On-line Adaptive Optimal Timing Control of Switched Systems

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***Abstract*— In this paper we consider the problem of optimizing over the switching times for a multi-modal dynamic system when the complete cost-to-go is not available. The instantaneous cost is assumed to be unavailable before run-time, but can be measured in real time. Since the cost-to-go for optimization is time-varying, this problem falls under the category of on-line optimization of switched systems. The goal of the paper is to present an iterative process to update the switching times for the system in such a way that they remain optimal with respect to the changing cost-to-go function. It is shown that the information required to update the switching times is the 'rate of change' of the instantaneous cost, which is in the form of the partial derivative of the instantaneous cost function.**

I. Introduction

Consider a switched-mode hybrid dynamical system (henceforth denoted as a switched system) described by the equation

$$\dot{x} = \begin{cases} f_1(x), & t \in [0, \tau_1] \\ f_2(x), & t \in (\tau_1, \tau_2] \\ \dots \\ f_{N+1}(x), & t \in (\tau_N, T], \end{cases} \quad (1)$$

where $\{\tau_i\}_{i=1}^N$ is a monotone-nondecreasing finite sequence of switching times with $\tau_1 \geq 0$, $\tau_N \leq T$, for a given final time T. The state $x \in \mathbb{R}^n$ and the initial condition $x(0) = x_0 \in \mathbb{R}^n$ is given. The functions $f_i : \mathbb{R}^n \to \mathbb{R}^n$ represent the various modes of the system and hence the sequence $\{f_i\}$ is called the *modal sequence*. It is assumed that these functions are continuously differentiable so that there exists a unique, well-defined solution to Equation (1) with the given initial condition x_0. A particular mode may be repeated in the modal sequence, and in this case, $f_i = f_j$ for some $i \neq j$. The 'standard' switch-time optimization problem can be defined as follows. Let $L(x,t) : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be an instantaneouscost cost defined over the state trajectory, and let J be a corresponding performance functional, defined via

$$J = \int_0^T L(x,t)dt. \quad (2)$$

We view J as a function of the switching times, denoted collectively by $\bar{\tau} := (\tau_1, \dots, \tau_N)^T$ (superscript 'T' indicating transpose), and hence we label it as $J(\bar{\tau})$. The problem of minimizing $J(\bar{\tau})$ over all switching-time sequences in some given feasible set has been extensively investigated in the past few years. Following its introduction in a general nonlinear hybrid-systems setting in [4], optimality conditions, based on the maximum principle, were derived in [9], [10], [12]. For the general case of nonlinear systems, several algorithms have been proposed and analyzed in [2], [6], [8], [10], [11], [13], [14], among others.

A particular kind of problems arises when the cost function $L(x,t)$ is not known a-priori, but rather the system learns about it incrementally as time advances. This is for instance the case when the system is tracking a target with unknown dynamics. In this case, it is impractical to attempt to optimize the cost functional J as defined in (2) and instead we consider the related problem of optimizing a suitable cost-to-go performance functional. This problem requires an on-line approach that corresponds to the evolution of the cost to go. We propose an on-line algorithm that adaptively adjusts an estimated value of the optimal switching-time vector as new information about the cost function becomes available. In other words, it attempts to track the solution point of the optimal cost-to-go problem as it learns about changes in the cost function. This approach may require an off-line solution of the problem at the initial time $t = 0$, which is assumed to be computable.

A similar approach has been developed for a different kind of real-time problems, corresponding to situations where the cost function is known a priori, but the state variable $x(t)$ cannot be measured and has to be estimated via an observer [3], [5]. However, despite the similarity of the approach, the problem considered here is more general, and this is reflected in the algorithms and their analysis, which are different from those in [3], [5].

We point out that the variable parameter of the algorithm consists of the switching times between the modes, but the sequence of the modes is assumed to be fixed. The more-difficult problem of optimizing J with respect to the modal sequence has been addressed in [2] in an off-line setting, and its extension to the on-line setting will be investigated at a future time based on the results derived in this paper.

The rest of the paper is organized as follows. Section II carries out an analysis for the case of a single mode-switching, and Section III discusses an extension of the analysis to a switched system with multiple switchings. The analysis of the two cases are identical but the multi-switching case involves more complicated notation. Therefore we present a detailed analysis only for the case of a single switching. Section IV presents a numerical example of a double-tank system whose input flow rate to the first tank is controlled by a valve with discrete modes. The problem at hand is to control the switching time of the system so that the average fluid level for the second tank tracks a reference

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value which is continuously varying in real-time. Finally, Section V concludes the paper and discuss directions for future research.

II. The Single-Switch case

Consider the system defined via Equation (1) for the case where $N = 1$, namely there is a single switching time. We can simplify the notation by deleting the subscript from τ_1, and denote the switching time by τ. In this case, according to (1), $\dot{x} = f_1(x)$ for $x \leq \tau$, and $\dot{x} = f_2(x)$ for $t > \tau$. Since we consider dynamic computations of the switching time, we will denote it by $\tau(t)$ to emphasize its dependence on t. Furthermore, we start by defining $\tau(t)$ to be a minimum point of the cost-to-go (defined later), and since this may vary continuously with t, we first describe $\{\tau(t)\}$ as a continuous process; our algorithm will be based on it and of course will perform its computation at a discrete set of times. In light of this, a simple translation of Equation (1) (with $N = 1$) that emphasizes the dependence of the switching time on t may not be well defined, and we have to carefully define the state trajectory $x(t)$ and the switching time $\tau(t)$ simultaneously. To this end, it is convenient to first define the state trajectory of the dynamics associated with the cost to go.

Given $x \in \mathbb{R}^n$, $\tau \in [0, T]$, and $t \in [0, T]$, define $\tilde{x}(s, t, x, \tau)$, $s \geq t$, to be the state trajectory forward from t, as a function of $s \geq t$, with the initial condition of x at the time $s = t$; that is, $\tilde{x}(s, t, x, \tau)$ is defined via the equation

$$\frac{\partial \tilde{x}}{\partial s} = \begin{cases} f_1(\tilde{x}), & \text{if } s \leq \tau \\ f_2(\tilde{x}), & \text{if } s > \tau, \end{cases}$$

with the initial condition $\tilde{x}(t, t, x, \tau) = x$. $\tilde{x}$ should be thought of as the predicted (or simulated) state trajectory starting at time $s = t$ and evolves until $s = T$.

Given $t \in [0, T]$, let $L(x, s, t)$ be a projected cost function of $x \in \mathbb{R}^n$ and future times $s \geq t$. It has the following meaning: At time t the system "has learned" about the cost function of the state variable x and the future times s, and this function is $L(x, s, t)$. For example, if it is desired to track a moving target whose position and velocity are known at time t, $L(x, s, t)$ may indicate the square-distance of the state (x) from the goal's projected position at future times (s) based on its position and velocity at time t. Generally, the function $L(x, s, t)$ forms the basis for the cost to go, and we assume that it is continuous, continuously differentiable in x for every $t \in [0, T]$ and $s \in [t, T]$, and piecewise continuously differentiable in t and s for every $x \in \mathbb{R}^n$. Lastly, we assume that $\frac{\partial L}{\partial x}(x, s, t)$ to be differentiable in t (the reason for this assumption to be needed should be clear later in this section).

For every $\tau \in [0, T]$, $t \in [0, T]$, and $x \in \mathbb{R}^n$, we define the cost-to-go performance functional, $J(t, x, \tau)$, by

$$J(t, x, \tau) = \int_t^T L\big(\tilde{x}(s, t, x, \tau), s, t\big) ds. \tag{3}$$

We define $\tau(t)$ as a local minimum point of the cost to go from time t and the initial state $x(t)$ at that time, where $x(t)$ is the "true" trajectory of the system. As mentioned earlier, we have to formally define the processes $\{\tau(t)\}$ and $\{x(t)\}$ simultaneously, and this is done in the following way: $x(t)$ satisfies the differential equation

$$\frac{dx}{dt} = \begin{cases} f_1(x(t)), & \text{if } t \leq \tau(t) \\ f_2(x(t)), & \text{if } t > \tau(t) \end{cases}$$

with a given boundary condition $x(0) = x_0$; and as for $\tau(t)$, as long as $t < \tau(t)$ then $\tau(t)$ is defined as a function that satisfies

$$\frac{\partial J}{\partial \tau}\big(t, x(t), \tau(t)\big) = 0, \tag{4}$$

with a given boundary condition $\tau(0) = \tau_0 \in [0, T]$ such that $\frac{\partial J}{\partial \tau}\big(0, x(0), \tau_0\big) = 0$. If $t \geq \tau(t)$, we define $\dot{\tau}(t) = 0$. We point out that Equation (4) is based on the standard necessary optimality condition in nonlinear programming. The condition $\dot{\tau}(t) = 0$ when $\tau(t) \geq t$ reflects the fact that once the switching time has occurred it become a part of the past and cannot be modified, in other words, once $t \geq \tau(t)$, $\tau(\xi)$ has a constant value $\forall\ \xi \geq t$.

Throughout this paper, we assume that the processes $\{x(t)\}$ and $\{\tau(t)\}$ are well defined in the above manner and $\tau(t)$ is continuous in t. Proving such a hypothesis from verifyable assumptions is beyond the scope of the general setting of this paper, but would be done for specific classes of problems in forthcoming publications.

Consider a time-point t such that $t < \tau(t)$. Since $\frac{\partial J}{\partial \tau}(t, x(t), \tau) = 0$, taking the total derivative with respect to t, it follows that

$$\frac{d}{dt}\frac{\partial J}{\partial \tau}\big(t, x(t), \tau(t)\big) = 0$$

as well. This total derivative has the following form,

$$\frac{\partial^2 J}{\partial t \partial \tau}\big(t, x(t), \tau(t)\big) + \frac{\partial^2 J}{\partial x \partial \tau}\big(t, x(t), \tau(t)\big)\dot{x}(t) + \frac{\partial^2 J}{\partial \tau^2}\big(t, x(t), \tau(t)\big)\dot{\tau}(t) = 0,$$

and hence,

$$\dot{\tau}(t) = -\Big(\frac{\partial^2 J}{\partial \tau^2}\big(t, x(t), \tau(t)\big)\Big)^{-1} \times \Big(\frac{\partial^2 J}{\partial t \partial \tau}\big(t, x(t), \tau(t)\big) + \frac{\partial^2 J}{\partial x \partial \tau}\big(t, x(t), \tau(t)\big)\dot{x}(t)\Big). \tag{5}$$

By the assumption that the process $\{\tau(t)\}$ is well defined and continuous, the second derivative $\frac{\partial^2 J}{\partial \tau^2}\big(t, x(t), \tau(t)\big)$ is positive. We next simplify the Right-hand-side (RHS) of Equation (5).

Define the function $\phi_t(\xi)$, $\xi \geq t$, by

$$\phi_t(\xi) = \frac{\partial J}{\partial \tau}\big(\xi, x(\xi), \tau(t)\big).$$

Then

$$\frac{d\phi_t}{d\xi}(\xi) = \frac{\partial^2 J}{\partial t \partial \tau}\big(\xi, x(\xi), \tau(t)\big) + \frac{\partial^2 J}{\partial x \partial \tau}\big(\xi, x(\xi), \tau(t)\big)\dot{x}(\xi), \tag{6}$$

and at $\xi = t$,

$$\frac{d\phi_t}{d\xi}(t) = \frac{\partial^2 J}{\partial t \partial \tau}\big(t, x(t), \tau(t)\big) + \frac{\partial^2 J}{\partial x \partial \tau}\big(t, x(t), \tau(t)\big)\dot{x}(t). \tag{7}$$

Plug this in (5) to obtain,

$$\dot{\tau}(t) \ = \ -\Big(\frac{\partial^2 J}{\partial \tau^2}\big(t, x(t), \tau(t)\big)\Big)^{-1}\frac{d\phi_t}{d\xi}(t). \qquad (8)$$

Let us next derive an expression for $\frac{d\phi_t}{d\xi}(t)$.

Given $t \in [0,T]$ and $x \in \mathbb{R}^n$, denote by $\Phi_{t,x}(s,s_0)$ the state transition matrix associated with the matrix-valued function (of s) $-\frac{\partial f_2}{\partial x}\big(\tilde{x}(s,t,x,\tau)\big)^T$. Consequently

$$\frac{\partial}{\partial s}\Phi_{t,x}(s,s_0) \ = \ -\frac{\partial f_2}{\partial x}\big(\tilde{x}(s,t,x,\tau)\big)^T\Phi_{t,x}(s,s_0), \qquad (9)$$

with $\Phi_{t,x}(s_0,s_0) = I$ (the identity matrix in $\mathbb{R}^n$).

Given $t \in [0,T)$ and $\Delta t > 0$ such that $t + \Delta t \leq T$.

Lemma 1: For all $s \geq t + \Delta t$ and for all $s_0 \geq t + \Delta t$,

$$\Phi_{t+\Delta t, \tilde{x}(t+\Delta t, t, x, \tau)}(s,s_0) \ = \ \Phi_{t,x}(s,s_0). \qquad (10)$$

Proof: Fix $t \in [0,T)$ and $\Delta t > 0$ such that $t+\Delta t \leq T$. $\forall\ s \in [t+\Delta t, T]$ and $\forall\ s_0 \in [t+\Delta t, T]$, equation (9) is in force. Moreover, the same equation holds with $t+\Delta t$ and $\tilde{x}(t+\Delta t, t, x, \tau)$ in lieu of t and x, respectively, meaning that

$$\frac{d}{ds}\Phi_{t+\Delta t,\tilde{x}(t+\Delta t,t,x,\tau)}(s,s_0) \ = \ -\Big(\frac{\partial f_2}{\partial x}\big(\tilde{x}(s, t+\Delta t, \tilde{x}(t+\Delta t, t, x, \tau), \tau)\big)\Big)^T \times \Phi_{t+\Delta t,\tilde{x}(t+\Delta t,t,x,\tau)}(s,s_0). \qquad (11)$$

Now $\forall\ s \geq t + \Delta t$,

$$\frac{\partial \tilde{x}}{\partial s}(s,t,x,\tau) \ = \ \begin{cases} f_1(\tilde{x}), & \text{if } s \leq \tau \\ f_2(\tilde{x}), & \text{if } s > \tau \end{cases}$$

with the boundary condition $\tilde{x}(t,t,x,\tau) = x$, and

$$\frac{\partial \tilde{x}}{\partial s}\big(s, t+\Delta t, \tilde{x}(t+\Delta t, t, x, \tau), \tau\big) \ = \ \begin{cases} f_1(\tilde{x}), & \text{if } s \leq \tau \\ f_2(\tilde{x}), & \text{if } s \geq \tau \end{cases}$$

with the boundary condition $\tilde{x}(t+\Delta t, t+\Delta t, \tilde{x}(t+\Delta t, t, x, \tau), \tau) = \tilde{x}(t+\Delta t, t, x, \tau)$. We see that $\tilde{x}(s,t,x,\tau)$ and $\tilde{x}(s, t+\Delta t, \tilde{x}(t+\Delta t, t, x, \tau), \tau)$ satisfy the same differential equation (in s) $\forall\ s \geq t+\Delta t$, and these two functions have the same value at $s = t+\Delta t$; consequently

$$\tilde{x}(s,t,x,\tau) \ = \ \tilde{x}(s, t+\Delta t, \tilde{x}(t+\Delta t, t, x, \tau), \tau) \qquad (12)$$

for all $s \in [t+\Delta t, T]$. Therefore, by (9) and (11), $\Phi_{t+\Delta t,\tilde{x}(t+\Delta t,t,x,\tau)}(s,s_0)$ and $\Phi_{t,x}(s,s_0)$ satisfy the same differential equation (in s), with the same value at $s = s_0$ (i.e., I), and this implies (10). ■

We now can characterize the rightmost term in the RHS of (8), and hence obtain a computable expression for $\dot{\tau}(t)$.

Proposition 1: Suppose that $t < \tau(t)$. Then the derivative term $\frac{d\phi_t}{d\xi}(\xi)$ at $\xi = t$ has the following form,

$$\frac{d\phi_t}{d\xi}(t) \ = \ \int_{\tau(t)}^{T}\frac{\partial^2 L}{\partial t \partial x}\big(\tilde{x}(s,t,x(t),\tau(t)),s,t\big)\ \big(\Phi_{t,x(t)}(\tau(t),s)\big)^T ds \ \times \Big(f_1\big(\tilde{x}(\tau(t),t,x(t),\tau(t))\big) - f_2\big(\tilde{x}(\tau(t),t,x(t),\tau(t))\big)\Big). \ (13)$$

Proof: Fix $t \in [0,T)$ and $x \in \mathbb{R}^n$, and consider $J(x,t,\tau)$ as a function of $\tau > t$. Define the costate $p(s) \in \mathbb{R}^n$, $s \in [\tau, T]$, via the following differential equation,

$$\dot{p}(s) \ = \ -\Big(\frac{\partial f_2}{\partial x}\big(\tilde{x}(s,t,x,\tau)\big)\Big)^T p(s) - \Big(\frac{\partial L}{\partial x}\big(\tilde{x}(s,t,x,\tau),s,t\big)\Big)^T$$

with the boundary condition $p(T) = 0$. By Equation (9),

$$p(s) \ = \ \int_s^T \Phi_{t,x}(s,\xi)\frac{\partial L}{\partial x}\big(\tilde{x}(\xi,t,x,\tau),\xi,t\big)^T d\xi, \qquad (14)$$

and at $s = \tau$, and replacing ξ by s in (14), we obtain,

$$p(\tau) \ = \ \int_\tau^T \Phi_{t,x}(\tau,s)\frac{\partial L}{\partial x}\big(\tilde{x}(s,t,x,\tau),s,t\big)^T ds. \qquad (15)$$

Next, by Reference [6],

$$\frac{\partial J}{\partial \tau}(t,x,\tau) \ = \ p(\tau)^T\Big(f_1\big(\tilde{x}(\tau,t,x,\tau)\big) - f_2\big(\tilde{x}(\tau,t,x,\tau)\big)\Big), \qquad (16)$$

and hence, and by (15),

$$\frac{\partial J}{\partial \tau}(t,x,\tau) \ = \ \int_\tau^T \frac{\partial L}{\partial x}\big(\tilde{x}(s,t,x,\tau),s,t\big)\Big(\Phi_{t,x}(\tau,s)\Big)^T ds \times\Big(f_1\big(\tilde{x}(\tau,t,x,\tau)\big) - f_2\big(\tilde{x}(\tau,t,x,\tau)\big)\Big). \qquad (17)$$

Use (17) with two specific values: $(t, x(t), \tau(t))$, and $(t+\Delta t, x(t+\Delta t), \tau(t))$, to obtain the following two equations,

$$\frac{\partial J}{\partial \tau}(t,x(t),\tau(t)) \ = \ \int_{\tau(t)}^T \frac{\partial L}{\partial x}\big(\tilde{x}(s,t,x(t),\tau(t)),s,t\big)\Big(\Phi_{t,x(t)}(\tau(t),s)\Big)^T ds \ \times \Big(f_1\big(\tilde{x}(\tau(t),t,x(t),\tau(t))\big) - f_2\big(\tilde{x}(\tau(t),t,x(t),\tau(t))\big)\Big), \ (18)$$

and

$$\frac{\partial J}{\partial \tau}(t+\Delta t, x(t+\Delta t, \tau(t)) \ = \ \int_{\tau(t)}^T \frac{\partial L}{\partial x}\big(\tilde{x}(s, t+\Delta t, x(t+\Delta t), \tau(t)), s, t+\Delta t\big) \times\Big(\Phi_{t+\Delta t, x(t+\Delta t)}(\tau(t),s)\Big)^T ds \times \Big(f_1\big(\tilde{x}(\tau(t), t+\Delta t, x(t+\Delta t), \tau(t))\big) - f_2\big(\tilde{x}(\tau(t), t+\Delta t, x(t+\Delta t), \tau(t))\big)\Big). \qquad (19)$$

Now suppose that $t+\Delta t < \tau(t)$. Then $\forall \xi \in [t, t+\Delta t]$, $\frac{dx}{d\xi} = f_1(x(\xi))$ and $\frac{\partial \tilde{x}}{\partial \xi}\big(\xi, t, x(t), \tau(t)\big) \ = \ f_1\big(\tilde{x}(\xi, t, x(t), \tau(t))\big)$ with $\tilde{x}(t,t,x(t),\tau(t)) = x(t)$, and hence, $\forall \xi \in [t, t+\Delta t]$,

$$\tilde{x}(\xi, t, x(t), \tau(t)) \ = \ x(\xi).$$

In particular, for $\xi = t + \Delta t$,

$$\tilde{x}(t+\Delta t, t, x(t), \tau(t)) \ = \ x(t+\Delta t). \qquad (20)$$

By Lemma 1 and Equation (20), $\forall\ s \geq t+\Delta t$ and $\forall\ s_0 \geq t+\Delta t$,

$$\Phi_{t+\Delta t, x(t+\Delta t)}(s,s_0) \ = \ \Phi_{t,x(t)}(s,s_0).$$

Therefore, and by (19),

$$\frac{\partial J}{\partial \tau}(t+\Delta t, x(t+\Delta t), \tau(t)) \ = \ \int_{\tau(t)}^{T} \frac{\partial L}{\partial x}\big(\tilde{x}(x, t+\Delta t, x(t+\Delta t), \tau(t)), s, t+\Delta t\big) \times \Big(\Phi_{t,x(t)}(\tau(t),s)\Big)^T ds \ \times \Big(f_1\big(\tilde{x}(\tau(t), t+\Delta t, x(t+\Delta t), \tau(t))\big) - f_2\big(\tilde{x}(\tau(t), t+\Delta t, x(t+\Delta t), \tau(t))\big)\Big). \quad (21)$$

Moreover, $\forall s \in [t+\Delta t, \tau(t)]$, $\frac{\partial}{\partial s}\tilde{x}(s, t+\Delta t, x(t+\Delta t), \tau(t)) = f_1\big(\tilde{x}(s, t+\Delta t, x(t+\Delta t), \tau(t)\big)$ and $\frac{\partial}{\partial s}\tilde{x}(s,t,x(t),\tau(t)) = f_1\big(\tilde{x}(s,t,x(t),\tau(t))\big)$; while the initial conditions of the respective terms in these two differentiable equations are $\tilde{x}(t+\Delta t, t+\Delta t, x(t+\Delta t), \tau(t)) = x(t+\Delta t)$ by definition, and $\tilde{x}(t+\Delta t, t, x(t), \tau(t)) = x(t+\Delta t)$ by Equation (20). Consequently, $\forall s \in [t+\Delta t, \tau(t)]$,

$$\tilde{x}(s, t+\Delta t, x(t+\Delta t), \tau(t)) \ = \ \tilde{x}(s,t,x(t),\tau(t)).$$

Using this in (21) we obtain,

$$\frac{\partial J}{\partial \tau}\big(t+\Delta t, x(t+\Delta t), \tau(t)\big) \ = \ \int_{\tau(t)}^{T} \frac{\partial L}{\partial x}\big(\tilde{x}(s,t,x(t),\tau(t)), s, t+\Delta t\big) \times \Big(\Phi_{t,x(t)}(\tau(t),s)\Big)^T ds \times \Big(f_1\big(\tilde{x}(\tau(t),t,x(t),\tau(t))\big) - f_2\big(\tilde{x}(\tau(t),t,x(t),\tau(t))\big)\Big). \quad (22)$$

By (22) and (18),

$$\frac{\partial J}{\partial \tau}\big(t+\Delta t, x(t+\Delta t), \tau(t)\big) - \frac{\partial J}{\partial \tau}\big(t, x(t), \tau(t)\big) \ = \ \int_{\tau(t)}^{T} \Big[\Big(\frac{\partial L}{\partial x}\big(\tilde{x}(s,t,x(t),\tau(t)), s, t+\Delta t\big) - \frac{\partial L}{\partial x}\big(\tilde{x}(s,t,x(t),\tau(t))s, t\big)\Big)\Big(\Phi_{t,x(t)}(\tau(t),s)\Big)^T\Big] ds \times \Big(f_1\big(\tilde{x}(\tau(t),t,x(t),\tau(t)\big) - f_2\big(\tilde{x}(\tau(t),t,x(t),\tau(t))\big)\Big). \quad (23)$$

Dividing both terms of Equation (23) by Δt, taking the limit $\Delta t \to 0$, and recalling the definition of $\phi_t(\xi)$, Equation (13) follows ■

Finally, we obtain an expression for $\dot{\tau}(t)$.

Proposition 2: For every $t \in [0,T)$ such that $t < \tau(t)$, we have that

$$\dot{\tau}(t) \ = \ -\Big(\frac{\partial^2 J}{\partial \tau^2}\big(t, x(t), \tau(t)\big)\Big)^{-1} \ \times \int_{\tau(t)}^{T} \frac{\partial^2 L}{\partial t \partial x}\big(\tilde{x}(s,t,x(t),\tau(t)), s, t\big)\Big(\Phi_{t,x(t)}(\tau(t),s)\Big)^T ds \times \Big(f_1\big(\tilde{x}(\tau(t),t,x(t),\tau(t))\big) - f_2\big(\tilde{x}(\tau(t),t,x(t),\tau(t))\big)\Big). \quad (24)$$

Proof: Follows directly from Equation (8) and Proposition 1. ■

We remark that Equation (24) is a differential equation in $\tau(t)$, and the algorithm that we use amounts to its numerical solution by the forward Euler method; this will be illustrated in Section IV for an example of a nonlinear system.

III. The Case of Multiple Switching Times

Consider the case where there are N switching times, τ_i, $i = 1, \ldots, T$, denoted collectively by the vector $\bar{\tau} := (\tau_1, \ldots, \tau_N)^T \in \mathbb{R}^N$. We assume that $0 \leq \tau_1 \leq \ldots \leq \tau_N \leq T$, and we define $\tau_0 := 0$ and $\tau_{N+1} := T$. For a fixed vector $\bar{\tau}$, the state equation is given via (1), but in the event that $\bar{\tau}$ is a function of time and hence denoted by $\bar{\tau}(t)$, we define it together with the state trajectory as it was done for the single-switching case.

Given a switching vector $\bar{\tau}$, $x \in \mathbb{R}^n$, $t \in [0,T]$, and $s \in [t,T]$, define the projected state trajectory $\tilde{x}(s,t,x,\bar{\tau})$ by the following differential equation,

$$\frac{\partial \tilde{x}}{\partial s} \ = \ f_i(\tilde{x}) \quad \forall\ s \in [\tau_{i-1}, \tau_i] \cap [t,T], \quad i = 1, \ldots, N+1,$$

with the initial condition $\tilde{x}(t,t,x,\bar{\tau}) = x$. Next, given $t \in [0,T]$ and $x \in \mathbb{R}^n$, let $L(x,s,t)$ be a cost function defined for $s \geq t$, and define the cost-to-go performance functional, $J(t,x,\tau)$, by

$$J(t,x,\bar{\tau}) \ = \ \int_t^T L\big(\tilde{x}(s,t,x,\bar{\tau}), s, t\big) ds. \quad (25)$$

Finally, Define $x(t)$ and $\bar{\tau}(t) = (\tau_1(t), ..., \tau_N(t))^T$ simultaneously in the following way: $x(t)$ satisfies the differential equation

$$\frac{dx}{dt} \ = \ f_i(x(t)), \quad t \in [\tau_{i-1}(t), \tau_i(t)), \quad i = 1, 2, .., N+1, \quad (26)$$

with a given boundary condition $x(0) = x_0$; and as for $\bar{\tau}(t)$, if $t < \tau_i(t)$ then

$$\frac{\partial J}{\partial \tau_i}\big(t, x(t), \bar{\tau}(t)\big) \ = \ 0 \quad (27)$$

with a given initial condition $\bar{\tau}_i(0) = \bar{\tau}_0$, and if $t \geq \tau_i(t)$ then $\dot{\tau}_i(t) = 0$. Furthermore, we assume that $x(t)$ and $\bar{\tau}(t)$ are continuous functions of t.

This definition reflects the fact that switching times that occurred prior to time t remain fixed and cannot be changed in the future. As a matter of fact, the number of future switching times may decline as time advances, and therefore the dimension of the optimization problem, namely the number of future switching times, may go down. In the forthcoming discussion we assume that $t < \tau_1(t)$ and we consider variations in N future switching times.

The main result of this section concerns an expression for $\dot{\bar{\tau}}(t)$ that is similar to Equation (24) for the single-switching case. Its proof is almost identical to that of Proposition 2, and hence it is not presented here.

Given $t \in [0,T]$ such that $t < \tau_1(t)$, and given $x \in \mathbb{R}^n$ and $i = 1, \ldots, N$, denote by $\Phi_{i,t,x}(s,s_0)$ the state transition matrix associated with the matrix-valued function (of s) $-\frac{\partial f_i}{\partial x}\big(\tilde{x}(s,t,x,\bar{\tau})\big)^T$. Furthermore, define the vector $p_i(t) \in \mathbb{R}^n$ in a backwards-recursive manner as follows: For $i = N-1, \ldots, 1$,

$$\begin{aligned} q_i(t)^T \; = \; & \Phi_{i,t,x(t)}(\tau_i(t), \tau_{i+1}(t)) q_{i+1}(t)^T \\ & + \int_{\tau_i(t)}^{\tau_{i+1}(t)} \Phi_{i,t,x(t)}(\tau_{i+1}(t), s) \\ & \times \left(\frac{\partial^2 L}{\partial t \partial x}\big(\tilde{x}(s,t,x(t),\bar{\tau}(t)), s, t\big) \right)^T ds, \end{aligned} \tag{28}$$

with the boundary condition

$$\begin{aligned} q_N(t) = \; & \int_{\tau_N(t)}^{T} \Phi_{i,t,x(t)}(\tau_{N+1}(t), s) \\ & \times \left(\frac{\partial^2 L}{\partial t \partial x}\big(\tilde{x}(s,t,x(t),\bar{\tau}(t)), s, t\big) \right)^T ds. \end{aligned} \tag{29}$$

Next, define $p_i(t) \in R$ by

$$\begin{aligned} p_i(t) \; = \; & \bigg(q_{i+1}(t)^T \Big(\Phi_{i,t,x(t)}(\tau_i(t), \tau_{i+1}(t)) \Big)^T + \\ & \int_{\tau_i(t)}^{\tau_{i+1}(t)} \frac{\partial^2 L}{\partial t \partial x}\big(\tilde{x}(s,t,x(t),\bar{\tau}(t)), s, t\big) \\ & \times \big(\Phi_{i,t,x(t)}(\tau_{i+1}(t), s) \big)^T ds \bigg) \times \\ & \Big(f_i\big(\tilde{x}(\tau_i(t), t, x(t), \bar{\tau}(t))\big) - f_{i+1}\big(\tilde{x}(\tau_i(t), t, x(t), \bar{\tau}(t))\big) \Big). \end{aligned} \tag{30}$$

Furthermore, let $\bar{p}(t) \in \mathbb{R}^N$ be the vector whose ith coordinate is p_i. We now present the main result of this section.

Proposition 3: If $t < \tau_1(t)$, then

$$\dot{\bar{\tau}}(t) \; = \; -\Big(\frac{\partial^2 J}{\partial \tau^2}\big(t, x(t), \bar{\tau}(t)\big) \Big)^{-1} \bar{p}(t). \tag{31}$$

It should be noted that again the computation of the vector $\bar{p}(t)$ only requires integration forward once.

IV. Numerical Example

Consider the double-tank system shown in Figure 1, where it is desired to control the fluid level in the second (lower) tank, so as to have it track a given reference signal, $r(t)$, by the input valve to the first (upper) tank. The reference signal is not known in advance, but its past values at any time t are known at that time. The valve can be either open or closed, corresponding to the respective modes of input flow to the first tank at the rate of either a given $u > 0$, or 0.

We denote the fluid level of the first and second tanks by $x_1(t)$ $x_2(t)$, respectively, and we consider $x(t) := (x_1(t), x_2(t))^T$ to be the state of the system. By Torricelli's Principle, the dynamics of this switched system can be described by the following equations:

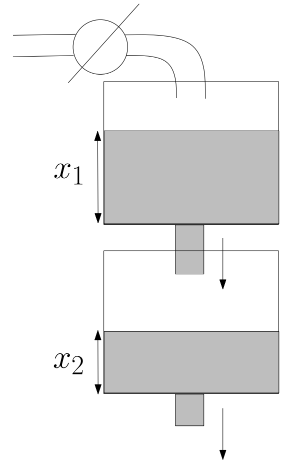

Fig. 1. The double tank process

$$\dot{x}(t) = f_1(x(t)) := \begin{bmatrix} -\alpha_1 \sqrt{x_1(t)} \\ \alpha_1 \sqrt{x_1(t)} - \alpha_2 \sqrt{x_2(t)} \end{bmatrix}, \tag{32}$$

or

$$\dot{x}(t) = f_2(x(t)) := \begin{bmatrix} -\alpha_1 \sqrt{x_1(t)} + u \\ \alpha_1 \sqrt{x_1(t)} - \alpha_2 \sqrt{x_2(t)} \end{bmatrix}, \tag{33}$$

where f_1 corresponds to the mode (mode 1) when the valve is closed, and f_2 corresponds to the mode (mode 2) when the valve is open. In this example we assume that initially the valve is open (Mode 2), and it can undergo a single mode-transition by becoming closed. Following the notation of Section II, $\tau(t)$ denotes the optimal cost-to-go at time t, which is the optimal future switching time.

Since the objective is to track the reference signal $r(t)$ by the fluid level in the second tank, we define the cost function $L(t, \tilde{x}(s,t,x(t),\tau(t)), s)$ by $L(t, \tilde{x}(s,t,x(t),\tau(t)), s) = \frac{1}{2}(\tilde{x}_2(s,t,x(t),\tau(t)) - r(t))^2$, and correspondingly, the cost-to-go functional is

$$J(t, x(t), \tau(t)) = \frac{1}{2}\int_t^T (\tilde{x}_2(s,t,x(t),\tau(t)) - r(t))^2 ds. \tag{34}$$

It can be seen that $\frac{\partial^2 L}{\partial t \partial x} = [0, -\dot{r}(t)]$, and therefore, Equation (24) has the form

$$\begin{aligned} \dot{\tau}(t) = -\Big(\frac{\partial^2 J}{\partial \tau^2}\big(t, x(t), \tau(t)\big) \Big)^{-1} \times [0, -\dot{r}(t)] \\ \times \int_{\tau(t)}^{T} \Big(\Phi_{t,x(t)}(\tau(t), s) \Big)^T ds \times \Big(f_1(\tilde{x}(\tau(t), t, x(t), \tau(t))) \\ - f_2(\tilde{x}(\tau(t), t, x(t), \tau(t))) \Big). \end{aligned} \tag{35}$$

We point out that the RHS of (35) requires the computation (or approximation) of the state transition matrix $\Phi_{t,x(t)}(\tau(t), s)$ for every $s \in [\tau(t), T]$, and this may require numerical integration.

In this example we set $\alpha_1 = \alpha_2 = 1$, $u = 2$, and T is set to 10. Furthermore, to ensure numerical accuracy, we use a relatively small step-size ($\Delta t = 0.005$) to update the

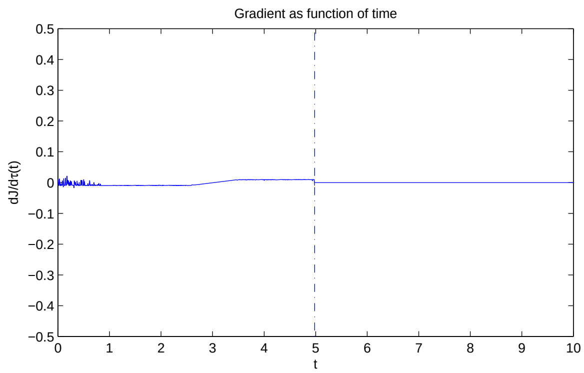

Fig. 2. Plot of the gradient $\frac{dJ}{d\tau}(t, x(t), \tau(t))$ as a function of time t. The vertical dotted line corresponds to the time when mode switch occured. After the mode switch, the gradient is set to 0.

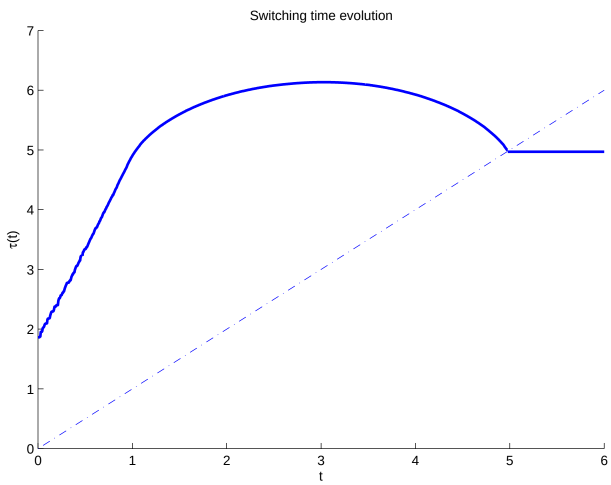

Fig. 3. Plot of the switching time evolution $\tau(t)$ as function of time. The diagonal line is the line $\tau(t) = t$ and mode switch takes place if $\tau(t)$ hits this line. The switching time does not change after this point since the mode switch has already occured.

switching time and the system. The reference signal is $r(t) = -\frac{1}{9}(t-3)^2 + 1.7$.

Using the method outlined in this paper, the resultant gradient $\frac{dJ}{d\tau}(t, x(t), \tau(t))$ for the cost-to-go (34) is shown in the Figure 2. The gradient is close to 0 for all time which is the desired result that corroborates the theoretical developments. The small deviations in the begining are due to numerical approximations.

Starting at optimal initial condition $\tau(0) = 1.86$ (corresponds to $r(0) = 0.7$), the evolution of the optimal switching time using (35) is shown in Figure 3. The diagonal line represents $\tau(t) = t$ and mode switch takes place when $\tau(t)$ hits this line. At this point, the system switchs from mode 2 to mode 1 and the system is no longer controlled.

V. Conclusion

In this paper we address the problem of adapting the optimal switching times of a switched system in an on-line environment where the instantaneous cost function is assumed to be continuously varying. We proposed an approach to obtain the trajectory of the optimal switching times with respect to the on-line cost-to-go function. We found that to maintain this optimal switching time evolution, on-line information in the form of *rate of change* of the instantaneous cost is required. Future work includes extension of this framework to the problem when the instantaneous cost function undergoes impulsive changes instead of continuous updates.

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