of information gained about $X$ by measurement $Z$, and there exists a positive integer $k$ such that

$$k = 2^{I(X;Y)} \in \mathbb{Z}^+,$$

then we can build a virtual communication channel $\mathcal{C}'$ using any factorization of $k$

$$k = k_1.k_2.\ldots.k_K, \qquad k_i \in \mathbb{Z}^+.$$

However, it is usually the case that $I(X;Y) \in \mathbb{R} \setminus \mathbb{Z}^+$ due to the choice of continuous, non-constant distributions, defined over intervals in the field $\mathbb{R}$. Therefore it may not always to be possible to construct the virtual channel, using this "trick". But, by the aforementioned argument, we assume that we can talk about sensors and communications channels interchangeably.

## IV. Complexity of Robot Formations

We now consider the problem of describing a complexity measure of multi-agent robot formations. As explained above, it would be appropriate to relate this measure to the number of local interactions between agents. In the previous section, we saw that these interactions are due to perception and communication, and that there is no fundamental difference between the two, from an information theoretic point of view. Therefore it makes sense to relate the complexity measure to the total amount of information flowing in the system. It should further be noted that this information exchange among agents is a dynamic quantity and depends on the distributed algorithm executed by the system.

A multi-agent formation is an evolving structure in both time and space. In space, it is dynamic due to the motion of the robots, which leads to the establishment of new interactions and the termination of old ones. This spatial relationship can be captured by a connectivity graph as explained in Section II. However, the establishment of a local interaction, does not mean that this interaction is present for all time. The information exchange at a particular time depends on *protocols* (e.g. [11], [20]), which may make the information interchange not only non-constant, but also non-deterministic. Therefore, it would be appropriate to refer to a quantity describing the time rate of information exchange. We call this quantity, the *information flow*, and refer to the complexity of a formation as the total information flow in the system.

In [20], complexity measures for multi-agent, distributed algorithms were given. Moreover, explicit scaling laws were proved with respect to the number of agents. However, the implicit assumption in that work is that all agents can communicate with all other agents directly. Therefore, the results are limited to complete graphs only, and can be said to measure the complexity of the communication strategies rather than the complexity of the formations. To remedy this, we provide an account of the structural complexity of formations that is applicable to formations in which agents have limited information about other agents.

### A. States, Measurements, and Packets

It would be appropriate to say something about the nature of the information flowing between agents. In multi-agent coordination strategies, the states that are being shared need not be physical states of a dynamical system associated with an agent or the environment. The states, about which a remote agent gains knowledge, via perception or communication, may moreover be discrete or continuous. However, the outputs of both sensors and communication channels are discrete quantities. If a discrete quantity $X \in \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \cdots \times \mathbb{Z}_{k_N}$, is coded and transmitted as $Z$, without modification over a communication channel (or acquired by perception without errors), and decoded reversibly, then there is no uncertainty associated with the communication (or measurement) and it describes the true state in its entirety. In this case $I(X;Z) = H(X) = H(Z)$. However if the state is defined over a continuous field, or the coding and transmission process is not reversible, then there is a loss of information in the perception and communication processes, as $I(X;Z) = H(X) - H(X|Z)$.

Let us now suppose that that an agent receives a sequence of measurements (or packets) about an evolving state $\{X(1), X(2), \ldots X(n)\}$ as $\{Z(1), Z(2), \ldots Z(n)\}$. The amount of information gained at time $n$ is $I(X; Z(n), Z(n-1), \ldots Z(1))$. This is because it may be possible to predict the true state using all previous observations, using an estimator [3]. However, if the random process $X(n)$ is white, i.e. previous values have no correlation with the latest one, then the information gained is $I(X(n); Z(n))$. If the process is Markovian, then $I(X(n); Z(n), Z(n-1))$, and so on [7]. Therefore, we will use $I(X;Z)$ to denote the information exchange, even though this quantity may actually be dependent on previous observations. Also without loss of generality, we will refer to the flow of continuous states $X \in \mathbb{R}^N$, only, but would not mean that only continuous quantities are being quantized and exchanged in the system.

### B. Protocols, Synchronous Algorithms, and Information Flows

Suppose $X_j \in \mathbb{R}^N$ is a state associated with an agent $j$, which agent $i$ wants to acquire by perception or communication. Let $Z_{j,i} \in \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \cdots \times \mathbb{Z}_{k_N} \subset \mathbb{R}^N$ be the measurement of a sensor $\mathcal{S}$ by agent $i$. Information about $X_j$ is also transmitted by agent $j$ over a communication channel $\mathcal{C}$ as $Y_{j,i} \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \cdots \times \mathbb{Z}_{p_N} \subset \mathbb{R}^N$, where $p_i \in \mathbb{N}$ for $1 \leq i \leq N$. If we consider $X_j, Z_{j,i}$ and $Y_{j,i}$ as random processes, then we can define the *information flow*, as the time rate of information exchange taking place at a certain agent, i.e.

$$F_{i,j}(t) = \frac{dI(X_j; Z_{j,i}, Y_{j,i})}{dt}. \quad (3)$$

There are several technical difficulties associated with the definition in Equation (3). The random processes are always discrete in time, because both the perception and communication process are discrete. In the most general case, the packets arrive (or measurements are taken) according to some *protocol*, which defines the time of arrival. The situation is further complicated by the fact that the information exchange may be completely asynchronous, both among different agents as well as between measurements and communication of the same state for one agent. The actual communication exchange takes place as a burst after possibly long unequal intervals. But, in this paper, we assume that the information flow for a single exchange should be considered as the information gained between two consecutive exchanges, averaged over the time interval.

With these considerations, we assume that if the information flow is well defined according to a particular protocol, then we can define the intrinsic structural complexity of a formation as follows.

*Definition 4.1 (Intrinsic Complexity of a Formation):* The intrinsic complexity of a formation $\mathcal{F} = (X_1, X_2, \ldots X_N) \in \mathcal{C}^N(\mathbb{R}^2)$ is defined as:

$$C(\mathcal{F}) = \sum_j \sum_{i \neq j} F_{i,j}(X_j),$$

where each $F_{i,j}$ is defined according to some given communication protocol.

The asynchrony in information exchange, and the presence of a particular protocol, lead to another interesting phenomenon. Intuitively, it seems reasonable to assume that the total complexity of the system is the sum of all local interactions. However, due to the presence of protocols and asynchrony, this complexity varies over time as the time averages may vary. It is easy to see that this variation is not contributed as much by asynchrony but by complex protocols of information exchange. We therefore assume here for simplicity that the multi-agent algorithm executed is synchronous, and leave the issue of asynchronous algorithms for future investigation. We will see below that this is a reasonable assumption to start with, and the results obtained are very useful as a first analysis of structural complexity. However the presence of protocols is not something that can be overlooked without a justification.

Since, the presence of protocols implies that every interaction is not active during a certain time period, the intrinsic complexity is bounded above by a quantity that assumes that all interactions are active for all time. This bound is in-fact a complexity associated with a *broadcast protocol*, defined below.

*Definition 4.2 (Synchronous Periodic Broadcast Protocol):* Suppose each agent $j$ transmits its state $X_j, j \neq i$ to all other agents as $Y_j$ after every $\Delta t$ seconds. The time $Y_j$ takes to reach agent $i$ is some integer multiple $k_{i,j}$ of $\Delta t$, where $k_{i,j}$ is the number of "hops" in the communication. Also, let the measurement $Z_{j,i}$ of remote state $X$ be periodically taken every $\Delta t$ seconds. Then this protocol of communication among agents is called the *Synchronous Periodic Broadcast Protocol*.

If $\Delta t$ is the minimum permissible time for information exchange in the system (due to either bandwidth, sensor update interval, or algorithm execution cycle), then we can easily see that protocols of synchronous information exchange that are more complicated than the broadcast protocol would result in a decrease of the total information flow. If we denote the complexity of a formation, associated with the broadcast protocol as $C_B(\mathcal{F})$, then

$$C_B(\mathcal{F}) \geq C_P(\mathcal{F}),$$

where $C_P(\mathcal{F})$ is the complexity for some arbitrary protocol. $C_B(\mathcal{F})$ is therefore the worst case complexity associated with a particular formation. The information flow of a remote state $X_j$ at agent $i$, according to this protocol, is

$$F_{i,j}(X_j) = \frac{I(X_j; Z_{j,i})}{\Delta t} + \frac{I(X_j; Y_j)}{k_{i,j}\Delta t} \text{ bits/sec}, i \neq j.$$

From the discussion in Section III, it is clear that it is always possible to create a virtual sensor $\mathcal{S}'$ such that $I(X_j; Y_j) = I(X_j; Z_j{}')$. Therefore, we will refer to the information flows with reference to sensors only, and write the information flow as

$$F_{i,j}(X_j) = \frac{I(X_j; \mathcal{Z}_{j,i})}{k_{ij}\Delta t}, \quad (4)$$

where $\mathcal{Z}_{j,i} = [Z_{j,i}, Z_j']$, in order to emphasize that we are referring to sensors only.

### *C. Complexity and Connectivity Graphs*

We now study the interesting relationship between the structural complexity defined above and an alternative description of complexity based on connectivity graphs of formations. The first interesting connection can be seen from the definition of the broadcast protocol. The number $k_{ij}$ defined as the number of hops in the communication between agents hints at the network topology between the agents. But, *the connectivity graphs defined in Section II is exactly this network topology.* Furthermore, it may be reasonable to ask if $k_{ij}$ is a unique number for any two agents, since the same information my be exchanged by different hopping paths. This corresponds to different paths in the connectivity graph. Since the information flow in Equation 4 depends on $k_{ij}$, it must be made clear what path are we using. But, since we are interested in distributed multi-agent algorithms, it cannot be assumed that global information about the network topology (i.e. the connectivity graph of the formation) is available all the time to all agents, so that the hopping paths are unique [4]. Instead, in the broadcast scenario, the information about $X_j$ reaches a remote agent $i$ via all possible hopping paths between them, so that

$$F_{i,j}(X_j) = \sum_{p=1}^{P_{ij}} \frac{I(X_j; \mathcal{Z}_{j,i})}{k_{p,ij}\Delta t}, \quad (5)$$

where $P_{ij}$ is the total number of paths, and $k_{p,ij}$ is the length of an individual path, $p$. If $k_{ij}$ is the smallest path between the agents, i.e. a geodesic in the corresponding connectivity graph, then

$$F_{i,j}(X_j) \leq deg(v_j)\frac{I(X_j; \mathcal{Z}_{j,i})}{k_{ij}\Delta t}. \quad (6)$$

Furthermore the complexity $C_B(\mathcal{F})$ is bounded above as

$$C_B(\mathcal{F}) \leq \sum_j \sum_{i \neq j} deg(v_j)\frac{I(X_j; \mathcal{Z}_{j,i})}{k_{ij}\Delta t}.$$

We now assume that the states exchanged by all agents are of the same type and encoded in the same way. Therefore $I(X_j; \mathcal{Z}_{i,j}) = \gamma$, i.e. the mutual information is constant for all $i, j$. Also, note that $k_{ij} = 1$ if $v_i, v_j$ make an edge in the connectivity graph i.e. when agent $j$ can be directly sensed (or communicated with) without an additional hop. We can also write this in standard graph theory notation as $v_j \in star(v_i)$ [8], [17]. Using this notation, we have:

$$C_B(\mathcal{F}) \leq \frac{\gamma}{\Delta t}\sum_i \left( \sum_{v_j \in star(v_i)} deg(v_j) + \sum_{v_j \not\in star(v_i)} \frac{deg(v_j)}{k_{ij}} \right).$$

It should further be noted that if $v_j \in star(v_i)$, the exact path of communication is *always* known, and the broadcast to other nodes is not necessary. Therefore we can make this bound tighter

$$C_B(\mathcal{F}) \leq \frac{\gamma}{\Delta t}\sum_i \left( deg(v_i) + \sum_{v_j \notin star(v_i)} \frac{deg(v_j)}{k_{ij}} \right),$$

where $\sum_{v_j \in star(v_i)} 1 = deg(v_i)$. Compare this to the complexity defined on a graph, in the context of molecular chemistry, given in Equation 1, and we get,

$$C_B(\mathcal{F}) \leq \frac{\gamma}{\Delta t} C(\Phi_N(\mathcal{F})),$$

where $\Phi_N(\mathcal{F})$ is the connectivity graph of the formation. This relationship leads to the following interesting observation:

*The complexity of the connectivity graph of a formation is a (tight) upper bound for the worst case complexity associated with an arbitrary protocol of communication in a multi-agent formation*.

Therefore the study of structural complexity of robot formations is closely related to the complexity of their connectivity graphs.

### *D. Simple and Complex Connectivity Graphs*

The complexity measure on connectivity graphs gives a good comparison between different formations. While it is difficult to produce an absolute order on all possible connectivity graphs, it distinguishes simple graphs from the more complex. We will prove below that the complete graph is the most complex connectivity graph for a fixed set of vertices, whereas a $\delta$-chain [24], which is a Hamiltonian path

[4]Network discovery may be possible eventually, but not guaranteed for all time, specially right after the formation graph switches to a new one.

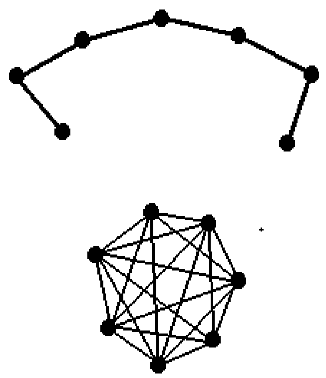

Fig. 3. $\delta$-chain and complete graph for 7 vertices.

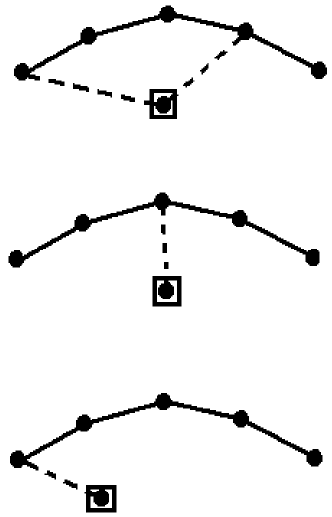

Fig. 4. Different ways to add a new vertex to $\delta_5$

on all vertices, is the least complex connected connectivity graph. (See fig 3.)

The conclusion that the complete graph is the most complex graph is not surprising and confirms to our intuition, as it has the maximum number of local interactions between any set of vertices. The characterization of the most simple graph is however an interesting result and gives the justification of the $\delta$-chaining algorithms that we have developed as a benchmark problem in our study of distributed algorithms [24], [25].

Consider a connectivity graph $G_N = (\mathcal{V}, \mathcal{E})$ on $N$ vertices, with the complexity measure

$$C(G_N) = \sum_{v_i \in \mathcal{V}} \left( deg(v_i) + \sum_{v_j \notin star(v_i)} \frac{deg(v_j)}{k_{ij}} \right).$$

If we add another vertex $v_{N+1}$ to $G_N$, we get a graph on $N+1$ vertices $G_{N+1}$. We can also form new edges between $v_{N+1}$ and vertices in $V$ so that the complexity of the new graph is perturbed as

$$C(G_{N+1}) = \sum_{v_i \in \mathcal{V}} (deg(v_i) + \Delta deg(v_i) + \cdots$$

$$\cdots + \sum_{v_j \notin star(v_i), v_j \in \mathcal{V}} \frac{deg(v_j) + \Delta deg(v_j)}{k_{ij} + \Delta k_{ij}} + \frac{deg(v_{N+1})}{k_{i,N+1}}) \cdots$$

$$\cdots + deg(v_{N+1}) + \sum_{v_m \notin star(v_{N+1})} \frac{deg(v_m) + \Delta deg(v_m)}{k_{mj} + \Delta k_{mj}},$$

where $\Delta deg(v_i)$ is the change of degree at vertex $v_i$ caused by the addition of a new vertex, and $\Delta k_{mj}$ is the corresponding decrease in the shortest path between vertexes $v_m$ and $v_j$.

It can be seen that adding a vertex always *increases* the complexity of the graph, as all perturbations are additive. It is therefore straightforward to capture the minimum or maximum perturbation that can be done by adding a vertex.

*Theorem 4.1:* If $G \in \mathcal{G}^c_{N,\delta}$, then the complexity of the connected graph $G$ is bounded above and below as

$$C(\delta_N) \leq C(G) \leq C(\mathcal{K}_N), \tag{7}$$

where $\delta_N$ is the $\delta$-chain on $N$ vertices, and $\mathcal{K}_N$ is the complete graph.

*Proof:* We prove the theorem by induction. Suppose it is true that $C(G) \leq C(\mathcal{K}_N)$ for $G \in \mathcal{G}_{N,\delta}$. Note that for any vertex $v_i$ in the graph, $deg(v_i) \leq N$. For $\mathcal{K}_N$, $deg(v_i) = N$ for all vertices. Therefore the maximum number by which any degree can be perturbed in $\mathcal{K}_N$ is 1. The perturbation will be maximized if all degrees are perturbed by 1. Similarly, in $\mathcal{K}_N$, $k_{ij} = 1$ for all pairs of vertices. The maximum perturbation will take place when the relation $k_{ij} = 1$ still holds for all pairs after addition of new vertex, i.e. all vertices are directly connected. It can be easily seen that this can only be accomplished by adding edges between all vertices in $K_N$ and the new vertex to make the graph $\mathcal{K}_{N+1}$. This proves that $C(G) \leq C(\mathcal{K}_N)$ for all $N$.

We repeat the induction argument for the lower bound as well. Suppose it is true that $C(\delta_N) \leq C(G)$ and we look at the perturbation equation of $\delta_N$ for minimum increase. (See fig 4.) Since all terms in the perturbation equation are non-decreasing, it would be least perturbed, if each individual term is minimally increased. In order to produce a connected graph, $deg(v_{N+1}) \geq 1$. (If connectedness was not required, we would have added another vertex with 0 degree). For minimum increase, set $deg(v_{N+1}) = 1$. This would also mean that $\Delta deg(v_i) = 0$ for all $v_i$ in $\delta_N$ except one. This corresponds to addition of exactly one edge to the old graph, $\delta_N$. However this edge can be added to any of the $N$ vertices. Note that this edge addition may disturb the shortest paths $k_{ij}$ between node pairs $v_i, v_j$. (The paths cannot be lengthened by edge addition). If that happens, terms of the form $deg(v)/k$ will get bigger. The only way to avoid this is to add the edges to either end of the chain. Therefore $\Delta k_{ij} = 0$ for all $1 \leq i, j \leq N$. This also maximizes $k_{i,N+1}$ for all $1 \leq i \leq N$ so that $deg(v_{N+1})/k_{i,N+1} = 1/k_{i,N+1}$ are minimized for all $i \leq N$. This shows that if the edge is added to a vertex which is not an end point, it results in an addition of degrees as well as a decrease in $k_{ij}$ for some vertices, again resulting in increase of complexity. Therefore, the optimal way to add the edge is to add the edge at its ends, which results in another delta chain $\delta_{N+1}$. ■

The consequence of this theorem is that the $\delta$-chain can be thought of as the simplest formation that can be formed over a fixed number of agents. This perhaps explains why humans like to make queues and birds fly in $V$-formations, both of which are essentially $\delta$-chains and require minimum coordination among individuals. We will use this result in the future to justify various $\delta$-chaining algorithms that are part of our current investigations of connectivity graphs.

## V. Conclusions

In this paper, we have presented a complexity measure for studying the structural complexity of robot formations. We have based this complexity measure on the number of local interactions in the system due to perception and communication. We showed that from an information theoretic point of view, perception and communication are fundamentally the same and should therefore not be discriminated when defining local interactions. We also showed that the broadcast protocol corresponds to the worst case complexity for a given formation and serves as an upper bound. We further noted that this upper bound is remarkably similar to the complexity measure of graphs defined in the context of molecular chemistry. This complexity measure on graphs was further explored to characterize the most complex and most simple graphs for a fixed number of vertices. We found that the complete graph and the $\delta$-chain are the extremal complexity graphs.

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