Pathwise Observability and Controllability are Decidable

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***Abstract*— This paper considers the problem of determining whether the observability (or controllability) matrix of a given discrete-time switched linear system reaches full rank for any mode sequence. We show that this problem is in fact decidable, and we provide finite upper bounds on the maximum sequence length at which this property may be satisfied.**

I. Introduction

In this paper, we consider discrete-time switched linear systems of the form:

$$\begin{array}{l} x_{k+1} = A(\theta_k)x_k + B(\theta_k)u_k, \ k \geq 1 \\ y_k = C(\theta_k)x_k, \end{array} \tag{1}$$

where x_k, u_k and y_k are in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively, and $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are real matrices of compatible dimensions. The mode θ_k assumes values in the set $\{1,\ldots,s\}$, so that the parameter matrices switch between s different known values. We furthermore assume that the mode sequence $\{\theta_k\}_{k=1}^{\infty}$ is arbitrary and independent of the initial state x_1 and input sequence $\{u_k\}_{k=1}^{\infty}$. The relevance of switched linear systems is now well established as they arise naturally, e.g., in multi-modal control systems and sensor and actuator failure models, and constitute an important class of hybrid systems.

Controllability and observability questions for the system in (1) have been extensively studied in the stochastic framework of Markov jump linear systems [5], [8], [11], [14], in the deterministic case [6], [7], [13], and in the hybrid systems literature [2], [3], [12]. In this paper, we examine a property of (1) concerning both observability and controllability, that can be motivated as follows: It is clear that given y_k, u_k and θ_k, $k = 1,\ldots,N$, it is possible to recover x_1 uniquely if and only if the following observability matrix has full rank n:

$$\begin{pmatrix} C(\theta_1) \\ C(\theta_2)A(\theta_1) \\ \vdots \\ C(\theta_N)A(\theta_{N-1})\cdots A(\theta_1) \end{pmatrix}. \tag{2}$$

If we further want to be able to recover x_1 for any sequence $\theta_1,\ldots,\theta_N$, then all such matrices must be of full rank. We define the property wherein such an N exists as *pathwise observability*, and we note that it has appeared in the literature for quite some time, starting with [8], where it was linked to a concept of stochastic observability, and where *pathwise controllability* was also considered and linked to the existence of steady-state solutions to the Markov jump linear quadratic problem. More recently, it was shown in [4], [10] that pathwise observability implied the existence of artificial stochastic parameters such that the corresponding Kalman filter results in an asymptotic observer for (1), and, in [1], an asymptotic observer was proposed for a special subclass of (1), whose convergence was established under similar assumptions. However, what has been missing is a way to check for pathwise observability. The direct way is to check the rank of all matrices (2) for increasing N until they all reach full rank, or until it is provably impossible for pathwise observability to hold. However, while it is well known [9], thanks to the Cayley-Hamilton Theorem, that this algorithm terminates at $N = n$ for standard (unimodal) linear systems, it has been unknown whether or not it terminates for switched linear systems. In this paper, we affirmatively answer this question, and we provide finite upper bounds on the maximum *index* of pathwise observability. Notice that these results imply that pathwise observability is decidable, and, as it turns out, by duality, that pathwise controllability is decidable as well.

This paper is organized as follows: We start off, in Section II, by establishing some definitions and by stating the main results. We devote Section III to the proofs. Finally, we study the dual problem of pathwise controllability in Section IV.

II. Definitions and Results

Without loss of generality, we restrict our analysis to autonomous switched linear systems of the form:

$$\begin{array}{l} x_{k+1} = A(\theta_k)x_k, \\ y_k = C(\theta_k)x_k, \end{array}$$

which we characterize by the set of pairs $\{(A(1),C(1)),\ldots,(A(s),C(s))\}$. We define a path θ of length N as a string $\theta_1\theta_2\cdots\theta_N$ over the set $\{1,\ldots,s\}$. We let the length of such a string be denoted by $|\theta| = N$. We also define the observability matrix of a path θ of length N as:

$$\mathcal{O}(\theta) \triangleq \begin{pmatrix} C(\theta_1) \\ C(\theta_2)A(\theta_1) \\ \vdots \\ C(\theta_N)A(\theta_{N-1})\cdots A(\theta_1) \end{pmatrix}.$$

We furthermore say that a path θ has rank r if and only if its observability matrix $\mathcal{O}(\theta)$ has rank r. Similarly, a path is observable if and only if its observability matrix has full rank n. Finally, we have the following definition:

Definition 1 (Pathwise Observability): The set of pairs $\{(A(1), C(1)), \ldots, (A(s), C(s))\}$ is pathwise observable if and only if there exists an integer N such that all paths of length N are observable. We refer to the smallest such integer as the index of pathwise observability. $\Diamond$

If a set of pairs is not pathwise observable (i.e. for all N, there exists an unobservable path of length N), it is said to be pathwise unobservable. We moreover need to define the pathwise r-rank property as follows:

Definition 2 (Pathwise r-rank): The set of pairs $\{(A(1), C(1)), \ldots, (A(s), C(s))\}$ is pathwise r-ranked if and only if there exists an integer N such that the rank of every path of length N is at least r. We refer to the smallest such integer as the index of pathwise r-rank. $\Diamond$

Note that the pathwise n-rank property is equivalent to pathwise observability. In this paper, we show that the pathwise r-rank property is decidable for all $r \leq n$, which includes pathwise observability. Now, before stating the main result of this paper, we need to define the following quantities:

$$N(s, n, 1) \triangleq 1$$
$$N(s, n, r) \triangleq G(r, N(s, n, r-1), s^{N(s,n,r-1)}, r), \; r \leq n,$$

where $G(r, g, p, k)$ is computed recursively as follows, by a double induction on k and p:

$$G(r, g, 1, r) \triangleq l + 1$$
$$G(r, g, p, k+1) \triangleq G(k+1, G(r, g, p, k), s^{G(r,g,p,k)}, k+1), \quad k = r, \ldots, n-1,$$
$$G(r, g, p+1, r) \triangleq 1 + \max_{k=r,r+1,\ldots,n} \{G(r, g, p, k)\}, \quad p = 1, \ldots, s^g - 1.$$

We also define:

$$N_c(s, n, 1) \triangleq 1,$$
$$N_c(s, n, r) \triangleq N_c(s, n, r-1) + s^{N_c(s,n,r-1)}, \; 2 \leq r \leq n.$$

We can now state the following theorem:

Theorem 1: Assume given a set of s pairs $\{(A(1), C(1)), \ldots, (A(s), C(s))\}$, where the dimension of the $A(\cdot)$ matrices is n.

1) If the set is pathwise r-ranked, then its index of pathwise r-rank is smaller than or equal to $N(s, n, r)$.
2) If furthermore the $A(\cdot)$ matrices are pairwise commuting, then the index of pathwise r-rank is bounded by the smaller number $N_c(s, n, r)$. $\Diamond$

The proof of this is given in Section III, and a corollary to Theorem 1 reads as follows:

Corollary 1: The pathwise r-rank property and pathwise observability are decidable. $\Diamond$

TABLE I

$N(s, n, n)$ FOR THE FIRST VALUES OF n AND s

n \ s	1	2	3
1	1	1	1
2	2	3	4
3	3	135	?

TABLE II

$N_c(s, n, n)$ FOR THE FIRST VALUES OF n AND s

n \ s	1	2	3	4
1	1	1	1	1
2	2	3	4	5
3	3	11	85	1029
4	4	2059	$\simeq 3.610^{40}$	$\simeq 3.310^{619}$

Proof: It clearly suffices to compute the rank of $\mathcal{O}(\theta)$ for every path θ of length $N(s, n, r)$ (resp. $N(s, n, n)$). A set of pairs is then pathwise r-ranked (resp. observable) if and only if the rank of $\mathcal{O}(\theta)$ for every such θ is at least r (resp. equals n). $\square$

We now define $\mathcal{N}(s, n, r)$ as the maximum index of pathwise r-rank over all pathwise r-ranked sets of pairs $\{(A(1), C(1)), \ldots, (A(s), C(s))\}$ with $n \times n$ $A(\cdot)$ matrices. Similarly, let $\mathcal{N}_c(s, n, r)$ be the maximum index of pathwise r-rank over all pathwise r-ranked sets of pairs $\{(A(1), C(1)), \ldots, (A(s), C(s))\}$ with pairwise commuting $n \times n$ $A(\cdot)$ matrices. By Theorem 1, the following hold:

$$\mathcal{N}(s, n, r) \leq N(s, n, r)$$
$$\mathcal{N}_c(s, n, r) \leq N_c(s, n, r).$$

Note that, so far, we have not taken into account p, i.e. the size of the measurements. It should be understood that the numbers $\mathcal{N}_c$ and $\mathcal{N}$ are maxima for all values of p. Likewise, N_c and N are upper bounds for all p. However, throughout the remainder of the paper, we will assume $p = 1$ for the sake of clarity, but the proofs can easily be modified to account for larger values of p.

Tables I and II contain the upper bounds $N(s, n, n)$ and $N_c(s, n, n)$, respectively, for the first few relevant values of n and s. Note that they grow extremely rapidly (in fact, $N(s, n, n)$ grows so fast that even $N(3, 3, 3)$ is unavailable), which does not really make pathwise observability easy to check. Imagine computing the rank of $s^{N(s,n,n)}$ different $N(s, n, n) \times n$ matrices, even for $s = 2$ and $n = 3$. For now, we simply point out that our upper bounds may be too conservative, and that we leave the task of reducing them to a future endeavor. In the mean time, note that finding the exact values of $\mathcal{N}(s, n, n)$ and $\mathcal{N}_c(s, n, n)$ is an even more difficult problem, to which the only solution we now have is to match the upper bounds to the actual index of pathwise observability of a particular set of pairs. It is indeed easy to see that $\mathcal{N}(1, n, n) = \mathcal{N}_c(1, n, n) = n$ and that $\mathcal{N}(s, 1, 1) = \mathcal{N}_c(s, 1, 1) = 1$. That $\mathcal{N}(s, 2, 2) = \mathcal{N}_c(s, 2, 2) = s+1$ follows from the fact that the set of pairs $\{(A, C(1)), \ldots, (A, C(s))\}$, where:

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \ \lambda \neq 1, 0 \\ C(i) &= (1\ \lambda^{-i}),\ i = 1, \ldots, s, \end{aligned}$$

is pathwise observable with index $s+1$. Since $N(s,2,2) = N_c(s,2,2) = s+1$, it follows that $\mathcal{N}(s,2,2) = \mathcal{N}_c(s,2,2) = s+1$.

III. Proof of Theorem 1

In this section, we prove Theorem 1. We begin by showing the result for commuting A's (part 2), because its proof is easier and makes use of the main observations in a much more direct way. But before that, we need to establish some preliminary notation.

A. Preliminaries

In order to make the development more straightforward, we begin with a few definitions. θ^1 and θ^2 being paths of length N_1 and N_2 respectively, $\theta^1\theta^2$ denotes their concatenation $\theta^1_1\theta^1_2\cdots\theta^1_{N_1}\theta^2_1\theta^2_2\cdots\theta^2_{N_2}$. Furthermore, $\theta^{(q)}$ is the path θ concatenated with itself $q-1$ times. Given a path θ, $\theta_{[i,j]}$ is its substring (or infix) $\theta_i\theta_{i+1}\cdots\theta_j$. By convention, we let $\theta_{[i,i-1]} = \epsilon$, the null string, for all $1 \leq i \leq |\theta|$. We also define the transition matrix $\Phi(\theta)$ of a path θ of length N as $\Phi(\theta) = A(\theta_N)\cdots A(\theta_1)$, and note that $\Phi(\theta^1\theta^2) = \Phi(\theta^2)\Phi(\theta^1)$ for any pair of paths θ^1 and θ^2. Again, by convention, we let $\Phi(\epsilon) = I$, the $n \times n$ identity matrix. Finally, let $\mathcal{O}(\theta)_{[i,j]}$ denote the submatrix of $\mathcal{O}(\theta)$ constituted by rows i through j of $\mathcal{O}(\theta)$:

$$\mathcal{O}(\theta)_{[i,j]} \triangleq \begin{pmatrix} C(\theta_i)A(\theta_{i-1})\cdots A(\theta_1) \\ \vdots \\ C(\theta_j)A(\theta_{j-1})\cdots A(\theta_1) \end{pmatrix},$$

and note that

$$\mathcal{O}(\theta)_{[i,j]} = \mathcal{O}(\theta_{[i,j]})\Phi(\theta_{[1,i-1]}), \tag{3}$$

for all i and j such that $1 \leq i \leq j \leq |\theta|$. For the sake of clarity, we will favor the notation of the right hand side of (3). For example, if $\theta = \theta^0\theta^1$, where $|\theta^0| = N_0$ and $|\theta^1| = N_1$, then

$$\mathcal{O}(\theta)_{[N_0+1, N_0+N_1]} = \mathcal{O}(\theta^1)\Phi(\theta^0). \tag{4}$$

Note that the right hand side of (4) is easier to read and makes much more explicit the fact that we are looking at the observability matrix of θ^1 "shifted forward by θ^0." Finally, we let range(M) denote the row range space of a matrix M.

B. Proof of Theorem 1, Part 2 (Pairwise Commuting A's)

The fact that $\mathcal{N}_c(1,n,n) = n$ is usually attributed to the Cayley-Hamilton Theorem [9]. However, trying to extend this approach to the switched case has led us nowhere. We therefore need to take another approach, and the following elementary observation actually provides an alternate way to show that $\mathcal{N}_c(1,n,n) = n$. Assume that the rank of the observability matrix does not increase at the kth measurement, i.e. that

$$CA^{k-1} = \sum_{i=1}^{k-1} \alpha_i CA^{i-1}$$

for some $k-1$ real scalars $\{\alpha_i\}_{i=1}^{k-1}$. Right multiplying this equation on both sides by any power of A (i.e. $A^{k'}$), we get

$$CA^{k-1+k'} = \sum_{i=1}^{k-1} \alpha_i CA^{i-1+k'},$$

which implies that the rank has stopped growing for good. $\mathcal{N}(1,n,n) = n$ then follows from the fact that the rank can grow at most n times. It turns out that this argument, along with the Pigeon-Hole Principle alone, is sufficient for establishing the finiteness of $\mathcal{N}_c(s,n,n)$. It should be clear by now that we will prove that pathwise observability is decidable by showing how to construct unobservable paths of arbitrary lengths whenever a system is not pathwise r-ranked at $N_c(s,n,r)$. Our observation translates into the following lemma in the switched case:

Lemma 1: Let θ^0, θ^1, and θ^2 be paths of lengths $N_0 \geq 0$, $N_1 > 0$ and $N_2 > 0$ respectively. Assume that:

$$\text{range}(\mathcal{O}(\theta^2)\Phi(\theta^0\theta^1)) \subset \text{range}(\mathcal{O}(\theta^1)\Phi(\theta^0)). \tag{5}$$

We then have

$$\text{range}(\mathcal{O}(\theta^2)\Phi(\theta^0\theta^3\theta^1)) \subset \text{range}(\mathcal{O}(\theta^1)\Phi(\theta^0\theta^3)) \tag{6}$$

for any path θ^3 of length $N_3 \geq 0$. ♢

Note that the range inclusion (5) holds between two submatrices of $\mathcal{O}(\theta)$, where $\theta = \theta^0\theta^1\theta^2$, and that (6) concerns $\theta' = \theta^0\theta^3\theta^1\theta^2$. In both cases, the submatrices involved are supported by θ^1 and θ^2, but the difference lies in the fact that $\theta^1\theta^2$ is shifted in θ' by a path θ^3. In other words, what Lemma 1 really tells us is that range inclusions within paths are conserved when the paths involved are both equally shifted. The proof is as follows:

Proof: Equation (5) implies the existence, for all $1 \leq k \leq N_2$, of N_1 scalars $\{\alpha_i\}_{i=1}^{N_1}$ such that:

$$C(\theta^2_k)\Phi(\theta^2_{[1,k-1]})\Phi(\theta^0\theta^1) = \sum_{i=1}^{N_1} \alpha_i C(\theta^1_i)\Phi(\theta^1_{[1,i-1]})\Phi(\theta^0).$$

Now, by commutativity of the $A(\cdot)$'s and therefore of the $\Phi(\cdot)$'s, and by recalling that $\Phi(\lambda^1\lambda^2) = \Phi(\lambda^2)\Phi(\lambda^1)$ for any pair of paths $\{\lambda^1, \lambda^2\}$, we get for all k, $1 \le k \le N_2$,

$$\begin{aligned} C(\theta^2_k)\Phi(\theta^2_{[1,k-1]})\Phi(\theta^0\theta^3\theta^1) &= C(\theta^2_k)\Phi(\theta^2_{[1,k-1]})\Phi(\theta^0\theta^1)\Phi(\theta^3) \\ &= \sum_{i=1}^{N_1} \alpha_i C(\theta^1_i)\Phi(\theta^1_{[1,i-1]})\Phi(\theta^0)\Phi(\theta^3) \\ &= \sum_{i=1}^{N_1} \alpha_i C(\theta^1_i)\Phi(\theta^1_{[1,i-1]})\Phi(\theta^0\theta^3), \end{aligned}$$

hence (6). □

The following lemma shows how to construct paths of bounded rank of arbitrary length:

Lemma 2: Let λ^0, λ^1 and λ^2 be paths of lengths $N_0 \ge 0$, $N_1 > 0$ and $N_2 > 0$ respectively. Assume that there exists a path λ^3 such that $\lambda^1\lambda^2 = \lambda^3\lambda^1$, and assume that

$$r = \text{rank}(\mathcal{O}(\lambda^1)\Phi(\lambda^0)) = \text{rank}(\mathcal{O}(\lambda^1\lambda^2)\Phi(\lambda^0)). \quad (7)$$

Then for any integer m, letting $\lambda' = \lambda^1\lambda^{2^{(m)}}$, we get

$$\text{range}(\mathcal{O}(\lambda')\Phi(\lambda^0)) \subset \text{range}(\mathcal{O}(\lambda^1)\Phi(\lambda^0)), \quad (8)$$

which implies that $\text{rank}(\mathcal{O}(\lambda')\Phi(\lambda^0)) = r$. ♢

Proof: First, note that for any integer k,

$$\lambda^1\lambda^{2^{(k)}} = \lambda^{3^{(k)}}\lambda^1,$$

which can be shown by induction. We next realize that by (7), we have $\text{range}(\mathcal{O}(\lambda^2)\Phi(\lambda^0\lambda^1)) \subset \text{range}(\mathcal{O}(\lambda^1)\Phi(\lambda^0))$. Therefore, by Lemma 1, we have that

$$\begin{aligned} &\text{range}(\mathcal{O}(\lambda^2)\Phi(\lambda^0\lambda^1\lambda^{2^{(k)}})) \\ &\quad = \text{range}(\mathcal{O}(\lambda^2)\Phi(\lambda^0\lambda^{3^{(k)}}\lambda^1)) \\ &\quad \subset \text{range}(\mathcal{O}(\lambda^1)\Phi(\lambda^0\lambda^{3^{(k)}})) \end{aligned} \quad (9)$$

for $0 \le k < m$ (simply let $\theta^0 = \lambda^1,\ \theta^1 = \lambda^1,\ \theta^2 = \lambda^2$ and $\theta^3 = \lambda^{3^{(k)}}$). Now, since $\mathcal{O}(\lambda^1)\Phi(\lambda^0\lambda^{3^{(k)}})$ is a submatrix of $\mathcal{O}(\lambda^1\lambda^{2^{(k)}})\Phi(\lambda^0)$, (9) yields

$$\begin{aligned} &\text{range}(\mathcal{O}(\lambda^2)\Phi(\lambda^0\lambda^1\lambda^{2^{(k)}})) \subset \\ &\qquad \text{range}(\mathcal{O}(\lambda^1\lambda^{2^{(k)}})\Phi(\lambda^0)), \end{aligned} \quad (10)$$

which, since $\mathcal{O}(\lambda^1\lambda^{2^{(k+1)}})\Phi(\lambda^0)$ contains both arguments of the *range* function in (10) as submatrices, implies that

$$\begin{aligned} &\text{range}(\mathcal{O}(\lambda^1\lambda^{2^{(k+1)}})\Phi(\lambda^0)) \subset \\ &\qquad \text{range}(\mathcal{O}(\lambda^1\lambda^{2^{(k)}})\Phi(\lambda^0)), \end{aligned} \quad (11)$$

for $0 \le k < m$. Finally, (11) yields (8) by induction on k, $0 \le k < m$, and by transitivity of the range inclusion partial ordering. □

We now establish the main lemma of this section:

Lemma 3: Let θ^0 and θ^1 be two paths of lengths $N_0 \ge 0$ and $N_c(s,n,r)$ respectively. If

$$\text{rank}(\mathcal{O}(\theta^1)\Phi(\theta^0)) < r,$$

then there exist paths θ^2 of arbitrary lengths N_2 such that

$$\text{range}(\mathcal{O}(\theta^2)\Phi(\theta^0)) \subset \text{range}(\mathcal{O}(\theta^1)\Phi(\theta^0)),$$

resulting in $\text{rank}(\mathcal{O}(\theta^2)\Phi(\theta^0)) < r$. ♢

Proof: The proof is by induction on r, for $r \le n$.

Assume that $\theta^1 = t$, $t \in \{1,\dots,s\}$, and that $C(t)\Phi(\theta^0) = 0$. Let $\theta^2 = t^{(N_2)}$. Then $\mathcal{O}(\theta^2)\Phi(\theta^0) = 0$, because $C(t)\Phi(\theta^0\theta^2_{[1,k]}) = C(t)\Phi(\theta^2_{[1,k]})\Phi(\theta^0) = C(t)\Phi(\theta^0)\Phi(\theta^2_{[1,k]}) = 0$ for all $k \le N_2$.

Now assume that Lemma 3 is true at $r-1$. We then have two cases:

First, assume there exists $i \in \{0,\dots,s^{N_c(s,n,r-1)}\}$ such that $\text{rank}(\mathcal{O}(\theta^1)_{[i+1,i+N_c(s,n,r-1)]}\Phi(\theta^0)) < r-1$. Defining $\lambda^0 = \theta^0\theta^1_{[1,i]}$ and $\lambda^1 = \theta^1_{[i+1,i+N_c(s,n,r-1)]}$, Lemma 3 at $r-1$ gives a path λ^2 of arbitrary length such that

$$\text{range}(\mathcal{O}(\lambda^2)\Phi(\lambda^0)) \subset \text{range}(\mathcal{O}(\lambda^1)\Phi(\lambda^0)).$$

Appending the matrix $\mathcal{O}(\theta^1_{[1,i]})\Phi(\theta^0)$ on top of both $\mathcal{O}(\lambda^2)\Phi(\lambda^0)$ and $\mathcal{O}(\lambda^1)\Phi(\lambda^0)$, and noting that since $\theta^1_{[1,i]}\lambda^1 = \theta^1_{[1,i+N_c(s,n,r-1)]}$, $\mathcal{O}(\theta^1_{[1,i]}\lambda^1)\Phi(\theta^0)$ is a submatrix of $\mathcal{O}(\theta^1)\Phi(\theta^0)$, we finally get

$$\text{range}(\mathcal{O}(\theta^1_{[1,i]}\lambda^2)\Phi(\theta^0)) \subset \text{range}(\mathcal{O}(\theta^1)\Phi(\theta^0)),$$

which concludes this case, since λ^2 is of arbitrary length.

Second, assume for all $i \in \{0,\dots,s^{N_c(s,n,r-1)}\}$, $\text{rank}(\mathcal{O}(\theta^1)_{[i+1,i+N_c(s,n,r-1)]}\Phi(\theta^0)) = r-1$, which implies that $\text{rank}(\mathcal{O}(\theta^1)\Phi(\theta^0)) = r-1$. Furthermore, since there are $s^{N_c(s,n,r-1)}$ different paths of length $N_c(s,n,r-1)$, and since the cardinality of $\{0,\dots,s^{N_c(s,n,r-1)}\}$ is $s^{N_c(s,n,r-1)}+1$, there exist, by virtue of the Pigeon Hole Principle, $i,j \in \{0,\dots,s^{N_c(s,n,r-1)}\}$, $i<j$, such that:

$$\theta^1_{[i+1,i+N_c(s,n,r-1)]} = \theta^1_{[j+1,j+N_c(s,n,r-1)]}.$$

Letting $\lambda^0 = \theta^0\theta^1_{[1,i]}$, $\lambda^1 = \theta^1_{[i+1,i+N_c(s,n,r-1)]}$, $\lambda^2 = \theta^1_{[i+N_c(s,n,r-1)+1,j+N_c(s,n,r-1)]}$, and $\lambda^3 = \theta^1_{[i+1,j]}$, we have $\lambda^1\lambda^2 = \lambda^3\lambda^1$. Moreover, (7) holds since, by assumption, the range of $\mathcal{O}(\theta^1)_{[i+1,i+N_c(s,n,r-1)]}\Phi(\theta^0)$ spans that of $\mathcal{O}(\theta^1)\Phi(\theta^0)$. Lemma 2 thus gives us a path λ' of arbitrary length such that

$$\text{range}(\mathcal{O}(\lambda')\Phi(\lambda^0)) \subset \text{range}(\mathcal{O}(\lambda^1)\Phi(\lambda^0)).$$

By the same argument as in Case 1, we have

$$\text{range}(\mathcal{O}(\theta^1_{[1,i]}\lambda')\Phi(\theta^0)) \subset \text{range}(\mathcal{O}(\theta^1)\Phi(\theta^0)),$$

which completes the proof. □

We can now prove part 2 of Theorem 1:

Proof: Assume that there exists a path θ^1 of length $N_c(s,n,r)$, but that $\text{rank}(\mathcal{O}(\theta^1)) < r$. Assuming $\theta^0 = \epsilon$, Lemma 3 directly implies the existence of paths of arbitrary length of rank strictly smaller than r. □

C. Proof of Theorem 1, Part 1 (The General Case)

We now address the general case, where we assume nothing about the $A(\cdot)$ matrices. The loss of commutativity destroys the previous results, even though it can be shown that the same upper bounds hold when the A matrices are all invertible, but not necessarily pairwise commuting. Nevertheless, Lemmas 1 and 2 easily carry over to the general case slightly modified, yielding the following two weaker lemmas whose proofs we omit to conserve space.

Lemma 4: Let θ^1 and θ^2 be paths of lengths $N_1 > 0$ and $N_2 > 0$ respectively. Assume that:

$$\text{range}(\mathcal{O}(\theta^2)\Phi(\theta^1)) \subset \text{range}(\mathcal{O}(\theta^1)).$$

We then have

$$\text{range}(\mathcal{O}(\theta^2)\Phi(\theta^3\theta^1)) \subset \text{range}(\mathcal{O}(\theta^1)\Phi(\theta^3))$$

for any path θ^3 of length $N_3 \geq 0$. $\Diamond$

Lemma 5: Let λ^1 and λ^2 be paths of lengths $N_1 > 0$ and $N_2 > 0$ respectively, assume that there exists a path λ^3 such that $\lambda^1\lambda^2 = \lambda^3\lambda^1$, and assume that

$$r = \text{rank}(\mathcal{O}(\lambda^1)) = \text{rank}(\mathcal{O}(\lambda^1\lambda^2)).$$

Then for any integer m, letting $\lambda' = \lambda^1\lambda^{2^{(m)}}$, we get

$$\text{range}(\mathcal{O}(\lambda')) \subset \text{range}(\mathcal{O}(\lambda^1)),$$

which implies that $\text{rank}(\mathcal{O}(\lambda')) = r$. $\Diamond$

We now unfortunately need a few more definitions. We say that a path θ is generated by the set $\{\lambda_1, \ldots, \lambda_p\}$ of paths of length l if for any $k \in \{0, \ldots, N-l\}$, where $N = |\theta|$, there exists some $i \in \{1, \ldots, p\}$ such that $\theta_{[N-k-l+1, N-k]} = \lambda_i$. The language generated by the set $\{\lambda_1, \ldots, \lambda_p\}$ is the language containing all paths generated by $\{\lambda_1, \ldots, \lambda_p\}$. We now define:

Definition 3 (Conditional Pathwise r-rank): The set of pairs $\{(A(1), C(1)), \ldots, (A(s), C(s))\}$ is L-conditionally pathwise r-ranked if and only if there exists an integer N such that all paths of length N in the language L are of rank r. The smallest such integer N_r is called the index of L-conditional pathwise r-rank. $\Diamond$

We now fix s and n, and define:

Definition 4: $\mathcal{G}(r, g, p, k)$ is the maximum index of L-conditional pathwise k-rank, over all languages L generated by p paths of length g, and such that g is larger than or equal to the index of L-conditional $r-1$-rank of L. $\Diamond$

We are now in measure to show the following lemma:

Lemma 6: $\mathcal{G}(r, g, p, k) \leq G(r, g, p, k)$ for $k = r, \ldots, n$, for $p = 1, \ldots, s^g$, and for all values of r and g required to compute $\mathcal{N}(s, n, r)$. $\Diamond$

Proof: The proof is by a double induction over k and p, and it suffices to show the following:

(i) $\mathcal{G}(r, g, 1, r) \leq G(r, g, 1, r)$.

(ii) If

$$\mathcal{G}(r, g, p, k) \leq G(r, g, p, k), \quad \text{and} \tag{12}$$

$$\mathcal{G}(k+1, G(r, g, p, k), s^{G(r,g,p,k)}, k+1) \leq G(k+1, G(r, g, p, k), s^{G(r,g,p,k)}, k+1), \tag{13}$$

$$\text{then} \quad \mathcal{G}(r, g, p, k+1) \leq G(r, g, p, k+1). \tag{14}$$

(iii) If $\mathcal{G}(r, g, p, k) \leq G(r, g, p, k)$, $k = r, \ldots, n$, then

$$\mathcal{G}(r, g, p+1, k) \leq 1 + \max_{k=r,r+1,\ldots,n} \{G(r, g, p, k)\}.$$

(i) Let λ be a path of length g such that $\text{rank}(\mathcal{O}(\lambda)) = r-1$. First, λ can generate a path θ of length larger than g if and only if λ is the constant path (i.e. $\lambda = t^{(k)}$ for some $t \in \{1, \ldots, s\}$ and some integer k). Let then θ be a path of length $g+1$ generated by λ such that $\text{rank}(\mathcal{O}(\theta)) = r-1$. Then any path $\theta' = t^{(m)}$ of arbitrary length satisfies $\text{rank}(O(\theta')) = r-1$ by Lemma 5.

(ii) It is clear, from the definition of $\mathcal{G}$, that

$$\mathcal{G}(r, g, p, k+1) \leq \mathcal{G}(k+1, \mathcal{G}(r, g, p, k), s^{\mathcal{G}(r,g,p,k)}, k+1).$$

Given that $\mathcal{G}(r, g, s^g, k)$ is nondecreasing in g and assuming (12), and then assuming (13) is true, we get

$$\mathcal{G}(r, g, p, k+1) \leq G(k+1, G(r, g, p, k), s^{G(r,g,p,k)}, k+1),$$

which yields (14) by definition of $G(r, g, p, k+1)$.

(iii) All we need to show is that

$$\mathcal{G}(r, g, p+1, r) \leq 1 + \max_{k=r,\ldots,n} \{\mathcal{G}(r, g, p, k)\},$$

because the conclusion follows from $\mathcal{G}(r, g, p, k) \leq G(r, g, p, k)$ and from $G(r, g, p+1, r) = 1 + \max_{k=r,\ldots,n}\{G(r, g, p, k)\}$ (by definition of G). Let θ be a path of length $N = 1 + \max_{k=r,\ldots,n}\{\mathcal{G}(r, g, p, k)\}$. Assume that it is generated by $p+1$ paths of length g, and that g is greater than the index of L-conditional $r-1$-rank of L, the language generated by those $p+1$ paths. Assume also that $\text{rank}(\mathcal{O}(\theta)) < r-1$. We then have two cases. First, assume that $\text{rank}(\mathcal{O}(\theta_{[1,g]})) < r-1$. Then, by definition of g, the system would not even be pathwise $r-1$-ranked. Second, assume that $\text{rank}(\mathcal{O}(\theta)) = \text{rank}(\mathcal{O}(\theta_{[1,g]})) = r-1$. We make the first remark that if $\theta_{[1,g]}$ appears elsewhere in θ, say $\theta_{[q+1,q+g]} = \theta_{[1,g]}$ for some q, then, by Lemma 5, $\theta_{[1,g]}\theta_{[g+1,g+q]}^{(m)}$ has rank $r-1$ for all m, and thus the system is not pathwise r-ranked. We can therefore assume that θ contains only one occurrence of $\theta_{[1,g]}$, and we note that if $\text{rank}(A(\theta_1)) = r_0$, then, given that $\mathcal{O}(\theta)_{[2,N]} = A(\theta_1)\mathcal{O}(\theta_{[2,N]})$,

$$\begin{aligned} \text{rank}(\mathcal{O}(\theta_{[2,N]})) &\geq \text{rank}(\mathcal{O}(\theta)_{[2,N]}) \\ &\geq \text{rank}(\mathcal{O}(\theta_{[2,N]})) - (n - r_0). \end{aligned} \tag{15}$$

Let $n_0 = r + (n - r_0)$. Now, if $\text{rank}(\mathcal{O}(\theta)) < r$, then $\text{rank}(\mathcal{O}(\theta)_{[2,N]}) < r$. If $n_0 \leq n$, then (15) gives $\text{rank}(\mathcal{O}(\theta_{[2,N]})) < n_0$, which, since $\theta_{[2,N]}$ is generated by only p paths of length g and $|\theta_{[2,N]}| \geq \mathcal{G}(r, g, p, n_0)$, implies the existence of a path θ' of arbitrary length such that range$(\mathcal{O}(\theta')) \subset$ range$(\mathcal{O}(\theta_{[2,N]}))$, and thus rank$(\mathcal{O}(\theta_1\theta')) < r$. If $n_0 > n$, then either rank$(\mathcal{O}(\theta_{[2,N]})) < n$ (see previous case) or range$(A(\theta_1)) \subset$ range$(\mathcal{O}(\theta))$, or in other words $A(\theta_1)$ annihilates any chances of increasing rank$(\mathcal{O}(\theta))$. $\square$

And finally, the proof of Theorem 1, part 1:

Proof: The proof is by induction on r, for $r \leq n$.

Clearly, $\mathcal{N}(s, n, 1) = 1 \leq N(s, n, 1) = 1$.

Now, assuming $\mathcal{N}(s, n, r-1) \leq N(s, n, r-1)$, we get

$$\begin{aligned} \mathcal{N}(s, n, r) &= \mathcal{G}(r, \mathcal{N}(s, n, r-1), s^{\mathcal{N}(s,n,r-1)}, r) & (16)\\ &\leq G(r, \mathcal{N}(s, n, r-1), s^{\mathcal{N}(s,n,r-1)}, r) & (17)\\ &\leq G(r, N(s, n, r-1), s^{N(s,n,r-1)}, r), & (18) \end{aligned}$$

hence $\mathcal{N}(s, n, r) \leq N(s, n, r)$ by definition of $N(s, n, r)$. Equation (16) follows from the definition of $\mathcal{N}$ and $\mathcal{G}$, (17) from Lemma 6, and (18) from the fact that $G(r, g, s^g, r)$ is nondecreasing in g and $\mathcal{N}(s, n, r-1) \leq N(s, n, r-1)$. $\square$

IV. Pathwise Controllability

Let us recall the model:

$$x_{k+1} = A(\theta_k)x_k + B(\theta_k)u_k, \; k \geq 1$$

whose controllability properties can be characterized by the set of pairs $\{(A(1), B(1)), \ldots, (A(s), B(s))\}$. Defining the controllability matrix of a path θ of length N as $\mathcal{C}(\theta) \triangleq (B(\theta_N) \; \ldots \; A(\theta_N)\cdots A(\theta_2)B(\theta_1))$, we note that

$$\mathcal{C}(\theta)' \triangleq \begin{pmatrix} B(\theta_N)' \\ B(\theta_{N-1})'A(\theta_N)' \\ \vdots \\ B(\theta_1)'A(\theta_2)'\cdots A(\theta_N)' \end{pmatrix}$$

happens to be equal to the observability matrix of the reversed path θ', where $\theta'_i = \theta_{N-i+1}$, obtained with the set of dual pairs $\{(A(1)', B(1)'), \ldots, (A(s)', B(s)')\}$. By defining *pathwise controllability* as pathwise observability of the set of dual pairs, all our previous results thus carry over to pathwise controllability, and we get:

Theorem 2: Assume given a set of s pairs $\{(A(1), B(1)), \ldots, (A(s), B(s))\}$, where the dimension of the $A(\cdot)$ matrices is n.

1) If the set is pathwise r-ranked, then its index of pathwise r-rank is smaller than or equal to $N(s, n, r)$.
2) If furthermore the $A(\cdot)$ matrices are pairwise commuting, then the index of pathwise r-rank is bounded by the smaller number $N_c(s, n, r)$. $\Diamond$

Corollary 2: The pathwise r-rank and pathwise controllability properties are decidable. $\Diamond$

V. Conclusion

In this paper, we have shown that pathwise observability and controllability are decidable. Unfortunately, the upper bounds given are too large to be of any practical significance, and it remains unknown whether they are actually reached.

VI. Acknowledgments

This work was sponsored by NSF through the ECS NSF-CAREER award (grant # 0237971).

VII. References

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