

Decentralized Degree Regularization for Multi-Agent Networks

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Abstract—Networked multi-agent systems are widely modeled as graphs where the agents are represented as nodes and edges exist between the agents that interact directly. In this setting, the degree of a node is the number of edges incident to it. For such systems, degree regularity (uniformity of degree across the nodes) typically provides desirable properties such as robustness and fast mixing time. As such, a key task is to achieve degree regularization in a decentralized manner. In this paper, we present a locally applicable rule that achieves this task. For any connected initial graph, the proposed reconfiguration rule preserves the graph connectivity and the total number of edges in the system while minimizing the difference between the maximum and the minimum node degrees.

I. INTRODUCTION

Multi-agent networks have been used to characterize a large number of natural and engineered systems such as protein networks, social networks, communication systems, transportation systems, power grids, and robotic swarms, to name a few. Such systems can be represented via their interaction graphs where the nodes correspond to the agents and the edges exist between the agents having direct interaction (e.g., [1], [2], [3]). Although the nature of these systems can be quite different, the impact of the interaction graph on the system behavior turns out to be a common feature. Therefore, various system properties such as *robustness* (how severely the system is influenced by perturbations of process/local components) and *mixing time* (how rapidly information spreads throughout the network) are usually analyzed through the topology of the interaction graph (e.g., [4], [5], [6], [7]). Robust interaction graphs with fast mixing times are desirable in numerous multi-agent applications including, but not limited to, flocking and swarming (e.g., [8]), sensor coverage (e.g., [9]), distributed estimation (e.g., [10]), and distributed control of robotic networks (e.g., [11]).

Node (or edge) connectivity is one of the fundamental robustness measures in graph theory. A graph is said to be *k-node (or edge) connected* if at least k nodes (or edges) must be removed to disconnect the graph. In general, graphs with higher connectivity have higher robustness to random failure of its components [4]. An arguably richer measure of robustness, and also mixing time, is the *expansion rate*. Expansion rate of a graph is quantified in terms of node, edge or spectral expansions. Node and edge expansions are refined notions of connectivity, whereas spectral expansion is given by the *spectral gap* of the adjacency matrix. Graphs with high

expansion rates are called *expanders*. Expanders are usually sparse yet well-connected, hence they are robust and they have fast mixing times. A detailed overview of expanders along with their numerous applications are presented in [12].

One class of graphs known to be excellent expanders is Ramanujan graphs [13], which are contained within the family of regular graphs. A graph is called a d -regular graph if each node has d edges incident to it. Moreover, a d -regular graph is Ramanujan if its spectral gap is greater than or equal to $d - 2\sqrt{d} - 1$. Regular graphs are one of the widely studied families of graphs due to their prevalence and typical properties such as robustness and fast mixing. For instance, it is known that almost every d -regular graph has a node connectivity d [14]. Furthermore, for $d \geq 3$, almost every d -regular graph is Ramanujan [15], [16]. In other words, for $d \geq 3$, if a graph is picked uniformly at random from the set of all possible d -regular graphs having n nodes, the probability of picking a Ramanujan graph approaches 1 as n increases.

Robustness and fast mixing in random regular graphs motivate the question of how a given system can be reconfigured so that the agents have similar (equal if possible) degrees. One possible way of achieving this by a central authority observing the whole graph structure is presented in [17], where random neighbors of maximally connected nodes are rewired to minimally connected nodes until a regular configuration is achieved. In this work, we study how such reconfiguration can be achieved in a decentralized fashion.

In this paper, we present a locally applicable rule that achieves the decentralized degree regularization. This reconfiguration rule only demands the nodes to have information available within their immediate neighborhoods. Furthermore, it preserves the graph connectivity and the total number of edges in the system. This rule induces stochastic dynamics of the graph topology, and we show that any feasible trajectory of the resulting system is a minimizing sequence for the difference between the maximum and the minimum node degrees in the system (with a probability 1). This difference converges to 0 if a regular graph is achievable with the available number of edges, and it converges to 1 otherwise.

The organization of this paper is as follows: Section II presents some preliminaries. Section III presents the decentralized degree regularization problem and the proposed solution. Section IV provides some simulation results. Finally, Section V concludes the paper.

II. PRELIMINARIES

In this section, we present some tools and definitions from graph theory that will be used in the remainder of this paper.

An undirected graph, $\mathcal{G} = (V, E)$, consists of a set of nodes, V , and a set of edges, E , given by unordered pairs of nodes. A graph is connected if there exists a path between any pair of nodes. A *path* is a sequence of nodes such that an edge exists between any two consecutive nodes in the sequence. A path is called a simple path if it contains no repeated nodes. Any two nodes are said to be adjacent if an edge exists between them. We refer to the set of nodes adjacent to any node, $i \in V$, as its *neighborhood*, \mathcal{N}_i , defined as

$$\mathcal{N}_i = \{j \mid (i, j) \in E\}. \quad (1)$$

For any node i , the number of nodes in its neighborhood is called its *degree*, d_i , i.e.,

$$d_i = |\mathcal{N}_i|, \quad (2)$$

where $|\mathcal{N}_i|$ denotes the cardinality of \mathcal{N}_i . For any graph \mathcal{G} , we use $\delta(\mathcal{G})$, $\Delta(\mathcal{G})$ and $\bar{d}(\mathcal{G})$ to denote the minimum, the maximum and the average degrees, respectively. A graph is said to be *d-regular*, if all the entries of its degree vector are equal to d . The degree irregularity of a graph can be measured via the difference of the maximum and the minimum node degrees in the system. Let $f(\mathcal{G})$ be a function defined as

$$f(\mathcal{G}) = \Delta(\mathcal{G}) - \delta(\mathcal{G}). \quad (3)$$

We refer to $f(\mathcal{G})$ as the *degree range*.

In our setting, nodes will seek to achieve the minimization of the degree range by rearranging their neighborhoods according to certain rules based on locally available information. In this context, we adopt the framework of graph grammars [18]. Graph grammars provide a systematic representation for local rules by encoding them in label (state) dependent local graph transformations. Some graph grammar preliminaries are presented next.

A *rule* is represented by an ordered pair of labeled graphs, $r = (g_l, g_r)$, where the labels represent the node states. As such, a rule is a change in the edge set that transforms graphs isomorphic to g_l to graphs isomorphic to g_r . The graphs g_l and g_r have the same node set, and the number of nodes in this set is called the rule size. A rule is said to be applicable to a graph \mathcal{G} , if \mathcal{G} has a subgraph isomorphic to g_l . A *grammar*, Φ , is a set of rules.

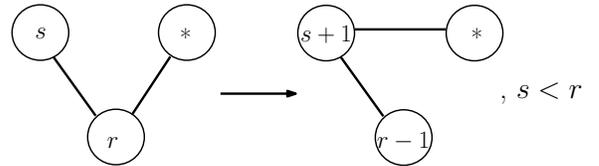
An initial graph \mathcal{G}_0 along with a grammar Φ defines a *system* represented as (\mathcal{G}_0, Φ) . Note that (\mathcal{G}_0, Φ) defines a non-deterministic dynamical system since, at any instant, many rules in Φ may be simultaneously applicable, possibly via intersecting subgraphs. A *trajectory*, τ , of a system (\mathcal{G}_0, Φ) is a sequence (finite or infinite) of graphs $\{\mathcal{G}_0, \mathcal{G}_1, \dots\}$, where each \mathcal{G}_{k+1} in the sequence is obtained from \mathcal{G}_k via the application of some rules in the grammar Φ . A trajectory is finite if and only if no rules are applicable to the final graph on the trajectory.

III. DECENTRALIZED DEGREE REGULARIZATION

The *decentralized degree regularization* problem is a problem of finding a decentralized scheme to minimize the degree range in a networked system while preserving its connectivity and the total number of edges. Preservation of connectivity is vital for various applications. Without connectivity, information and interactions cannot spread throughout the network, and certain parts behave independently as separate systems. Preservation of the total number of edges bounds the number of resources used in the system. Each edge represents an interaction typically requiring a power consumption, an information exchange, a measurement, or a physical link. As such, sparsity (having a small number of edges) is an important feature of networked systems. For instance, one can always come up with a local rule leading to a complete graph, which has the ultimate robustness and mixing time, but not a good option for large systems due to the lack of sparsity. Hence, we assume that the initial configuration of a given system is built using a certain number of available resources (edges), and we search for a rather desirable reallocation of these edges. In particular, we search for a set of locally applicable rules (a graph grammar) and define the decentralized degree regularization problem as follows.

Definition (Decentralized Degree Regularization Problem): Find a graph grammar, Φ , such that any trajectory $\tau = \{\mathcal{G}_0, \mathcal{G}_1, \dots\}$ of (\mathcal{G}_0, Φ) , where $\mathcal{G}_0 = (V, E_0)$ is a connected graph, is (with probability 1) a minimizing sequence for the degree range, $f(\mathcal{G}_k)$, subject to \mathcal{G}_k being connected and $|E_k| = |E_0|$ for every $\mathcal{G}_k \in \tau$.

As a solution to the decentralized degree regularization problem, we propose a grammar, Φ^* , that consists of a single rule. In this grammar, we set the label of a node to be its degree, and we define Φ^* as



where s , r and $*$ denote the degrees of the corresponding nodes. The rule requires $s < r$, whereas $*$ is arbitrary and it is unaffected by the application of the rule.

In accordance with Φ^* , nodes behave as follows: Let i and j be two adjacent nodes, and let r and s denote their degrees respectively. If $s < r$, then a link is formed between j and an arbitrary neighbor of i , say h , that is not currently linked with j . At the same time, the link between i and h is terminated. Note that Φ^* only requires information available within local neighborhoods. If each node knows the degrees of its immediate neighbors, then whenever Φ^* is applicable, at least one node will be able to detect it.

One of the important properties for distributed systems is concurrency. In a concurrent system, multiple subsystems may operate simultaneously at any time instant. In graph

grammars, concurrency is modeled by the commutativity of rule applications. That is to say, if an application of a rule needs the output of an other rule application, then these events need to happen sequentially. Since Φ^* consists of a single rule, at any instant it can be simultaneously executed at distinct locations on the graph. In order to represent the concurrency in the system, for any trajectory $\tau = \{\mathcal{G}_0, \mathcal{G}_1, \dots\}$, let $C_k = \{c_1, c_2, \dots, c_m\}$ denote the set of distinct ordered node triplets that are involved in the concurrent applications of Φ^* to \mathcal{G}_k . The ordered triplets in C_k can be represented as

$$c_p = \{s'_p, r'_p, *'_p\}, \quad \forall p = 1, 2, \dots, |C_k|, \quad (4)$$

where $s'_p, r'_p, *'_p \in V$ denote the nodes involved in the corresponding application of Φ^* with labels (degrees) s , r , and $*$, respectively. Note that for every $c_p, c_q \in C_k$, if $p \neq q$, then $c_p \cap c_q = \emptyset$.

We start our analysis of Φ^* by first showing that the resulting trajectories satisfy the constraints of the decentralized degree regularization problem.

Proposition 3.1 *Let $\mathcal{G}_0 = (V, E_0)$ be a connected graph, and $\tau = \{\mathcal{G}_0, \mathcal{G}_1, \dots\}$ be an arbitrary trajectory of the system (\mathcal{G}_0, Φ^*) . Then, \mathcal{G}_k is connected for every $\mathcal{G}_k \in \tau$.*

Proof: (Induction) Let \mathcal{G}_k and \mathcal{G}_{k+1} be two consecutive graphs in τ , and let \mathcal{G}_k be connected. Furthermore, let C_k denote the set of distinct node triplets, each of them given as in (4), involved in concurrent applications of Φ^* at instant k . If \mathcal{G}_k is connected, then for every node pair, i and j , there exists a finite simple path P from i to j . If P does not have a subsequence of the form $\{r'_p, *'_p\}$ (or $\{*'_p, r'_p\}$), for any $p = 1, 2, \dots, |C_k|$, then P is a valid path on \mathcal{G}_{k+1} as well. Otherwise, if P contains such subsequences, replace those $\{r'_p, *'_p\}$ (or $\{*'_p, r'_p\}$) in P with $\{r'_p, s'_p, *'_p\}$ (or $\{*'_p, s'_p, r'_p\}$) to obtain a valid path between i and j on \mathcal{G}_{k+1} . Hence, if $\mathcal{G}_0 = (V, E_0)$ is connected, then every $\mathcal{G}_k \in \tau$ is connected. ■

Proposition 3.2 *Let $\mathcal{G}_0 = (V, E_0)$ be a graph, and $\tau = \{\mathcal{G}_0, \mathcal{G}_1, \dots\}$ be an arbitrary trajectory of the system (\mathcal{G}_0, Φ^*) . Then, $|E_k| = |E_0|$ for every $\mathcal{G}_k \in \tau$.*

Proof: Φ^* contains a single rule that preserves the number of edges in the system. Hence it is not possible to change the number of edges in the system via Φ^* , and the number of edges remain constant along any trajectory τ . ■

Note that if the number of edges is constant along any trajectory of the system (\mathcal{G}_0, Φ^*) , then so is the average degree of nodes, \bar{d} .

Corollary 3.3 *The average degree, \bar{d} , is constant along any trajectory of the system (\mathcal{G}_0, Φ^*) .*

Next, we present the equilibrium points for Φ^* . These equilibrium points are the graphs such that Φ^* is not applicable anywhere on them. We are particularly interested in

equilibrium points that are connected graphs, since we have already shown that Φ^* maintains connectivity.

Proposition 3.4 *Let $\mathcal{G} = (V, E)$ be a connected graph. \mathcal{G} is an equilibrium point for Φ^* if and only if \mathcal{G} is a regular graph.*

Proof:

\Rightarrow : (Contradiction) Let \mathcal{G} be an irregular graph. Since \mathcal{G} is connected, if \mathcal{G} is irregular, then there exists $i, j \in V$ such that $(i, j) \in E$ and $d_i > d_j$. Note that $d_i = |\mathcal{N}_i \setminus \mathcal{N}_j| + |\mathcal{N}_i \cap \mathcal{N}_j| + 1$ and $d_j = |\mathcal{N}_j \setminus \mathcal{N}_i| + |\mathcal{N}_i \cap \mathcal{N}_j| + 1$. If $d_i > d_j$, then $|\mathcal{N}_i \setminus \mathcal{N}_j| - |\mathcal{N}_j \setminus \mathcal{N}_i| > 0$ implying $|\mathcal{N}_i \setminus \mathcal{N}_j| > 0$. Hence, for such i and j , node i always has a neighbor that is not adjacent to node j . Consequently, Φ^* is applicable to \mathcal{G} and it is not an equilibrium point.

\Leftarrow : If \mathcal{G} is a regular graph, then by definition all the nodes have the same degree. So, there is no pair $i, j \in V$ such that $(i, j) \in E$ and $d_i > d_j$. Consequently, Φ^* is not applicable anywhere on the graph \mathcal{G} , and \mathcal{G} is an equilibrium point. ■

Note that for a given initial configuration, \mathcal{G}_0 , the maintenance of connectivity and the total number of edges define a feasible set of graphs. In general, depending on \mathcal{G}_0 , this feasible set may not contain an equilibrium point for Φ^* . In particular, if the average degree of the initial configuration, \bar{d}_0 , is not an integer, then the feasible set does not contain any regular graph. Hence, we want to address two questions: 1) if the feasible set includes equilibrium points, will (\mathcal{G}_0, Φ^*) converge to an equilibrium point?, 2) how does (\mathcal{G}_0, Φ^*) behave if the feasible set does not contain any equilibrium point?

We address these two questions by inspecting how the degree range behaves along the possible trajectories of (\mathcal{G}_0, Φ^*) . For any trajectory, $\tau = \{\mathcal{G}_0, \mathcal{G}_1, \dots\}$, let $\tau_f = \{f(\mathcal{G}_0), f(\mathcal{G}_1), \dots\}$ be the degree range sequence along the trajectory τ . Firstly, we will prove the convergence of τ_f by showing that it is bounded below and it is monotonically decreasing.

Proposition 3.5 *Let \mathcal{G} be a graph, and let $f(\mathcal{G})$ be its degree range. Then, $f(\mathcal{G}) \geq 0$. Furthermore, $f(\mathcal{G}) = 0$ if and only if \mathcal{G} is a regular graph.*

Proof: By definition $\Delta(G) \geq \delta(G)$, and we have $f(\mathcal{G}) = \Delta(G) - \delta(G) \geq 0$. Moreover, $\delta(G) \leq d_i \leq \Delta(G)$ for all $i = 1, 2, \dots, n$. Hence, if $\Delta(G) - \delta(G) = 0$, we obtain $d_i = \delta(G) = \Delta(G)$ for all $i = 1, 2, \dots, n$. For such a case, each node has the same degree, and the corresponding graph is regular. ■

Proposition 3.6 *Let \mathcal{G}_k and \mathcal{G}_{k+1} be consecutive graphs on a trajectory τ of the system (\mathcal{G}_0, Φ^*) . Then, their degree ranges satisfy*

$$f(\mathcal{G}_{k+1}) - f(\mathcal{G}_k) \leq 0. \quad (5)$$

Proof: We prove this by showing that, under Φ^* , Δ is monotonically decreasing, and δ is monotonically increasing.

For the sake of contradiction, let us assume that Δ increases using Φ^* . Then, a node with maximum degree participates in an application of Φ^* as the s -labeled node. However, the rule requires $s < r$, and by definition we have $\Delta \geq d_i$ for any node i in the system. Hence, we reach a contradiction. Similarly, let us assume that δ decreases using Φ^* . Then, a node with minimum degree participates in an application of Φ^* as the r -labeled node. However, the rule requires $s < r$ and by definition we have $d_i \geq \delta$ for any node i in the system. Hence, again we reach a contradiction. Consequently, we obtain $f(\mathcal{G}_{k+1}) - f(\mathcal{G}_k) \leq 0$. ■

Corollary 3.7 *Let $\tau = \{\mathcal{G}_0, \mathcal{G}_1, \dots\}$ be an arbitrary trajectory of (\mathcal{G}_0, Φ^*) . Then, the degree range sequence $\tau_f = \{f(\mathcal{G}_0), f(\mathcal{G}_1), \dots\}$ converges to a finite integer τ_f^* .*

Proof: In light of Proposition 3.5, τ_f is bounded below. Furthermore, Proposition 3.6 states that it is monotonically decreasing. Hence, one can conclude that τ_f is a convergent sequence, and since $f(\mathcal{G})$ is an integer valued function, it converges to a finite integer τ_f^* . ■

Next, we will prove that any τ_f converges to the minimum possible value that can be achieved using $|E_0|$ edges. Note that (\mathcal{G}_0, Φ^*) is a non-deterministic dynamical system since, at any instant, a node may have multiple neighbors having degrees smaller than its own degree. Furthermore, for each such neighbor, the node may have multiple neighbors that can participate as $*$ to the application of the rule. In such cases, one combination is randomly chosen among the possible rule applications by assigning a non-zero probability to each option. In this setting, the resulting dynamics is non-deterministic, and each time step k , the set denoting the concurrently applied rules, C_k , is a random variable. We will prove that, under the stochastic dynamics of (\mathcal{G}_0, Φ^*) , τ_f almost surely converges to the minimum feasible value.

Let $\Pr(m; \mathcal{G})$ denote the probability that in m time steps (\mathcal{G}, Φ^*) reaches a graph with a smaller degree range. Note that, since f is monotonically decreasing on any trajectory of (\mathcal{G}, Φ^*) , we have

$$\Pr(m; \mathcal{G}) \leq \Pr(m'; \mathcal{G}), \quad \forall m' \geq m. \quad (6)$$

Lemma 3.8 *Let $\tau = \{\mathcal{G}_0, \mathcal{G}_1, \dots\}$ be an arbitrary trajectory of (\mathcal{G}_0, Φ^*) . For any $\mathcal{G}_{k'} \in \tau$, let $\tau_{k'} = \{\mathcal{G}_{k'}, \mathcal{G}_{k'+1}, \dots\}$ be a contiguous subsequence of τ such that the degree range is constant along $\tau_{k'}$. If there exists a finite m^* such that $\Pr(m^*; \mathcal{G}_k) > 0$ for all $\mathcal{G}_k \in \tau_{k'}$, then $\tau_{k'}$ almost surely has a finite length.*

Proof: Let $\Pr(\tau_{k'}^\infty)$ denote the probability of having an infinitely long contiguous subsequence, $\tau_{k'}^\infty$, along which the degree range is constant. Since the degree range never decreases within some finite time along $\tau_{k'}^\infty$, for any finite positive integer m , we have

$$\Pr(\tau_{k'}^\infty) \leq \prod_{i=0}^{\infty} (1 - \Pr(m; \mathcal{G}_{k'+i})). \quad (7)$$

If there exists a finite m^* such that $\Pr(m^*; \mathcal{G}_k) > 0$ for all $\mathcal{G}_k \in \tau_{k'}^\infty$, then we can plug $m = m^*$ into (7) and have the infinite product equal to 0. Hence, if such m^* exists, we obtain $\Pr(\tau_{k'}^\infty) = 0$. ■

Lemma 3.9 *For any graph \mathcal{G} , if there exists a graph with a smaller degree range, \mathcal{G}' , that can be reached from \mathcal{G} using Φ^* in m time steps, then $\Pr(m'; \mathcal{G}) > 0$ for all $m' \geq m$.*

Proof: If there exists such a \mathcal{G}' , then there is a non-zero probability that (\mathcal{G}, Φ^*) will reach \mathcal{G}' in m steps, i.e., $\Pr(m; \mathcal{G}) > 0$. In light of (6), we have $\Pr(m'; \mathcal{G}) > 0$ for all $m' \geq m$. ■

Next, we present a simple algorithm to show that for any \mathcal{G} , if $f(\mathcal{G}) \geq 2$, then a graph \mathcal{G}' having $f(\mathcal{G}') < f(\mathcal{G})$ can always be reached in finite time steps via Φ^* . To this end, we consider a shortest simple path, $P^* = \{i^*, j^*, \dots, q^*\}$, such that $d_{i^*} = \Delta$ and $d_{q^*} = \delta$. In other words, for any $P = \{i, j, \dots, q\}$, if $d_i = \Delta$ and $d_q = \delta$, then P^* satisfies

$$|P^*| \leq |P|. \quad (8)$$

Algorithm I

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1 : input:  $\mathcal{G} = (V, E)$  s.t.  $f(\mathcal{G}) \geq 2$ 
1 : initialize:  $m = 0$ ,  $\mathcal{G}^- = \mathcal{G}$ ,  $df = 0$ 
2 : while  $df = 0$ 
3 :   Find a  $P^* = \{i^*, j^*, \dots, q^*\}$  on  $\mathcal{G}^-$ 
4 :   Find an  $i \in V$  s.t.  $(i, i^*) \in E^-$ ,  $(i, j^*) \notin E^-$ 
5 :    $E^+ = (E^- \setminus \{(i, i^*)\}) \cup \{(i, j^*)\}$ 
6 :    $\mathcal{G}^+ = (V, E^+)$ 
7 :    $df = f(\mathcal{G}^+) - f(\mathcal{G}^-)$ 
8 :    $\mathcal{G}^- = \mathcal{G}^+$ 
9 :    $m = m + 1$ 
10 : end while
11 : return  $m$ 

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Proposition 3.10 *For any connected graph \mathcal{G} with a degree range $f(\mathcal{G}) \geq 2$, Algorithm I returns an integer m satisfying $\Pr(m; \mathcal{G}) > 0$.*

Proof: For any graph, one can always find a shortest simple path, P^* , between a node with maximum degree and a node with minimum degree. We will first show that the rewiring in Algorithms I (line 5) is a valid application of Φ^* . Note that for such a $P^* = \{i^*, j^*, \dots, q^*\}$, we always have $d_{j^*} < d_{i^*}$. Since d_{i^*} is the maximum degree in the system, it is always greater than or equal to the degree of an other node. Furthermore, d_{j^*} can not be equal to d_{i^*} since this would contradict (8) and give us a shorter path, $\{j^*, \dots, q^*\}$, between a node with maximum degree and a node with minimum degree. Since $d_{j^*} < d_{i^*}$, one can always find a node i as described in line 4 of Algorithm I, and the rewiring in line 5 is a valid application of Φ^* .

At each iteration of the *while* loop in Algorithm I, \mathcal{G}^+ has either fewer nodes having a degree $\Delta(\mathcal{G})$, or a shorter P^* compared to \mathcal{G}^- . If $d_{j^*} < d_{i^*} - 1$ we have the first case, and if $d_{j^*} = d_{i^*} - 1$ we have the second case. However, getting a shorter P^* also eventually reduces the number of nodes having degree $\Delta(\mathcal{G})$, since if $|P^*| = 1$, then we have a valid application of Φ^* to which nodes having degrees $\Delta(\mathcal{G})$ and $\delta(\mathcal{G})$ participate and end up having degrees $\Delta(\mathcal{G}) - 1$ and $\delta(\mathcal{G}) + 1$, respectively. For any graph \mathcal{G} with $f(\mathcal{G}) \geq 2$, this implies a reduction in the number of agents having a degree $\Delta(\mathcal{G})$. Hence, eventually we get a graph \mathcal{G}^+ where each node has a degree smaller than $\Delta(\mathcal{G})$. Furthermore, since the minimum degree is monotonically increasing, for the final \mathcal{G}^+ we have $f(\mathcal{G}^+) < f(\mathcal{G})$. In light of Lemma 3.9, this implies that for the value, m , returned by Algorithm I, we have $\Pr(m; \mathcal{G}) > 0$. ■

Theorem 3.11 *Let $\tau = \{\mathcal{G}_0, \mathcal{G}_1, \dots\}$, be an arbitrary trajectory of (\mathcal{G}_0, Φ^*) for a connected \mathcal{G}_0 . Then, the degree range sequence, $\tau_f = \{f(\mathcal{G}_0), f(\mathcal{G}_1), \dots\}$, converges to an integer τ_f^* that almost surely satisfies*

$$0 \leq \tau_f^* \leq 1. \quad (9)$$

Proof: From Corollary 3.7, we know that τ_f converges to an integer τ_f^* . Let $\tau_{k'}$ be a contiguous subsequence of τ such that f is constant along $\tau_{k'}$. Let $f(\mathcal{G}_{k'})$ denote this constant value. If $f(\mathcal{G}_{k'}) \geq 2$, then for any $\mathcal{G}_k \in \tau_{k'}$, Algorithm I provides a value m_k satisfying $\Pr(m_k; \mathcal{G}_k) > 0$. Let m^* be the maximum of those m_k . Using m^* , in light of Lemma 3.8, we can say that almost surely $\tau_{k'}$ has a finite length. Since (\mathcal{G}_0, Φ^*) is in equilibrium only when a regular configuration is attained ($f(\mathcal{G}) = 0$), a finite length $\tau_{k'}$ with $f(\mathcal{G}_{k'}) \geq 2$ implies a reduction in f at the end of $\tau_{k'}$. Using this along with Proposition 3.5, we conclude that τ_f almost surely converges to an integer value $0 \leq \tau_f^* \leq 1$. ■

Note that in solving the decentralized degree regularization problem for general graphs, (9) is the best result that can be achieved. This is because $f(\mathcal{G}) = 0$ is not always feasible when the initial number of edges is preserved throughout the dynamics. Clearly, if the average degree in the system, \bar{d} , is not an integer, then it is not possible to form a regular graph with the available number of edges. Next, we focus on the cases where it is possible to form a regular graph, i.e. $\bar{d} \in \mathbb{N}$.

Lemma 3.12 *Let \mathcal{G} be an irregular graph, let \bar{d} be the average degree of \mathcal{G} , and let $f(\mathcal{G})$ be the degree range of \mathcal{G} . If $\bar{d} \in \mathbb{N}$, then $f(\mathcal{G}) \geq 2$.*

Proof: For any graph $\mathcal{G} = (V, E)$, one can always represent its degree vector, d , as

$$d = \delta(\mathcal{G})\mathbf{1} + \tilde{d}, \quad (10)$$

where $\mathbf{1}$ is a vector having all its entries equal to 1. Since there is at least one node having the minimum degree, \tilde{d} has

at least one entry being equal to 0. The average degree, \bar{d} , satisfies

$$\bar{d} = \frac{1}{n}\mathbf{1}^T d = \delta(\mathcal{G}) + \frac{1}{n}\mathbf{1}^T \tilde{d}. \quad (11)$$

Using (11), we get

$$\bar{d} - \delta(\mathcal{G}) = \frac{1}{n}\mathbf{1}^T \tilde{d}. \quad (12)$$

Note that if $\bar{d} \in \mathbb{N}$ and \mathcal{G} is irregular, we have $\bar{d} - \delta(\mathcal{G}) \geq 1$, and (12) implies

$$\frac{1}{n}\mathbf{1}^T \tilde{d} \geq 1. \quad (13)$$

Since at least one entry of \tilde{d} is equal to 0, \tilde{d} can have at most $n - 1$ positive entries. In light of (13), the sum of those positive entries are greater than or equal to n . Hence, at least one of them is greater than 1, and $f(\mathcal{G}) \geq 2$. ■

Theorem 3.13 *Let \mathcal{G}_0 be a connected graph having an average degree \bar{d} . If $\bar{d} \in \mathbb{N}$, then an arbitrary trajectory of (\mathcal{G}_0, Φ^*) almost surely converges to a \bar{d} -regular graph.*

Proof: Let $\tau = \{\mathcal{G}_0, \mathcal{G}_1, \dots\}$, be an arbitrary trajectory of (\mathcal{G}_0, Φ^*) , and let $f(\mathcal{G})$ be the function given in (3). Then, in light of Theorem 3.11, the sequence $\tau_f = \{f(\mathcal{G}_0), f(\mathcal{G}_1), \dots\}$ almost surely converges to an integer $0 \leq \tau_f^* \leq 1$. Furthermore, in light of Lemma 3.12, for a graph \mathcal{G} having an integer average degree \bar{d} , $f(\mathcal{G}) \neq 1$. Hence, we conclude that if $\bar{d} \in \mathbb{N}$, then almost surely $\tau_f^* = 0$ and the trajectory converges to a \bar{d} -regular graph. ■

We combine Theorem 3.11 and Theorem 3.13 to give the following corollary for the convergence of $\tau_f = \{f(\mathcal{G}_0), f(\mathcal{G}_1), \dots\}$ along the possible trajectories of (\mathcal{G}_0, Φ^*) .

Corollary 3.14 *Let $\tau = \{\mathcal{G}_0, \mathcal{G}_1, \dots\}$, be an arbitrary trajectory of (\mathcal{G}_0, Φ^*) for a connected \mathcal{G}_0 . Then, the degree range sequence $\tau_f = \{f(\mathcal{G}_0), f(\mathcal{G}_1), \dots\}$ almost surely converges to an integer τ_f^* such that*

$$\tau_f^* = \begin{cases} 0 & \text{if } \bar{d} \in \mathbb{N} \\ 1 & \text{otherwise,} \end{cases} \quad (14)$$

where \bar{d} denotes the average degree of \mathcal{G}_0 .

IV. SIMULATION RESULTS

In this section, we present some simulation results for the proposed decentralized degree regularization scheme. We randomly generate a connected initial graph using 20 nodes and 30 edges. Note that with this many nodes and edges, a 3-regular graph can be formed. Starting from the initial configuration, nodes concurrently update their neighborhoods according to Φ^* , and the system converges to a 3-regular configuration. Some graph configurations along an arbitrary trajectory of the system (\mathcal{G}_0, Φ^*) are depicted in Fig. 1, whereas the degree range, $f(\mathcal{G}_k)$, along the same trajectory is shown in Fig. 2. As it can be seen in Fig. 2, the initial graph

has a degree range equal to 4. This value rapidly drops to 3, and then to 2 within the first 10 steps. Then, the degree range stays at 2 for a relatively longer period as the incidences that can reduce the degree range get fewer. During this period, for most of the rule applications, the degrees of the participating nodes satisfy $r - s = 1$ and the result is a degree swapping. Yet, rule applications involving nodes with maximum and minimum degrees, where $r - s = 2$, eventually occur and the system reaches a 3-regular configuration at $k = 116$.

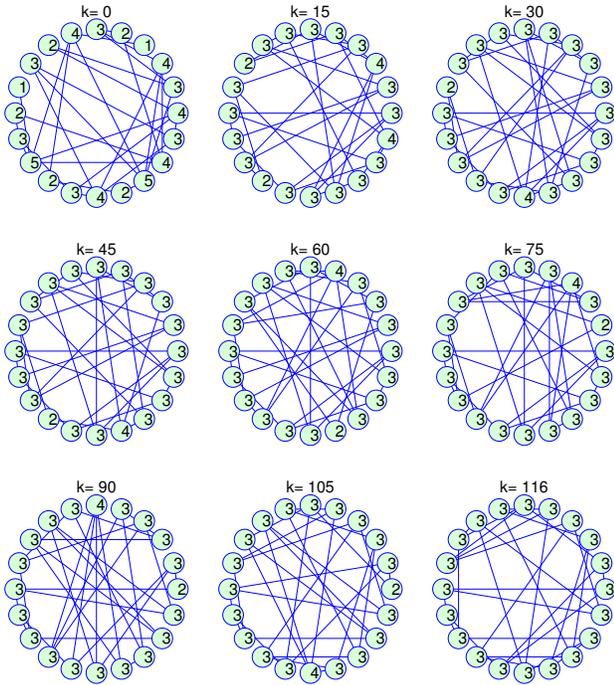


Fig. 1. \mathcal{G}_k at some instants along an arbitrary trajectory of (\mathcal{G}_0, Φ^*) . On this trajectory, the initial graph ($k=0$) converges to a 3-regular graph ($k=116$) via some concurrent applications of Φ^* in 116 time steps. On each \mathcal{G}_k , nodes are labeled with their degrees.

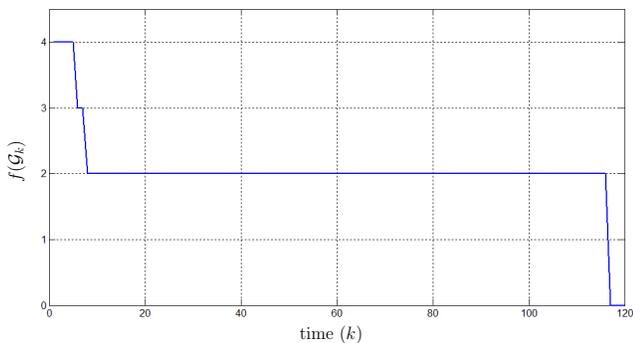


Fig. 2. Degree range, $f(\mathcal{G}_k)$, along the trajectory depicted in Fig. 1.

V. CONCLUSION

In this paper, we posed a decentralized degree regularization problem and presented a solution. The decentralized

degree regularization problem is motivated by the earlier results in the literature showing that, for $d \geq 3$, random d -regular graphs are almost surely Ramanujan, and as such they are excellent expanders.

Our solution to the decentralized degree regularization problem is a graph grammar, Φ^* , consisting of a single rule. Application of Φ^* requires each node to only know the degrees of its neighbors. According to Φ^* , if any node has a neighbor with fewer connections, then one of its other neighbors is rewired to the one with fewer connections. The total number of edges and the connectivity are preserved under the resulting dynamics. Furthermore, starting with any connected initial graph, with a probability 1, the degree range converges to 0 if a regular graph is reachable, and it converges to 1 otherwise. Some simulation results for the proposed scheme were also presented in the paper.

For any connected initial graph, the scheme presented in this paper provides almost sure convergence to a connected regular graph if a regular configuration is feasible using the available number of edges. However, the resulting graph may not always be an excellent expander since the non-Ramanujan configurations may also be attractive. As a future work, we want to improve the proposed scheme to avoid possible convergence to such undesired equilibrium points.

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