A Hybrid Bellman Equation for Bimodal Systems

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Abstract. In this paper we present a dynamic programming formulation of a hybrid optimal control problem for bimodal systems with regional dynamics. In particular, based on optimality-zone computations, a framework is presented in which the resulting hybrid Bellman equation guides the design of optimal control programs with, at most, N discrete transitions.

1 Introduction

Optimal control of hybrid systems is certainly not a new topic. For example, the hybrid maximum principle has been well-studied [3,6], and the community now has a clear grasp of what constitutes necessary optimality conditions for very general classes of hybrid systems. Moreover, a number of results of a more computational flavor have complemented the work on the maximum principle, in which specialized classes of systems are considered. (See for example [1,4,7].)

The contribution in this paper fits squarely in between the hybrid maximum principle work and the more computationally flavored work, in that we produce a Bellman equation for hybrid systems, along the lines of [2,5], that can be easily solved once a so-called optimality zone computation has been performed to seed the computation.

2 The Bimodal Hybrid System

2.1 Geometric Framework

Given two open, connected, and simply connected regions D_1, D_2 such that $D_1 \cap D_2 = \emptyset$, forming a partition of the compact state space X in the sense that

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$X = (D_1 \cup \partial D_1) \cup (D_2 \cup \partial D_2)$, where the boundaries $\partial D_1, \partial D_2$ are assumed to be finite unions of closed, smooth codimension one submanifolds of X.

The vector fields $f_i(x, u),\ i = 1, 2$ associated with each region are further assumed to satisfy a transversality condition in the sense that for $i = 1, 2$, (i) the vector field $f_i(x, u)$ is non-tangential to the boundary ∂D_i at any point in the relative interior of each component of D_i; (ii) at points x on ∂D_i at which smooth components intersect, the vector field $f_i(x, u)$ is non-tangential to each of the tangent spaces of the intersecting components, for all control values.

2.2 Specifications of Executions

The controlled continuous dynamics of the bimodal hybrid system in any of the two regions, on any bounded time interval, are given by:

$$\dot{x}(t) = \begin{cases} f_1(x(t), u(t)),\ x(t) \in D_1 \\ f_2(x(t), u(t)),\ x(t) \in D_2, \end{cases} \quad x(0) \in D_1 \cup D_2,$$

where f_i is continuously differentiable in x (for all u) on the closure of $D_i,\ i = 1, 2$ (and hence uniformly continuous and uniformly Lipschitz in x on the closure of D_i) for each i. The solutions are interpreted in the Caratheodory sense, and the initial condition $x(0)$ of an admissible execution satisfies $x(0) \in D_1 \cup D_2$.

3 Optimal Control Formulation

3.1 The Hybrid Bellman Equation

Given an initial condition $x(0) \in D_i$, the control input $u \in \mathcal{U}$ gives rise to a trajectory x^u passing through a sequence of N regions (regions 1 and 2 repeatedly). We let $i(x^u)$ denote this index, i.e. $i(x^u) = N$. Corresponding to this index there is an ordered set of half open intervals $\{[t_{i_k}, t_{i_{k+1}});\ 0 \leq k \leq N-1\}$, such that $x^u(t) \in \overline{D}_i$ for $t \in [t_{i_k}, t_{i_{k+1}})$.

The hybrid optimization problem addressed in this paper is the following:

$$\mathcal{P}_N : \inf_{u \in \mathcal{U}} \int_0^T \ell(x(t), u(t)) dt$$

subject to the constraints that $\dot{x} = f_i(x, u)$ (when $x \in D_i$), $x(0) = x_0,\ x(T) = x_T$, and $|i(x^u)| \leq N$, for a given upper limit on the total number of switches $N \leq \infty$.

Given $x_1, x_2 \in \overline{D}_i$ we let $c_i(x_1, x_2, \Delta)$ denote the infimum of the costs associated with driving the system from x_1 to x_2 during a time horizon Δ without leaving D_i (except possibly at time 0 or Δ). Our ambition is to produce a hybrid Bellman equation describing the cost-to-go dynamics, and for this we define $V_i^M(x_1, \tau)$ as the cost of going from x_1 to x_T in time τ, using exactly M switches, starting with mode i. In other words, by defining the complementary indicator i^c as $i^c = 2$ if $i = 1$ and $i^c = 1$ if $i = 2$, we get

$$V_i^M(x_1, \tau) = \inf_{t \in [0, \tau], x_2 \in \partial \mathcal{D}} \left\{ c_i(x_1, x_2, t) + V_{i^c}^{M-1}(x_2, \tau - t) \right\}.$$

This relation holds as long as $M \geq 1$. If $M = 0$ then we get the following direct simplification

$$V_i^0(x_1, \tau) = c_i(x_1, x_T, \tau).$$

It should be noted already at this point that $V^M(x_1, \tau) = \infty$ for all τ if M is even and x_1 and x_T belong to different regions, or if M is odd and they belong to the same region.

Since we do not want to insist on an *a priori* given number of intersections of the switching surface $\partial\mathcal{D}$, we need to relate V^N to the original problem. If we let $x_0 \in D_i$ then the optimal cost associated with the original problem $W_i^N(x_0, T)$ is given by

$$W_i^N(x_0, T) = \min_{0 \leq k \leq N} V_i^k(x_0, T).$$

3.2 Optimality Zones

The Bellman equation from the previous section immediately allows an interpretation in terms of optimality zones [3]. In fact, it can be noted that except for the initial and final pieces of the trajectory, from x_0 to the first intersection of the switching surface and from the last intersection to x_T, the trajectory is simply given by a concatenation of trajectories from points on the switching surface. This observation leads to a computational framework in which a large computational burden is needed initially when preparing the so-called optimality zones, but once that price is paid, fast solutions are possible.

4 Examples

We first consider an example in which a planar system $x \in \mathbb{R}^2$ is driven between the boundary points $x(0) = (-1, 0)^T,\ x(T) = (1, 0)^T$, under the system dynamics

$$\dot{x} = \begin{cases} \begin{pmatrix} -0.3 & 0.05 \\ -0.5 & 0 \end{pmatrix} x + \begin{pmatrix} 0.1 \\ 1 \end{pmatrix} u, & \begin{pmatrix} 1 & 1 \end{pmatrix} x < 0 \\ \begin{pmatrix} 0.8 & 1 \\ -3 & -5 \end{pmatrix} x + \begin{pmatrix} -0.3 \\ 3 \end{pmatrix} u, & \begin{pmatrix} 1 & 1 \end{pmatrix} x > 0. \end{cases}$$

Moreover, the final time is $T = 1$, with the maximum number of intersection being given by $N = 20$.

The numerical solution is obtained by discretizing the area over which the optimality zone is computed, and we let the space-time domain be discretized into 50 temporal steps (over $(0, T)$) and 40 spatial steps (over each dimension of the state space.) The particular cost function under consideration is the control energy of the control signal (in the L_2-sense), and the resulting optimal solution is given in Figure 1a. In this case, the optimal solution was obtained when only one crossing of the switching surface took place, with the corresponding optimal cost being $W_1^{20}(x_0, 1) = V_1^1(x_0, 1) \approx 22.91$.

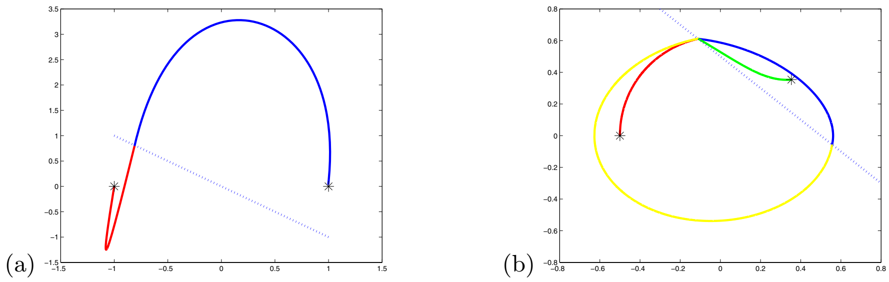

Fig. 1. One switch is optimal (a). Three switches are optimal (b).

In order to highlight the fact that multiple switches may be to prefer, we now consider another linear situation in which

$$\dot{x} = \begin{cases} \begin{pmatrix} \varepsilon_{11} & \omega_1 \\ -\omega_1 & \varepsilon_{12} \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \ \begin{pmatrix} 1 & 1 \end{pmatrix} x < 0 \\ \begin{pmatrix} \varepsilon_{21} & \omega_2 \\ -\omega_2 & \varepsilon_{22} \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \ \begin{pmatrix} 1 & 1 \end{pmatrix} x > 0. \end{cases}$$

In fact, by making ε_{ij} small, we have a (slightly disturbed) oscillation in the system and if we let $\omega_1 = \pi/4$, $\omega_2 = \pi/2$, $T = \pi/2\omega_1 + \pi/2\omega_2 + 3\pi/2\omega_1 + \pi/4\omega_2 = 9.5$, we get that, using the initial and final conditions $x_0 = (-1/2, 0)^T$, $x_T = (1/\sqrt{8}, 1/\sqrt{8})^T$, a zero control effort would result in a three-switch situation if $\varepsilon = 0$. Using exactly the same numerical parameters and costs as in the previous example, with small but non-zero ε_{ij}, we still get that the three switch-situation is optimal, as seen in Figure 1b, with $W_1^{20}(x_0, 9.5) = V_1^3(x_0, 9.5) \approx 0.014 < V_1^1(x_0, 9.5) \approx 0.043$.

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