Responsiveness and Manipulability of Formations of Multi-Robot Networks

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***Abstract*— This paper unifies the previously defined instantaneous measures of responsiveness in multi-agent systems, namely those of stiffness and manipulability. These concepts are used for characterizing how a networked multi-agent system responds to perturbation of agents or to exogenous movements of its leader nodes. The resulting unified notion of a system's responsiveness provides us with a precise characterization of the effect of different choices of leader nodes and interaction topologies have on how easy or hard it is to control the network.**

I. Introduction

During the last decade, our understanding of how to design decentralized controllers for networked, multi-agent systems has increased significantly. Controllers have for example been designed for maintaining formation in coordinated mobile systems, for achieving coverage in sensor networks, and for establishing communication links in mobile ad-hoc networks. However, what is still not well-understood is how such networked systems should be structured in order to make them amenable to human control.

One can imagine a number of ways in which human inputs can be incorporated into the design; from control of virtual nodes [1], [2] and manipulation of the team's boundaries [3], [4], to the introduction of incompressible flows in the system [5]. In this paper, we follow the idea that control inputs can be injected into the network through dedicated leader-nodes, while these inputs are propagated to the remaining nodes through the dynamic interaction protocol, e.g., [6], [7]. In particular, we are interested in what effect the movements of the leader nodes have on the remaining nodes.

One manner in which this impact can be measured is in terms of the system's controllability properties [6]. However, controllability is a point-to-point property in that it establishes whether or not the agents, viewed as an ensemble, can be transferred from one configuration to another. Rather than investigating such point-to-point properties, in this paper we study instead the *instantaneous* effect that the leader nodes' movements have on the remainder of the network.

Such instantaneous measures have been previously proposed in the literature. For example, the classic notion of rigidity can be thought of as measuring whether or not links of fixed lengths between agents in the network can ensure that the system behaves like a rigid body [8]. This notion was generalized in [9] to systems where the underlying graph structures are rigid, yet the edge lengths are asymptotically maintained through actual control laws, resulting in *stiffness* and the related *rigidity* indices. Similarly, in [10], the notion of *manipulability* indices was introduced in the multi-robot context in a manner similar to its traditional use for robotic manipulators as a measure of how hard/easy it is to move a manipulator arm in a given direction. In this paper we unify these previous notions of instantaneous network responses through the introduction of *responsiveness* as a measure of how well the system responds to the movements of the leaders.

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II. Preliminaries and Background

A. Multi-Agent Formations and the Rigidity Matrix

Let $x(t) = [x_1(t)^\top, ..., x_N(t)^\top]^\top \in \mathbb{R}^{Nd}$ be a configuration of N agents, where agent $i \in \{1, ..., N\}$ has state $x_i(t) \in \mathbb{R}^d$ at time t. Consider that M pairs of agents have communication links and constitute an information-exchange network. Using a graph-based representation, $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ denotes the underlying graph of the network. Here $\mathsf{V} = \{\mathsf{v}_1, ..., \mathsf{v}_N\}$ is the node set, where the node v_i corresponds to agent i; and, $\mathsf{E} = \{\{\mathsf{v}_{i_1}, \mathsf{v}_{j_1}\}, ..., \{\mathsf{v}_{i_M}, \mathsf{v}_{j_M}\}\}$ is the edge set, where each edge represents a communication link between two agents. Then we refer to (x, G) as the "formation" of the network [11], which is also called the "framework" in the mathematics literature (e.g., [8]).

Assume the situation where the connected agents do not change their relative distances and the trajectories of $x_i(t)$ are smooth and differentiable in t. This gives us the following constraints on agents' motion:

$$\frac{d}{dt}\frac{1}{2}||x_i - x_j||^2 = (x_i - x_j)^\top(\dot{x}_i - \dot{x}_j) = 0 \quad \forall \{\mathsf{v}_i, \mathsf{v}_j\} \in \mathsf{E},$$

which can be summarized as

$$R\dot{x} = 0,$$

where $R(x, \mathsf{G}) \in \mathbb{R}^{M \times Nd}$ is called the rigidity matrix of the formation (x, G) [8], [11]. R consists of $M \times N$ blocks of $1 \times d$ row vectors, and its (k, i_k) and (k, j_k) blocks are $(x_{i_k} - x_{j_k})^\top$ and $-(x_{i_k} - x_{j_k})^\top$ (or $-(x_{i_k} - x_{j_k})^\top$ and $(x_{i_k} - x_{j_k})^\top$), respectively, and other blocks are zeros, where the agents i_k and j_k are connected by edge $k \in \{1, ..., M\}$.

The edges constrain the motion of the agents. However, the translational and rotational freedom, $d + \binom{d}{2} = \frac{d(d+1)}{2}$, always remain [8]: $\mathrm{rank}(R) \leq Nd - \frac{d(d+1)}{2}$. Suppose that all the N agents are not contained in any hyperplane within $\mathbb{R}^d$ (i.e., "generic" case [8]). The formation is said to be "rigid" if and only if the equality holds.

We here, without loss of generality, assign the last N_ℓ indices to leaders and remaining $N_f(= N - N_\ell)$ indices to followers. Hence, the vector $x(t)$ is divided into two parts: $x(t) = [x_f(t)^\top, x_\ell(t)^\top]^\top$, where $x_f \in \mathbb{R}^{N_f d}$ and $x_\ell \in \mathbb{R}^{N_\ell d}$ are the states of the followers and the leaders, respectively. Using the submatrices $R_f(x) \in \mathbb{R}^{M \times N_f d}$ and $R_\ell(x) \in \mathbb{R}^{M \times N_\ell d}$ of the rigidity matrix such that $R(x) = [R_f(x)|R_\ell(x)]$, we have

$$R_f(x)\dot{x}_f(t) + R_\ell(x)\dot{x}_\ell(t) = 0. \tag{1}$$

By definition, $R(x)$ has a null space corresponding to the translational freedom of the agents' (infinitesimal) motion, i.e., $R(x)(\mathbf{1}_N \otimes I_d) = 0$, where $\mathbf{1}_k$ is the k-dimensional column vector whose entries are all 1, and I_k is the $k \times k$ identity matrix. Hence $R_f(x)(\mathbf{1}_{N_f} \otimes I_d) = -R_\ell(x)(\mathbf{1}_{N_\ell} \otimes I_d)$, and in the case of $N_\ell = 1$, we have

$$R_f(x)(\mathbf{1}_{N_f} \otimes I_d) = -R_\ell(x). \tag{2}$$

B. Edge-Tension Energy

To formulate the dynamics of the agents later, we introduce an edge-tension energy

$$\mathcal{E}(x) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{E}_{ij}(x_i(t), x_j(t)),$$

where

$$\mathcal{E}_{ij}(x_i, x_j) = \begin{cases} \frac{1}{2}\{e_{ij}(||x_i - x_j||)\}^2 & \{\mathsf{v}_i, \mathsf{v}_j\} \in \mathsf{E} \\ 0 & \text{otherwise.} \end{cases}$$

Here, $e_{ij} : \mathbb{R}^+ \to \mathbb{R}$ is a strictly increasing, twice differentiable function such that $e_{ij}(d_{ij}) = 0$ and $e'_{ij}(d_{ij}) \neq 0$, where $d_{ij} > 0$ is the desired distance between agents i and j, and $e'_{ij}(r) \triangleq \frac{de_{ij}(r)}{dr}$. A typical example of e_{ij} is

$$e_{ij}(||x_i - x_j||) = c_{ij}\{||x_i - x_j|| - d_{ij}\}, \tag{3}$$

where $c_{ij} > 0$ is a weight assigned to the communication link between agents i and j.

Let $D(\mathsf{G}) \in \{-1, 0, 1\}^{N \times M}$ be an incidence matrix [2] of graph G. The first and the second derivatives of the edge-tension energy are given as follows (see [10] for the details of the derivation):

$$\begin{aligned} \frac{\partial \mathcal{E}(x)}{\partial x}^\top &= (L_w(x) \otimes I_d)x = ((DW_1(x)D^\top) \otimes I_d)x, \\ \frac{\partial^2 \mathcal{E}(x)}{\partial x^2} &= L_w(x) \otimes I_d + R(x)^\top W_2(x) R(x), \end{aligned}$$

where $L_w(x) \triangleq DW_1(x)D^\top$; $W_1(x)$ and $W_2(x)$ are $M \times M$ diagonal matrices whose diagonal elements are

$$\begin{aligned} [W_1(x)]_{kk} &= w_{i_k j_k}(||x_{i_k} - x_{j_k}||), \\ [W_2(x)]_{kk} &= \frac{w'_{i_k j_k}(||x_{i_k} - x_{j_k}||)}{||x_{i_k} - x_{j_k}||}, \\ & \quad k = 1, ..., M, \quad \{\mathsf{v}_{i_k}, \mathsf{v}_{j_k}\} : \text{edge } k, \end{aligned}$$

where, letting $e''_{ij}(r) \triangleq \frac{d^2 e_{ij}(r)}{dr^2}$,

$$w_{ij}(r) \triangleq \frac{e_{ij}(r)e'_{ij}(r)}{r},$$

$$w'_{ij}(r) \triangleq \frac{dw_{ij}}{dr} = \frac{\{e'_{ij}(r)^2 + e_{ij}(r)e''_{ij}(r)\}r - e_{ij}(r)e'_{ij}(r)}{r^2}.$$

Note here that the edges are assumed to be indexed consistently for $W_1(x)$ and the incidence matrix D, and for $W_2(x)$ and the rigidity matrix $R(x)$, respectively.

Remark 1 *If all the desired distances are satisfied at $x = x^*$, then $\frac{\partial \mathcal{E}}{\partial x}\big|_{x=x^*} = 0$ and*

$$H \triangleq \left.\frac{\partial^2 \mathcal{E}(x)}{\partial x^2}\right|_{x=x^*} = R(x^*)^\top W_2(x^*) R(x^*).$$

By the definition of the function e_{ij} and the fact that $||x_i - x_j|| = d_{ij} > 0$ at $x = x^*$, $W_2(x^*)$ is always positive definite.

Example 1 *If the edge-tension energy is given by (3), then $e'_{ij}(r) = c_{ij}$, $e''_{ij}(r) = 0$, $[W_1(x)]_{kk} = c^2_{i_k j_k}(1 - d_{i_k j_k}/||x_{i_k} - x_{j_k}||)$, and $[W_2(x)]_{kk} = c^2_{i_k j_k} d_{i_k j_k}/||x_{i_k} - x_{j_k}||^3$. Hence, when the desired distances are satisfied at $x = x^*$, the k-th diagonal elements of the weight matrices become*

$$[W_1(x^*)]_{kk} = 0, \quad [W_2(x^*)]_{kk} = \left(\frac{c_{i_k j_k}}{d_{i_k j_k}}\right)^2.$$

III. Responsiveness and Manipulability

A. Dynamics of Followers

Suppose the desired distances are satisfied at the original configuration x^*. Then, assume that the leaders' positions are moved instantaneously with a small enough increment δx_ℓ. In response to this input from the leaders, suppose the response of the followers is obtained according to the following formation control strategy:

$$\dot{x}_f = -\frac{\partial \mathcal{E}(x)}{\partial x_f}^\top = -((D_f W_1(x) D^\top) \otimes I_d)x,$$

where $D_f = [I_{N_f}|O]D$. Under this strategy, the followers try to recover their original desired distances.

Using the Taylor expansion of $\frac{\partial \mathcal{E}(x_f + \delta x_f, x_\ell + \delta x_\ell)}{\partial x_f}$ around $x = x^*$ and noting Remark 1, the first-order approximation of the followers' response is described by

$$\dot{\delta x}_f(t) = -H_{ff}\delta x_f(t) - H_{f\ell}\delta x_\ell, \tag{4}$$

where $H_{ff} \triangleq \frac{\partial^2 \mathcal{E}(x)}{\partial x_f^2}\big|_{x=x^*}$ and $H_{f\ell} \triangleq \frac{\partial^2 \mathcal{E}(x)}{\partial x_f \partial x_\ell}\big|_{x=x^*}$, which are the submatrices of the Hessian H in (1):

$$H = \left[\begin{array}{c|c} H_{ff} & H_{f\ell} \\ \hline H_{\ell f} & H_{\ell\ell} \end{array}\right].$$

Since we focus on the response around $x = x^*$, let us use the simplified notations $R_f \triangleq R_f(x^*)$, $R_\ell \triangleq R_\ell(x^*)$, and $W_2 \triangleq W_2(x^*)$. By Remark 1, we have

$$H_{ff} = R_f^\top W_2 R_f, \quad H_{f\ell} = R_f^\top W_2 R_\ell. \tag{5}$$

In what follows, we focus on the single-leader case (i.e., $N_\ell = 1$) for simplicity unless otherwise stated and see the convergence of the response of the followers. We note that the most part of the following discussion still holds in $N_\ell > 1$ cases if inputs to the network are given by the *feasible leader motion* [10], i.e., constrained motion in which followers can fully recover the desired distances.

B. Approximate Manipulability

The notion of manipulability is originally introduced in the field of robot-arm manipulators [12], [13]. Let θ be the joint angles and $r = f(\theta)$ be the state of the end-effector. Then the index of manipulability is defined by the response of the end-effector δr to the inputs on joint angles $\delta\theta$:

$$\frac{\delta r^\top Q_r \delta r}{\delta\theta^\top Q_\theta \delta\theta},$$

where $Q_r, Q_\theta \succ 0$ are the weight matrices. Although the manipulability of robot-arm manipulators can be obtained through the kinematic relation $\delta r = \frac{\partial f}{\partial \theta}\delta\theta$, this cannot be directly applied to leader-follower networks that involve dynamics of agents. Therefore, the approximate notion of manipulability was proposed in [10] by using the convergence point of the followers: $\delta x_f^* = \lim_{t\to\infty} \delta x_f(t) = J\delta x_\ell$, where $J \triangleq -R_f^\dagger R_\ell$ with $R_f^\dagger$ the Moore-Penrose pseudo inverse of R_f, which can be related to the minimum norm solution of (1) with respect to $\dot{x}_f$. Here, one can see that J is analogous to the Jacobian matrix in robot-arm manipulators. Specifically, the approximate manipulability is defined as

$$m(x, \mathsf{G}, \delta x_\ell) = \frac{||\delta x_f^*||^2}{||\delta x_\ell||^2} = \frac{\delta x_\ell^\top J^\top J \delta x_\ell}{\delta x_\ell^\top \delta x_\ell}, \tag{6}$$

where the weight matrices in the original definition in [10] are assumed to be the identity matrices. In what follows, we simply refer to this approximate manipulability index as the "manipulability".

Since the matrix J is a function x and G, the manipulability index enables us to evaluate the instantaneous response of a network in terms of its agent configurations, network topologies, and input directions (i.e., leader's movements); however, it is not entirely useful to compare rigid formations. Consider the situation where the configuration of agents is fixed and the edge set of the underlying graph can be changed, for example, by adding or deleting communication links. Then the manipulability index does not change its value as long as the formation is rigid.

Proposition 1 *Given agent configuration x and arbitrary rigid formations (x, G_1) and (x, G_2), where G_1 and G_2 share the same vertex set V but can have different edge sets, i.e., $\mathsf{G}_1 = (\mathsf{V}, \mathsf{E}_1)$ and $\mathsf{G}_2 = (\mathsf{V}, \mathsf{E}_2)$. The values of the manipulability index are identical under the same input δx_ℓ regardless of the choice of E_1 and E_2:*

$$m(x, \mathsf{G}_1, \delta x_\ell) = m(x, \mathsf{G}_2, \delta x_\ell).$$

See Appendix A for the proof.

To compare arbitrary formations including rigid formations in terms of followers' response to leader's movements, we will consider the convergence process of the followers instead of analyzing only the convergence point.

C. Responsiveness of Multi-Agent Formations

For a given formation satisfying all the desired distance constraints, to analyze the response of the followers to the input δx_ℓ to the network, namely, the leader's instantaneous displacement, we define the following index.

Definition 1 *The* responsiveness *of the followers at time $t \geq 0$ is the ratio of the norm of the followers' response to the norm of the input to the network:*

$$\nu(t, x, \mathsf{G}, \delta x_\ell) = \frac{||\delta x_f(t)||^2}{||\delta x_\ell||^2}. \tag{7}$$

We here exploit (4) to analyze the network response.

Lemma 1 *Suppose $N_\ell = 1$. Let $H_{ff} = V\Lambda V^\top$ be the eigenvalue decomposition of H_{ff}, where $V = [v_1, ..., v_r] \in \mathbb{R}^{N_f d\times r}$ is a column orthogonal matrix (i.e., $V^\top V = I_r$), and $\Lambda = \mathbf{Diag}(\lambda_1, ..., \lambda_r)$ is a diagonal matrix whose diagonal elements are non-zero eigenvalues. Let $\tilde{v}_k \triangleq (\mathbf{1}_{N_f}^\top \otimes I_d)v_k \quad (k = 1, ..., r)$. The followers' response is given by*

$$\begin{aligned}\delta x_f(t) &= V\mathbf{Diag}(1 - e^{-\lambda_k t})V^\top(\mathbf{1}_{N_f} \otimes I_d)\delta x_\ell \\ &= \sum_{k=1}^{r}(1 - e^{-\lambda_k t})(\tilde{v}_k^\top \delta x_\ell)v_k.\end{aligned} \tag{8}$$

Proof: Since $\delta x_f(0) = 0$, $\delta x_f(t)$ is given by the zero-state response

$$\delta x_f(t) = -\left(\int_0^t e^{-H_{ff}(t-s)} ds H_{f\ell}\right)\delta x_\ell.$$

By (2) and (5), we have $H_{f\ell} = -H_{ff}(\mathbf{1}_{N_f} \otimes I_d)$. Meanwhile, $e^{-V\Lambda V^\top t} - I_n = \sum_{k=1}^\infty \frac{t^k}{k!}V(-\Lambda)^k V^\top = V(e^{-\Lambda t} - I_r)V^\top$. Using these facts with $V^\top V = I_r$, we obtain

$$\begin{aligned}\delta x_f(t) &= \int_0^t e^{-V\Lambda V^\top(t-s)} ds V\Lambda V^\top(\mathbf{1}_{N_f} \otimes I_d)\delta x_\ell \\ &= V\left(\int_0^t e^{-\Lambda(t-s)} ds\right)\Lambda V^\top(\mathbf{1}_{N_f} \otimes I_d)\delta x_\ell,\end{aligned}$$

and the lemma follows. ■

Then we obtain the following useful proposition.

Proposition 2 *The responsiveness $\nu(t)$ is given by the Rayleigh quotient*

$$\begin{aligned}\nu(t, x, \mathsf{G}, \delta x_\ell) &= \frac{\delta x_\ell^\top J(t)^\top J(t)\delta x_\ell}{\delta x_\ell^\top \delta x_\ell} \\ &= \sum_{k=1}^{r}(1 - e^{-\lambda_k t})^2(\tilde{v}_k^\top \hat{\delta x_\ell})^2,\end{aligned} \tag{9}$$

where $J(t) \triangleq V\mathbf{Diag}(1 - e^{-\lambda_k t})V^\top(\mathbf{1}_{N_f} \otimes I_d)$, and $\hat{\delta x_\ell} \triangleq \delta x_\ell / ||\delta x_\ell||$ is the normalized input.

Proof: The proof follows directly by (7) and (8). ■

Since (9) has a form of the Rayleigh quotient similar to the manipulability index, the following corollary holds.

Corollary 1 *The maximum and minimum values of responsiveness over all the directions of δx_ℓ are given by*

$$\max_{\delta x_\ell} \nu(t) = \lambda_{\max}(J(t)^\top J(t)), \ \min_{\delta x_\ell} \nu(t) = \lambda_{\min}(J(t)^\top J(t)),$$

where the arguments of the maximum and minimum are given by the corresponding eigenvectors.

Proof: By the properties of the Rayleigh quotient. ■

In the remaining of this section, we remark several facts to see the connection of the responsiveness to the notions of manipulability and stiffness (rigidity indices).

1) Connection to Manipulability: Since $J(t)$ in (9) converges to J in (6), i.e., $\lim_{t\to\infty} J(t) = VV^\top(\mathbf{1}_{N_f} \otimes I_d) = R_f^\dagger R_f(\mathbf{1}_{N_f} \otimes I_d) = -R_f^\dagger R_\ell = J$, we have the following remark:

Remark 2 *The manipulability is characterized by the eigenvectors of H_{ff} as*

$$\lim_{t\to\infty} \nu(t, x, \mathsf{G}, \delta x_\ell) = m(x, \mathsf{G}, \delta x_\ell) = \sum_{k=1}^{r} (\tilde{v}_k^\top \hat{\delta x_\ell})^2. \quad (10)$$

Proposition 2 and this remark lead to the following corollary regarding the range of $\nu(t)$.

Corollary 2 *The responsiveness takes $0 \le \nu(t) < N_f$.*

Proof: $\nu(0) = 0$ follows by $\delta_f(0) = 0$. We know that, by (9), $\nu(t)$ increases monotonously in the increase of $t \in [0, \infty)$ and converges to the manipulability m from below. Hence, $\nu(t) < m \le \lambda_{\max}(J^\top J) = \lambda_{\max}((\mathbf{1}_{N_f}^\top \otimes I_d)VV^\top(\mathbf{1}_{N_f} \otimes I_d)) \le \lambda_{\max}((\mathbf{1}_{N_f}^\top \otimes I_d)(\mathbf{1}_{N_f} \otimes I_d)) = \lambda_{\max}(N_f I_d) = N_f$. ■

2) Connection to Stiffness and Rigidity Indices: The *stiffness matrix* and the *rigidity indices* are defined for rigid formations in 2-d plane to measure the robustness of a multi-agent system in maintaining a given formation [9]. Assume that the agents are perturbed by an infinitesimal displacement, δx_0, from their original configuration that satisfies given desired distances. While the original definition of the stiffness matrix is based on a spring-mass analogy, the matrix coincides with the Hessian of the edge-tension energy $\mathcal{E}$ if the notion of stiffness is applied to the formation control $\dot{x} = -\frac{\partial \mathcal{E}(x)}{\partial x}^\top$ [9].

The rigidity indices are defined based on the eigenvalues of the stiffness matrix. Therefore, in this context, it captures the rate of convergence to the desired configurations of the agents. Consider the case $N = N_f$ and $N_\ell = 0$ in (4):

$$\dot{\delta x} = -H\delta x, \quad (11)$$

where the initial configuration is perturbed as $\delta x(0) = \delta x_0$. Suppose that the eigenvalues of H are sorted in descending order as $\lambda_1 \ge \cdots \ge \lambda_r > 0$. Given a rigid formation, (x, G), with a generic configuration, x. If $d = 2$ then $r = \mathrm{rank}(H) = 2N-3$, and H has three zero eigenvalues; here, the smallest non-zero eigenvalue (i.e., the fourth-smallest eigenvalue) is $\lambda_r(H)$. To capture the rate of convergence in (11), the indices $\lambda_r(H)$ and $(\frac{1}{r}\sum_{k=1}^{r} \lambda_k(H)^{-1})^{-1}$ are proposed in [9] as the worst-case rigidity index (WRI) and the mean rigidity index (MRI), respectively.

Consequently, we can make the following remark as the relevance of the responsiveness to the rigidity indices.

Remark 3 *The rate of the convergence in (10) is dictated by the smallest non-zero eigenvalue of H_{ff}.*

From Remark 2 and 3, we see that the responsiveness is the unified notion of both the manipulability and stiffness in the following sense: In (9), the responsiveness notion combines the effective input directions determined by the eigenvectors $\{v_1, ..., v_r\}$ of H_{ff}, which characterizes the manipulability of the network, with the rate of convergence given by the eigenvalues $\{\lambda_1, ..., \lambda_r\}$ of H_{ff}, which characterizes the stiffness of the network.

IV. Optimization of Leader-Follower Network

A. Most Effective Leader Position

Provided we are given the freedom to adjust the position of the leader, how can we find the optimal leader's position in terms of maximizing the manipulability in the worst-case scenario? We will introduce the gradient-based method to solve this problem. For convenience, let us define the operator $\mathfrak{D}$ as $\mathfrak{D}A \triangleq \partial A/\partial\theta$, where A is a scalar or a matrix which depends on some variable θ.

Lemma 2 *For the single leader case ($N_\ell = 1$), the following identity holds:*

$$H_{ff}^\dagger \mathfrak{D}\left(H_{ff}^\dagger H_{f\ell}\right) = H_{ff}^\dagger H_{ff}^\dagger \left((\mathfrak{D}H_{f\ell}) - (\mathfrak{D}H_{ff})H_{ff}^\dagger H_{f\ell}\right).$$

See Appendix B for the proof.

Lemma 3 ([14]) *Suppose X is a real symmetric matrix. The i-th largest eigenvalue of X is denoted by $\lambda_i(X)$ and the corresponding normalized eigenvector by v_i. Then,*

$$\mathfrak{D}\lambda_i(X) = v_i^\top (\mathfrak{D}X)\, v_i.$$

Proposition 3 *For the single leader case ($N_\ell = 1$), if the minimum manipulability index $\lambda_{\min}(J^\top J) > 0$, then its gradient with respect to the leader's position x_ℓ is*

$$\nabla\lambda_{\min}(J^\top J) = 2\sum_{j\in\mathcal{N}(\ell)} (X_{jj} + X_{jj}^\top + Y_j + Y_j^\top)(x_j - x_\ell),$$

where

- *v is the normalized eigenvector associated with the eigenvalue $\lambda_{\min}(J^\top J)$;*
- *X_{jj} is the d-by-d block of the matrix $X \in \mathbb{R}^{N_f d\times N_f d}$,*

$$X = R_f^\dagger R_\ell v v^\top R_\ell^\top R_f^{\dagger\top} H_{ff}^\dagger;$$

- *Y_j is the d-by-d block of the matrix $Y \in \mathbb{R}^{d\times N_f d}$,*

$$Y = v v^\top R_\ell^\top R_f^{\dagger\top} H_{ff}^\dagger;$$

Proof: In the single leader case, since W_2 is positive definite (Remark 1), we always have $H_{ff}^\dagger H_{f\ell} = R_f^\dagger R_\ell$.

Therefore, without loss of generality we can assume $W_2 = I_M$. By Lemma 3,

$$\begin{aligned}\mathfrak{D}\lambda_{\min}(J^\top J) &= v^\top \left(\mathfrak{D}(J^\top J)\right) v = 2v^\top \left(J^\top \mathfrak{D} J\right) v \\ &= 2\,\mathrm{trace}\left(vv^\top J^\top \mathfrak{D} J\right) \\ &= 2\,\mathrm{trace}\left(vv^\top H_{f\ell}^\top H_{ff}^\dagger \mathfrak{D}\left(H_{ff}^\dagger H_{f\ell}\right)\right).\end{aligned}$$

Using Lemma 2 and the fact that $H_{ff}^\dagger H_{f\ell} = R_f^\dagger R_\ell$ for the single leader case, we have

$$\mathfrak{D}\lambda_{\min}(J^\top J) = 2\,\mathrm{trace}\left(Y\mathfrak{D}H_{f\ell}\right) - 2\,\mathrm{trace}\left(X\mathfrak{D}H_{ff}\right).$$

If we view X and Y as constants, it is not hard to see that

$$\begin{aligned}\mathrm{trace}\left(Y\mathfrak{D}H_{f\ell}\right) &= \sum_{j\in\mathcal{N}(\ell)} \mathrm{trace}\left(Y_j \mathfrak{D}\left(-(x_j - x_\ell)(x_j - x_\ell)^\top\right)\right) \\ &= -\sum_{j\in\mathcal{N}(\ell)} \mathfrak{D}\left((x_j - x_\ell)^\top Y_j (x_j - x_\ell)\right) \\ &= \sum_{j\in\mathcal{N}(\ell)} (x_j - x_\ell)^\top \left(Y_j + Y_j^\top\right) \mathfrak{D}(x_\ell - x_j).\end{aligned}$$

Similarly,

$$\mathrm{trace}\left(X\mathfrak{D}H_{ff}\right) = -\sum_{j\in\mathcal{N}(\ell)} (x_j - x_\ell)^\top \left(X_{jj} + X_{jj}^\top\right) \mathfrak{D}(x_\ell - x_j).$$

Now we replace $\mathfrak{D}$ with $\frac{\partial}{\partial x_\ell}$. Note that $\frac{\partial}{\partial x_\ell}(x_\ell - x_j) = I_d$. Therefore,

$$\frac{\partial \lambda_{\min}(J^\top J)}{\partial x_\ell} = 2\sum_{j\in\mathcal{N}(\ell)} (x_j - x_\ell)^\top \left(X_{jj} + X_{jj}^\top + Y_j + Y_j^\top\right),$$

which implies the desired conclusion. ■

The gradient information enables us to use the gradient descent method to search for a locally optimal leader's position which maximizes the minimum (worst-case) manipulability.

B. Most Efficient Link Resource Allocation

In what follows we consider the instance given by Example 1. We can view the weights c_{ij} of the communication links as some resource (e.g. signal power). Thus, it is interesting to see that given constant amount of resource, what is the best link resource allocation scheme that maximizes the convergence rate of the responsiveness as mentioned in Remark 3. This can be formulated as the following optimization problem,

$$\begin{aligned}&\underset{c_{ij}}{\text{maximize}} \quad \lambda_r(H_{ff}) && (12)\\ &\text{subject to} \quad \sum_{\{i,j\}\in\mathsf{E}} c_{ij}^2 \le C, \quad \text{and} \quad c_{ij} = 0 \ \text{ for } \ \{i,j\} \notin \mathsf{E},\end{aligned}$$

where $\lambda_r(H_{ff})$ denotes the smallest non-zero eigenvalue of H_{ff} ($r = \mathrm{rank}(H_{ff})$). C is a constant describing the total amount of available resource. Problem (12) is actually a convex program, and the convexity ensures that the numerical solution of the link resource allocation problem can be computed very efficiently.

Note that the link resource allocation problem cannot be well formulated for maximizing the manipulability in the single leader case. This is because the manipulability only depends on the binary topology of the graph, not the link weights. In contrast, the responsiveness makes it possible to formulate the problem as in (12) by taking the convergence rate into account.

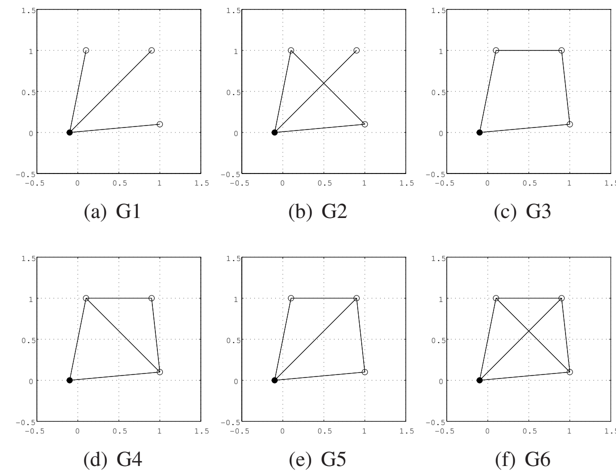


Fig. 1. Formations used in the examples. The black and white circles depict the leader and the followers, respectively, and the lines depict the communication links between agents.

V. Example

This section first demonstrates the basic characteristics of the responsiveness, and then shows some examples of the optimization problems. Throughout this section, we consider six formations, (x, G_i) $(i = 1, ..., 6)$, shown in Fig. 1, and we simply use Gi $(i = 1, ..., 6)$ to denote each formation since their agent configurations are identical. Here, G1 to G3 are non-rigid formations, and G4 to G6 are rigid formations. The given edge-tension energy has the form of (3) with $c_{ij} = d_{ij}$ $\forall \{\mathsf{v}_i, \mathsf{v}_j\} \in \mathsf{E}_k$; that is, $W_2 = I_M$ (see also Example 1).

A. Responsiveness

First, we chose the leader's displacement $\delta x_\ell = (-\epsilon, -\epsilon)^\top$ as the input to the network, where we suppose ϵ is small enough so that the first-order approximation in (4) is valid. [1] Table I shows the responsiveness at $t = 2$ and $t = 10$, and it also shows the manipulability, WRI, and MRI. Fig. 2 shows the change of the responsiveness $\nu(t)$ in time. We first observe that in each formation, the responsiveness converges to the value of manipulability (Remark 2). We also see that, while the manipulability can be used to compare non-rigid formations, it takes the same value for G4 to G6 since these formations are rigid (Proposition 1). Meanwhile, rigidity indices (WRI and MRI) successfully evaluate rigid formations in terms of the convergence rate; however, the they are not defined in non-rigid formations and cannot be used for comparing formations such as G1 to G3.

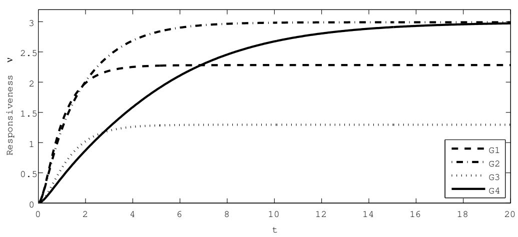
On the other hand, from Fig. 2(a) we observe that the responsiveness can compare arbitrary formations, for example, G2 (non-rigid) and G4 (rigid). An interesting observation

[1] By the definition of the responsiveness and manipulability, their values are not affected by the scale of δx_ℓ.

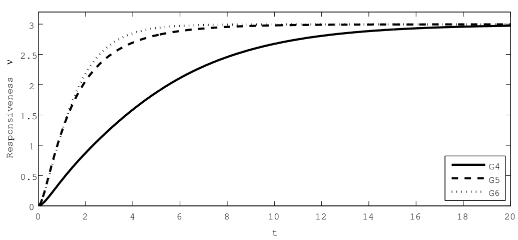
TABLE I

COMPARISON OF THE RESPONSIVENESS INDEX, THE MANIPULABILITY INDEX, AND THE RIGIDITY INDICES. WHILE THE RIGIDITY INDICES ARE NOT DEFINED FOR NON-RIGID FORMATIONS, THE SMALLEST NON-ZERO EIGENVALUE λ_r AND $(\frac{1}{r}\sum_{k=1}^{r}\lambda_k^{-1})^{-1}$ ARE SHOWN FOR WRI AND MRI, RESPECTIVELY (SHOWN IN THE PARENTHESES).

Formation	Is rigid?	r	Resp. $\nu(t=2)$	$\nu(t=10)$	Manip. m	WRI (λ_r)	MRI
G1	No	3	1.9856	2.2824	2.2825	(1.0400)	(1.3151)
G2	No	4	2.0092	2.9838	2.9912	(0.6257)	(1.2288)
G3	No	4	1.0212	1.2960	1.2961	(0.9581)	(1.2457)
G4	Yes	5	0.8717	2.6740	2.9953	0.2710	0.8025
G5	Yes	5	2.0583	2.9794	2.9953	0.4771	1.0279
G6	Yes	5	2.1791	2.9946	2.9953	0.8918	1.5220



(a) Non rigid (G1, G2, G3) and rigid (G4)



(b) Rigid formations (G4, G5, G6)

Fig. 2. Responsiveness $\nu(t)$ under leader input $\delta x_\ell = (-\epsilon, -\epsilon)^\top$. (a) Non-rigid formations G1 (dashed line), G2 (dashed-dotted), G3 (dotted), and rigid formation G4 (solid). (b) Rigid formations G4 (solid), G5 (dashed), and G6 (dotted).

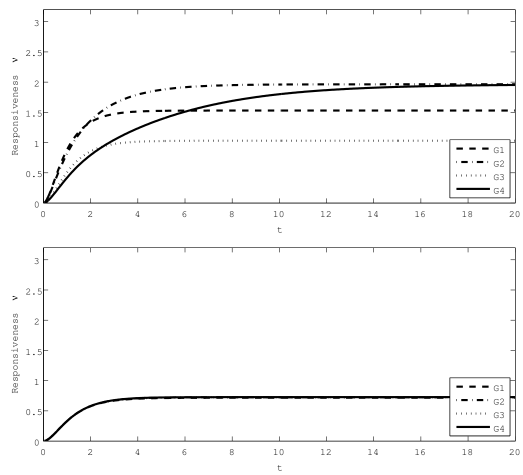


Fig. 3. Responsiveness under different input directions: $\delta x_\ell = (-\epsilon, 0)^\top$ (top) and $(-\epsilon, \epsilon)^\top$ (bottom), where formations G1 to G4 are used.

here is that a non-rigid formation (e.g., G2) can outperform a rigid formation (e.g., G4) in terms of the responsiveness if t is not so large. In addition, the responsiveness provides a framework to evaluate the effectiveness of not only network topologies and agent configurations but also input directions of the leader. In fact, from Fig. 3, which shows a comparison of the responsiveness under different input directions, we see that the responsiveness successfully captures the effective input directions in terms of the followers' response (see also Corollary 1 for the most/least effective inputs).

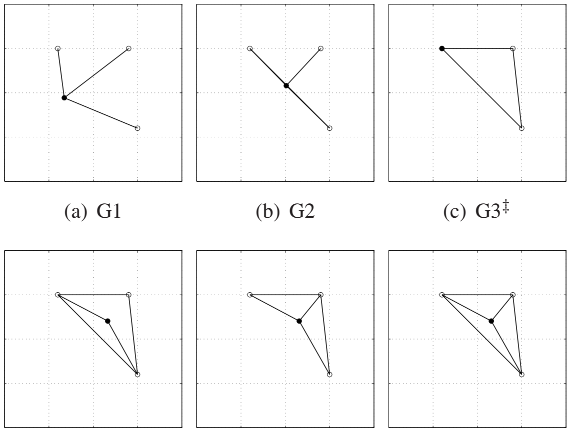
B. Leader-Follower Network Optimization

We now see some numerical examples of formation optimizations as is discussed in Section IV. Fig. 4 depicts the optimal leader's position in each instance where the topology is given as in Fig. 1 and the positions of the followers are assumed fixed. It is worth noting that a legitimate solution for optimal leader's position does not always exist. For example, in G3, the optimal leader's position tends to one of the followers, but the asymptotic result does not give a valid solution (the two agents collide). Another interesting observation is that for every rigid formation, the optimal leader's position is always the centroid of the remaining of the agents, in which case the manipulability is isotropic and equal to N_f. Due to space limit, the theoretical analysis of these results will be presented in our future work.

Fig. 5 shows the optimal link resource allocation schemes under the same set of graph topologies. In each instance, we set C to be the number of edges so that the average values of c_{ij}^2 is identity. The small number beside each link in Fig. 5 is the value of c_{ij}^2 for that link. We should notice that in the optimal scenarios, the amount of resource allocated to the links that are connected to the leader is not necessarily larger than that to the other links.

VI. CONCLUSION

In this paper, we unify the previously defined measures of instantaneous network responses in multi-agent systems, namely those of *stiffness* and *manipulability*, through the notion of *responsiveness*. The responsiveness index is a measure of how well the system responds to the movements of leaders, and therefore it enables us to evaluate the effect of different choices of network topologies and agent configurations given input directions. Numerical examples show the proposed index can be used to compare variety of topologies, and two optimization problems that find the optimal leader's position and the optimal allocation of link resource are addressed. Future work includes the adaptive

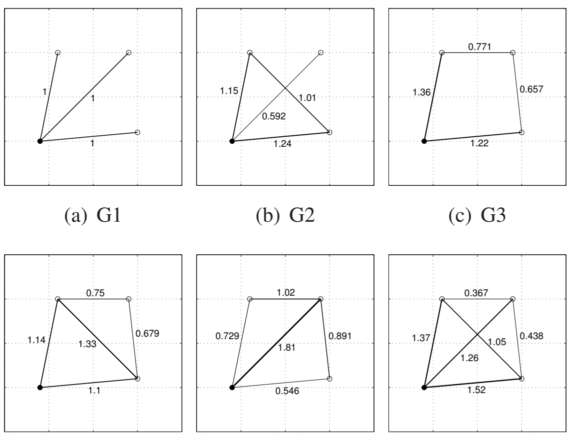


(a) G1 (b) G2 (c) G3‡

(d) G4 (e) G5 (f) G6

Fig. 4. Optimal leader position for maximizing the minimum manipulability.

‡ This is the asymptotic configuration, but not a valid solution.



(a) G1 (b) G2 (c) G3

(d) G4 (e) G5 (f) G6

Fig. 5. Optimal link resource allocation schemes for fastest responsiveness convergence.

optimization of network properties so as to improve the effect of human inputs through the leaders.

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APPENDIX

A. Proof of Proposition 1

Proof: Let $J_i(x, \mathsf{G}_i) = -{R_{if}}^\dagger R_{i\ell}$ $(i = 1, 2)$, where $R_i = [R_{if} | R_{i\ell}]$ is the rigidity matrix of the formation (x, G_i). Here $R_{if}(\mathbf{1}_{N_f} \otimes I_d) = -R_{i\ell}$ $(i = 1, 2)$ holds by (2). It is also clear that $\mathrm{null}(R_{1f}) = \mathrm{null}(R_{2f})$ under any rigid formations as the rotational freedom around the leader always remains; the projection matrices onto the row space of R_{1f} and R_{2f} are identical: $R_{1f}^\dagger R_{1f} = R_{2f}^\dagger R_{2f}$. Hence, $J_1 = -{R_{1f}}^\dagger R_{1\ell} = {R_{1f}}^\dagger R_{1f}(\mathbf{1}_{N_f} \otimes I_d) = {R_{2f}}^\dagger R_{2f}(\mathbf{1}_{N_f} \otimes I_d) = -{R_{2f}}^\dagger R_{2\ell} = J_2$, and the proposition follows. ■

B. Proof of Lemma 2

Proof: Recall that for the single leader case, the column space of $H_{f\ell}$ is contained in the column space of $H_{f\!f}$. Since $H_{f\!f}$ is real symmetric, $H_{f\!f} H_{f\!f}^\dagger = H_{f\!f}^\dagger H_{f\!f}$. Furthermore, $H_{f\!f}^\dagger H_{f\!f}$ is the projection matrix onto the column space of $H_{f\!f}$. The above facts imply the following,

$$H_{f\!f}^\dagger H_{f\!f}^\dagger H_{f\!f} = H_{f\!f}^\dagger, \quad H_{f\!f} H_{f\!f}^\dagger H_{f\ell} = H_{f\ell}.$$

According to [15] and the above identities, we have

$$\begin{aligned} H_{f\!f}^\dagger (\partial H_{f\!f}^\dagger) H_{f\ell} &= H_{f\!f}^\dagger \Big(-H_{f\!f}^\dagger (\partial H_{f\!f}) H_{f\!f}^\dagger + H_{f\!f}^\dagger H_{f\!f}^\dagger (\partial H_{f\!f}) \\ &\quad - H_{f\!f}^\dagger H_{f\!f}^\dagger (\partial H_{f\!f}) H_{f\!f} H_{f\!f}^\dagger + (\partial H_{f\!f}) H_{f\!f}^\dagger H_{f\!f}^\dagger \\ &\quad - H_{f\!f}^\dagger H_{f\!f} (\partial H_{f\!f}) H_{f\!f}^\dagger H_{f\!f}^\dagger \Big) H_{f\ell} \\ &= -H_{f\!f}^\dagger H_{f\!f}^\dagger (\partial H_{f\!f}) H_{f\!f}^\dagger H_{f\ell}. \end{aligned}$$

Finally,

$$\begin{aligned} H_{f\!f}^\dagger \partial (H_{f\!f}^\dagger H_{f\ell}) &= H_{f\!f}^\dagger (\partial H_{f\!f}^\dagger) H_{f\ell} + H_{f\!f}^\dagger H_{f\!f}^\dagger \partial H_{f\ell} \\ &= H_{f\!f}^\dagger H_{f\!f}^\dagger \left((\partial H_{f\ell}) - (\partial H_{f\!f}) H_{f\!f}^\dagger H_{f\ell} \right), \end{aligned}$$

which concludes the proof. ■