A Hybrid Bellman Equation for Systems with Regional Dynamics

Angela Schöllig, Peter E. Caines, Magnus Egerstedt, and Roland Malhamé

***Abstract*— In this paper, we study hybrid systems with regional dynamics, i.e., systems where transitions between different dynamical regimes occur as the continuous state of the system reaches given switching surfaces. In particular, we focus our attention on the optimal control problem associated with such systems, and we present a Hybrid Bellman Equation for such systems that provide a characterization of global optimality, given an upper bound on the number of switches. Not surprisingly, the solution will be hyrbid in nature in that it will depend on not only the continuous control signals, but also on discrete decisions as to what domains the system should go through in the first place. A number of examples are presented to highlight the operation of the proposed approach.**

***Index Terms*— Hybrid systems, optimal control, Bellman equation, dynamic programming, finite automata**

I. Introduction

During the last decade, a vast body of research on hybrid control systems has been produced, drawing its relevance from the fact that hybrid models are becoming more and more common. This trend is driven by the fact that many modern application domains involve complex systems, in which sub-system interconnections, mode-transitions, and heterogeneous computational devices are present. Optimal control of hybrid systems is certainly not a new topic. For example, the hybrid maximum principle has been well-studied, and the community now has a clear grasp of what constitutes necessary optimality conditions for very general classes of hybrid systems [1]–[5]. Moreover, a number of results of a more computational flavor have complemented the work on the maximum principle, in which specialized classes of systems are considered. See for example [6]–[10]. These computational contributions typically fall in one of two camps, namely the camp in which the switching times are available to the controller as a design parameter [8], [10], or, the camp in which a more restrictive class of model dynamics (e.g. piecewise linear or affine discrete-time models) is considered, for which mixed-integer programming techniques can be used [6], [7].

In this paper, we continue the development begun in [11], where only bimodal systems were considered, and take the point of view that the hybrid nature of the system is inherent in that transitions between different dynamical regimes are triggered as the state intersects certain surfaces in the state space. As such, the novelty lies in a treatment of global optimality conditions for hybrid systems with regional dynamics, through a Hybrid Bellman Equation.

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II. The Multimodal System

In this section, we introduce a general description of the regional dynamics system considered in our work.

A. Regions, Dynamics and Executions

1) Regions and Dynamics: The starting point of our approach is a given compact state space X which is divided into q open, connected, and simply connected regions $D_i,\ i \in \mathcal{I} = \{1, 2, \ldots, q\}$, such that $X = \bigcup_{i=1}^{q}(D_i \cup \partial D_i)$, where $D_i \cap D_j = \emptyset,\ \forall i, j \in \mathcal{I},\ i \neq j$. The boundaries ∂D_i are assumed to be finite unions of closed, smooth codimension one submanifolds s_i^k of X

$$\partial D_i = \bigcup_{k=1}^{n_i} s_i^k, \quad i \in \mathcal{I},\ n_i \in \mathbb{N},$$

where each boundary ∂D_i is composed of the n_i submanifolds s_i^k, $1 \leq k \leq n_i$. We next define a *switching manifold* $m_{(i,j)}$ by

$$m_{(i,j)} = \partial D_i \cap \partial D_j, \quad i, j \in \mathcal{I}\,. \tag{1}$$

With each region D_i a time-invariant vector field $f_i(x, u)$ is associated which uniquely describes the continuous dynamics in the corresponding partition:

$$\dot{x}(t) = f_i(x(t), u(t)) \quad \text{if } x(t) \in D_i.$$

The function $u(\cdot) \in \mathcal{U}$ is the *continuous-time* control input of the hybrid system, where $\mathcal{U} = \mathcal{U}\left(U, L_\infty\left([0, T]\right)\right)$ denotes the set of all bounded measurable functions on the interval $[0, T]\,,\ T < \infty$, taking values in the set U.

The discrete state space $Q = \{q_i \mid i \in \mathcal{I}\}$ of the system is in one-to-one correspondence with the set of regions $D_i,\ i \in \mathcal{I}$. When the continuous (valued) state $x(t)$ lies in the interior of some D_i the corresponding discrete state is $q(t) = q_i$; when $x(t)$ lies in a boundary segment $m_{(i,j)}$ the interpretation of the possible discrete state values $q(t-) = q_i$ or $q(t-) = q_j$ is that the continuous state has arrived at $x(t)$ along a trajectory which most recently lay in D_i, or respectively, D_j, and a discrete event switch of the discrete state to, say, $q(t) = q_i$, indicates that the system trajectory will evolve on a time interval with initial instant t in D_i under the ith vector field.

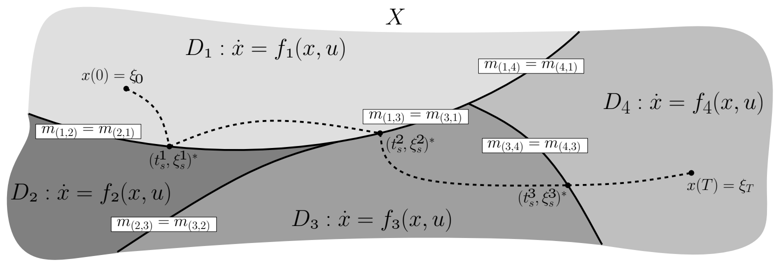


Fig. 1. Example of a System with Regional Dynamics

In the following, the *(controlled) discrete dynamics* of the system are specified in a formal way. Later, in Section II-B, we will introduce an untimed automaton which represents the discrete transitions defined below. A controlled discrete transition is defined at the continuous state x and is denoted by

$$DSC: \quad q_j = \Gamma(q_i, e_{ij}), \tag{2}$$

where $i, j \in \mathcal{I}$, in case (i) $x \in m_{(i,j)}$, (ii) there exists a continuous control $u \in U$ such that the oriented vector at x given by the vector field $f_j(x,u)$ meets the open set D_j within all neighborhoods of x, and (iii) the controlled input event $e \in E$ takes the value e_{ij}. Thereby, E denotes the finite set of transition labels containing the labels of all transitions which are possible in the given partitioned state space X. It is assumed that if at any $x \in m_{(i,j)},\ i,j \in \mathcal{I}$ the condition above is satisfied; then, the condition holds for all other continuous states in $m_{(i,j)}$. Hence, the discrete transition events $e \in E$ are well defined.

Assumption 1: In order to assure existence and uniqueness of the executions in each constituent region $D_i,\ i \in \mathcal{I}$ the vector fields $f_i(x,u)$ are assumed to be continuously differentiable in x (for all u) on the closure of D_i (and hence uniformly continuous and uniformly Lipschitz in x on the closure of D_i). Furthermore, we assume that the vector field $f_i(x,u),\ i \in \mathcal{I}$, satisfies the "transversality" condition in the sense that

(i) $f_i(x,u)$ is non-tangential to the boundary ∂D_i at any point $x \in \partial D_i$ for all choices of u and
(ii) at points $x \in s_i^k \cap s_j^l \neq \emptyset,\ i,j \in \mathcal{I},\ k \in \{1, \ldots, n_i\},\ l \in \{1, \ldots, n_j\}$ the vector field $f_i(x,u)$ is non-tangential to each of the tangent spaces of the intersecting components s_i^k and s_j^l at the point x, for all choices of u.

The solutions are interpreted in the Carathéodory sense, and the initial condition ξ_0 of an admissible execution satisfies $\xi_0 \in D_i,\ i \in \mathcal{I}$. □

2) Executions: In order to precisely describe the transition behavior, that is to say the possible executions, and, in particular, to specify the dynamics of the multimodal system on the boundaries $\partial D_i,\ i \in \mathcal{I}$, we use the notation x_k, where $\dot{x}_k(t) = f_k(x_k(t), u(t)),\ k \in \mathcal{I}$, to emphasize which dynamical regime f_k determines the execution of the trajectory at a given point $x_k(t) \in X$.

Now, given a continuous starting state $\xi_0 \in D_i,\ i \in \mathcal{I}$ and a discrete start state $q_i \in Q$ the *continuous-time* control input $u(\cdot) \in \mathcal{U}$ gives rise to a trajectory evolving according to $\dot{x}_i = f_i(x_i, u)$. If there is a finite time t_s such that the state enters a *switching manifold*

$$m_{(i,j)} = \partial D_i \cap \partial D_j, \quad j \in \mathcal{I}\,;$$

then, it is the case that $\xi_s = \lim_{t \to t_s} x_i(t) \in m_{(i,j)}$ and there are two different possibilities of further execution corresponding to the joint hybrid dynamics specified earlier.

(i) One possible action is that the trajectory *passes through* the switching manifold and henceforth evolves in region j as $\dot{x}_j = f_j(x_j, u)$, with the initial condition $x_j(t_s) = \xi_s$ until a further possible intersection with a switching surface $m_{(j,k)},\ k \in \mathcal{I}$. In this case, the switching at $m_{(i,j)}$ corresponds to a transition from domain i to j.
(ii) However, we also count it as a switch if the trajectory after arriving at $m_{(i,j)}$ under the dynamics $f_i(x_i,u)$ *bounces back* into region i where the dynamical regime is again $\dot{x}_i = f_i(x_i,u)$, with initial condition $x_i(t_s) = \xi_s$ (until a further possible intersection).

The decision about which way will be taken when arriving at a switching manifold $m_{(i,j)}$ can be seen as a *discrete* control input $e \in E$, where E is again the finite set of transition labels. A transition label $e \in E$ specifies the region of the further execution of a trajectory arrived at a switching manifold $m_{(i,j)},\ i,j \in \mathcal{I}$. Clearly, the set of possible transitions at a given switching point $\xi_s \in m_{(i,j)}$ depends on the location of ξ_s in the state space X and also on the family of vector fields defined at ξ_s.

Hence, a hybrid execution is uniquely determined by a given initial condition $\xi_0 \in D_i,\ i \in \mathcal{I}$, a discrete start state $q_i \in Q$ the continuous-time control input $u(\cdot) \in \mathcal{U}$, and a *compatible* discrete control sequence

$$S(\overline{\tau}, w) = \left(\left(t_s^1, e_1\right), \left(t_s^2, e_2\right), \ldots, \left(t_s^M, e_M\right)\right), \tag{3}$$

where $\overline{\tau} = (t_s^1, t_s^2, \ldots, t_s^M),\ 0 < t_s^1 < t_s^2 < \cdots < t_s^M < T$ is the strictly increasing (finite or infinite) sequence of switching times and $w = e_1 e_2 \ldots e_M$ is the corresponding string of appropriate discrete control inputs $e_i \in E,\ i \in \{1, 2, \ldots, M\}$. M denotes the number of switches.

The hybrid execution description above has continuous and discrete values states, where the latter have right continuous trajectories in $\mathbb{N}$ which are piece-wise constant. In Section II-B, an automaton representing the connected regions will be used to determine *possible* switching sequences w of a given partitioned state space X. In conclusion, the admissible control actions available are the continuous control $u(\cdot) \in \mathcal{U}$ and the discrete control input sequence $S(\overline{\tau}, w)$. Both control inputs are part of our optimization problem which is explicitly defined in Section III.

The resulting controlled dynamics of a multimodal hybrid system evolving in a partitioned state space X can be pictured as in Figure 1.

Remark 1: Some former approaches, e.g. [12] and [13], define the transition behavior in a different way. Their models do not include the case of "bouncing back" at a switching manifold. However, in our model, cf. Section IV, these (and other) kinds of switches can be inhibited on a higher level of control.

B. The Transition Automaton

In this part, we present an important step in solving the optimal control problem associated with the multimodal hybrid system introduced in Section II-A. Roughly speaking, the optimal control problem under consideration is stated as follows: Given a specific cost function, our goal is to determine the *optimal path* of going from a given initial state $\xi_0 = x_{i_0}(0) \in D_{i_0}$ to a fixed final state $\xi_T \in D_{i_T}$ during a time horizon T, where T is also specified a priori. In order to solve this optimization problem and even in order to formulate this problem in a precise manner, a discrete representation of the geometric structure is needed which explicitly specifies the connections between the constituent regions D_i, $i \in \mathcal{I}$. A *deterministic automaton* is introduced which contains information about the regional composition in the way that a transition between the *discrete* states i and j, i.e., between region D_i and D_j, is only possible if the switching surface $m_{(i,j)} \neq \emptyset$. Furthermore, information about the initial region D_{i_0}, $\xi_0 \in D_{i_0}$ and the final region D_{i_T}, $\xi_T \in D_{i_T}$ is incorporated in the automaton. Consequently, the automaton answers the question: Which ways, i.e., which sequences of transitions, are possible in order to get from ξ_0 to ξ_T?

The formal definition of the automaton, following the notation of [14], is given by the six-tupel

$$A = (D, E, g, h, d_0, D_m), \tag{4}$$

where

- the set of discrete states is $D = \mathcal{I}$,
- the set of events is given by $E = \{e_{ij} \mid i, j \in \mathcal{I}\}$,
- the transition function is defined as $g(i, e_{ij}) = j$ if $m_{(i,j)} \neq \emptyset$ and is not defined for all other cases,
- the initial state $d_0 = i_0$, where $\xi_0 \in D_{i_0}$, and
- the set of marked states is given by $D_m = \{i_T \mid \xi_T \in D_{i_T}\}$.

The function h is defined by the specifications above.

In this formalism, the states $D = \mathcal{I}$ represent the discrete state space $Q = \{q_i \mid i \in \mathcal{I}\}$, i.e., the different regions D_i, $i \in \mathcal{I}$. An event e_{ij} characterizes the transition, i.e., the switch, from region D_i to region D_j. The set of events E corresponds to the previous defined set of transition labels (Section II-A) and the transition function g shows which transitions $e_{ij} \in E$ are possible for a given state i.

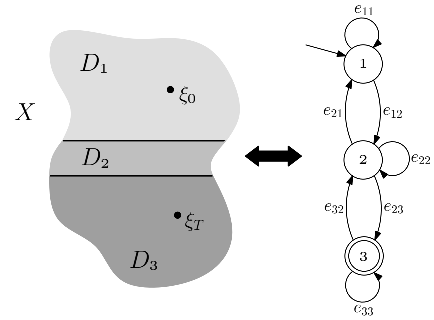


Fig. 2. Example for the Transition from a Given Geometric Structure to the Corresponding Automaton

Note that the geometric shape of a region $D_i, i \in \mathcal{I}$ is reduced to a discrete state i and the switching manifold $m_{(i,j)}$, $i, j \in \mathcal{I}$ to a single event e_{ij}. Furthermore, it is important to observe that the automaton does contain information about the initial state $\xi_0 \in D_{i_0}$ and the final state $\xi_T \in D_{i_T}$, but does not contain any time-valued information.

A simple example for the transition from a given geometric structure to the corresponding automaton is depicted in Figure 2.

Our further approach will be based on the language generated by the introduced automaton A, equation (4). In particular, the marked language $L_m(A)$ will be used in order to formulate and solve the optimization problem. The marked language $L_m(A)$, cf. [14], contains *all strings*, i.e., all sequences of transitions, leading from state i_0, where $\xi_0 \in D_{i_0}$, to state i_T, where $\xi_T \in D_{i_T}$. In order to restrict the strings $w \in L_m(A)$ to a certain length M, i.e., to find all ways from ξ_0 to ξ_T using exactly M switches, we define an additional language

$$L_M = \left\{ e \in E^* \;\middle|\; |e| = M \right\},$$

where E^* represents the set of all finite strings of elements of E and $|\cdot|$ denotes the length of a string, i.e., the number of events contained in the string. The length of the empty string ϵ is $|\epsilon| = 0$. The intersection

$$L(M, A) = L_m(A) \cap L_M \tag{5}$$

gives us the desired set of strings leading from state i_0 to i_T and consisting of *exactly M events*. In addition, let us introduce a further language which will be useful in deriving the Hybrid Bellman Equation.

$$F_{M,A}(K) = L_K \cap \textit{suff}(L(M, A)) \tag{6}$$

is the set of all suffices of $L(M, A)$ consisting of K events. The set $\textit{suff}(L) = \{s \in E^* \mid \exists\, v \in E^* \text{ with } v \cdot s \in L\}$ denotes the suffix closure of the language L, where the notation $v \cdot s$ symbolizes the concatenation of string v and s. Note that $F_{M,A}(0) = \{\epsilon\}$ only consists of the empty string ϵ.

III. The Optimization Problem

Given an *upper bound* $N \in \mathbb{N}_0$ *on the total number of switches* the hybrid optimization problem is the following:

$$\mathcal{P}_N : \inf_{\substack{u(\cdot)\in\mathcal{U},\\ S(\overline{\tau},w)}} \int_0^T \ell(x(t), u(t))dt \qquad (7)$$

subject to, for $0 \le M \le N$,

- the geometric structure
$$w = e_{i_1 j_1} e_{i_2 j_2} \dots e_{i_M j_M} \in L(M, A),$$
- the discrete dynamics
$$q_{j_k} = \Gamma\left(q_{i_k}, e_{i_k j_k}\right)$$
at a switching time t_s^k, where $1 \le k \le M$, yielding to the discrete-state dynamics $q(t) = q_{i_k},\ t \in \left[t_s^{k-1}, t_s^k\right)$ with $0 < k \le M+1$ and $i_{M+1} = j_M$,
- the continuous-state dynamics
$$\dot{x}_{i_k}(t) = f_{i_k}(x_{i_k}(t), u(t)), \quad t \in \left[t_s^{k-1}, t_s^k\right),$$
where $0 < k \le M+1$ and $i_{M+1} = j_M$,
- and the corresponding initial and final conditions
$$\begin{aligned} x(0) &= x_{i_1}(t_s^0) &&= \xi_0 &&\in D_{i_0}, \\ x_{i_{k+1}}(t_s^k) &= \lim_{t\to t_s^k} x_{i_k}(t) &&= \xi_s^k &&\in m_{(i_k,j_k)}, \\ x(T) &= x_{i_{M+1}}(t_s^{M+1}) &&= \xi_T &&\in D_{i_T}, \end{aligned}$$
if $0 < k \le M$.

Note that $\xi_0, \xi_T \notin \partial D_i,\ i \in \mathcal{I}$ (Section II-B).

IV. The Hybrid Bellman Equation

In order to solve the optimization problem (7) a main step of our approach is the creation of a hierarchical structure, i.e., the creation of different levels of abstraction. Using this structure the optimal control problem can be solved partially on each level of control. On the highest level, the given geometric framework is taken into account. The presented deterministic automaton (Section II-B) provides us a language which specifies all switching sequences possible in the given composition of regions. Furthermore, we will show that on this level of control arbitrary switching rules can be incorporated in the automaton. On a lower level, based on the language generated by the automaton a Hybrid Bellman Equation is derived. This equation provides us the optimal switching points $(t_s^i, \xi_s^i),\ \ i \in \{1, 2, \dots, M\}$ together with the corresponding discrete control sequence $S(\overline{\tau}, w)$ for a given number M of switchings. In addition, we obtain the associated cost $V^M(\xi_0, q_{i_0}, \xi_T, q_{i_T}, T)$. In order to solve the original problem (7) a minimization over all $V^M(\xi_0, q_{i_0}, \xi_T, q_{i_T}, T)$, $0 \le M \le N$ is necessary leading to an optimal number of switches M^* associated with the optimal sequence of switching points and discrete control inputs. With this result, the optimal path connecting the calculated switching points and the initial and end point, $\xi_0 \in D_{i_0}$ and $\xi_T \in D_{i_T}$, respectively, is obtained by solving – on the lowest level of control – a standard (non-hybrid)

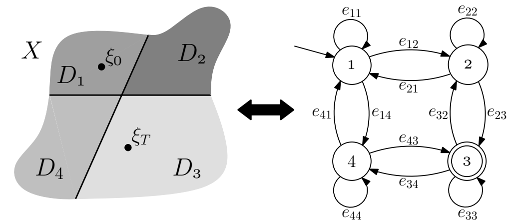


Fig. 3. Example for Incorporating Additional Switching Rules

state-constrained optimal control problem separately for each two consecutive pairs:

$$\Big((0, \xi_0),\ (t_s^1, \xi_s^1)^*,\ (t_s^2, \xi_s^2)^*,\ \dots,\ (t_s^{M^*}, \xi_s^{M^*})^*,\ (T, \xi_T)\Big).$$

In the following, the different levels of abstraction and the inherent procedures are described in greater detail.

At the highest level of abstraction, the structure of the partitioned state space X is considered. The deterministic automaton introduced in Section II-B contains information about the connections between the regions and about possible ways to get from $\xi_0 \in D_{i_0}$ to $\xi_T \in D_{i_T},\ \ i_0, i_T \in \mathcal{I}$. However, it is important to emphasize that it is also possible to incorporate arbitrary switching rules on this level of abstraction. One example is given in Figure 3. In this case, the transitions between region 1 and 3, and 2 and 4, respectively, are obviated by the constructed automaton. Moreover, supervisory control [14] and similar operations can be applied to disable, for example, such transitions. The language $L(M, A)$, equation (5), associated with the automaton provides sequences of switchings leading from q_{i_0} (indicating the continuous state lies in D_{i_0}) to q_{i_T} (corresponding to D_{i_T} containing the terminal state). These give the *accessibility relations (i.e. words) with a prescribed number of events (letters) which correspond to potential accessibility relations between specified discrete states along trajectories with a prescribed number of switchings.*

At the base (continuous) system level, these global accessibility relations will be used to constrain the infimization in the Hybrid Bellman Equation in the characterization of the optimal switching states $\xi_s^i,\ \ i \in \{1, 2, \dots, M\}$, discrete control sequences, $S(\overline{\tau}, w)$, and optimal continuous control functions. The approach is based upon the fundamental (Dynamic Programming) Principle of Optimality which informally states that along an optimal hybrid trajectory the execution of the continuous state x between two consecutive switching points (t_s^j, ξ_s^j), (t_s^{j+1}, ξ_s^{j+1}) is optimal. In order to produce a Hybrid Bellman Equation describing the cost-to-go dynamics let *end* : $E^* \setminus \{\epsilon\} \rightarrow \mathcal{I}$ denote the mapping

$$\textit{end}(s) = \textit{end}(e_{i_1 j_1} e_{i_2 j_2} \dots e_{i_k j_k}) = j_k\,,$$

which will be used to specify the region D_{j_k} associated with the last index of the last transition label $e_{i_k j_k}$ of the string s. Additionally, we define $c(\xi_1, q_{i_1}, \xi_2, q_{i_2}, \Delta),\ \xi_1, \xi_2 \in X$, as the infimum of the costs associated with driving the system from $\xi_1 \in D_{i_1} \cup \partial D_{i_1}$ to $\xi_2 \in D_{i_1} \cup \partial D_{i_1}$, $i_1 \in \mathcal{I}$ over a time interval Δ *without leaving* $D_{i_1} \cup \partial D_{i_1}$ *and without a* *switching taking place*. Here, $q_{i_1}, q_{i_2} \in Q$ are the discrete states associated with ξ_1 and ξ_2, respectively. Clearly, the cost $c(\xi_1, q_{i_1}, \xi_2, q_{i_2}, \Delta)$ is infinite if ξ_2 is inaccessible from ξ_1 by trajectories remaining in $D_{i_1} \cup \partial D_{i_1}$ and along which no switch occurs. The Hybrid Bellman Equation can be formulated using the notation $V^M(\xi_1, q_{i_1}, \xi_2, q_{i_2}, T)$ to describe the infimum of the costs of going from $\xi_1 \in X$ to $\xi_2 \in X$ in time T using *exactly M switches* determined by $e \in L_M$ starting in region D_{i_1}, where the feasible input words e are determined from the untimed system automaton with state set Q, event set E and transition function g.

The main theorem of the paper is established using the Principle of Optimality and is expressed using the notations above as follows:

Theorem 1: Assume that all hypotheses for the existence and uniqueness of regional dynamics hybrid systems hold and that all infima exist in the definition of the hybrid value functions $V^{\cdot}(\cdot,\cdot,\cdot,\cdot,\cdot)$, for all admissible argument values, whenever the expressions are finite. Then

$$V^K(\xi_1, q_{i_1}, \xi_2, q_{i_2}, T) = \inf_{t \in (0,T)} \inf_{\xi \in m_{(i_1,j)}}$$

$$\left\{ \inf_{j \in \mathcal{I}} \left\{ c(\xi_1, q_{i_1}, \xi, q_j, t) + V^{K-1}(\xi, q_j, \xi_2, q_{i_2}, T-t) \right\} \right\}$$

such that

$$\begin{aligned} e &= e_{i_1 j}\,, & s &\in F_{M,A}(K-1)\,, \\ \mathit{end}(s) &= i_2\,, & e \cdot s &\in F_{M,A}(K)\,. \end{aligned}$$

This relation holds for $0 < K \leq M$. When $K = 0$, the initial condition of the recursive scheme is given by

$$V^0(\xi_1, q_{i_1}, \xi_2, q_{i_2}, T) = c(\xi_1, q_{i_1}, \xi_2, q_{i_2}, T). \qquad \square$$

Since we do not insist on an *a priori* given number of switches M, $0 \leq M \leq N$, we need to relate $V^K(\xi_1, q_{i_1}, \xi_2, q_{i_2}, T)$ to the original problem. Recalling that $\xi_0 \in D_{i_0}$ and $\xi_T \in D_{i_T}$, $i_0, i_T \in \mathcal{I}$, the optimal cost associated with the original problem $W^N(\xi_0, q_{i_0}, \xi_T, q_{i_T}, T)$ is given by

$$W^N(\xi_0, q_{i_0}, \xi_T, q_{i_T}, T) = \min_{0 \leq k \leq N} V^K(\xi_0, q_{i_0}, \xi_T, q_{i_T}, T)\,.$$

For a more detailed treatment of the presented approach including the computational procedure associated with the derived recursive equation, cf. Theorem 1, the reader is referred to [15]. Moreover, note that stochastic approaches to similar problems were proposed in [16], [17].

V. Examples

Finally, two examples illustrate the theoretical results of the previous sections. First, a bimodal system is chosen showing the phenomenon of "bouncing back" at a switching manifold. Second, an example of a state space partitioned into three regions is considered

Example 1: The planar state space $X \subset \mathbb{R}^2$ is divided into two regions $D_1 = \{x \mid (1 \;\; 2)\, x > 4.5\}$ and $D_2 = \{x \mid (1 \;\; 2)\, x < 4.5\}$. The system is driven between the points $\xi_0 = (-1 \;\; 1)^T \in D_2$ and $\xi_T = (0 \;\; 3.5)^T \in D_1$ with an unconstrained input $u(t) \in \mathbb{R}$, where the transition behavior is determined by the automaton in Figure 4 and the system dynamics are given by

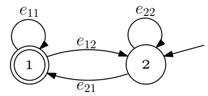


Fig. 4. Transition Automaton Considered in Example 1

$$\dot{x} = \begin{cases} \begin{pmatrix} 0 & 0.25 \\ -3 & -0.5 \end{pmatrix} x + \begin{pmatrix} -10 \\ 100 \end{pmatrix} u, & x \in D_1 \\ \begin{pmatrix} 0.5 & 0.1 \\ -10 & -0.5 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, & x \in D_2. \end{cases}$$

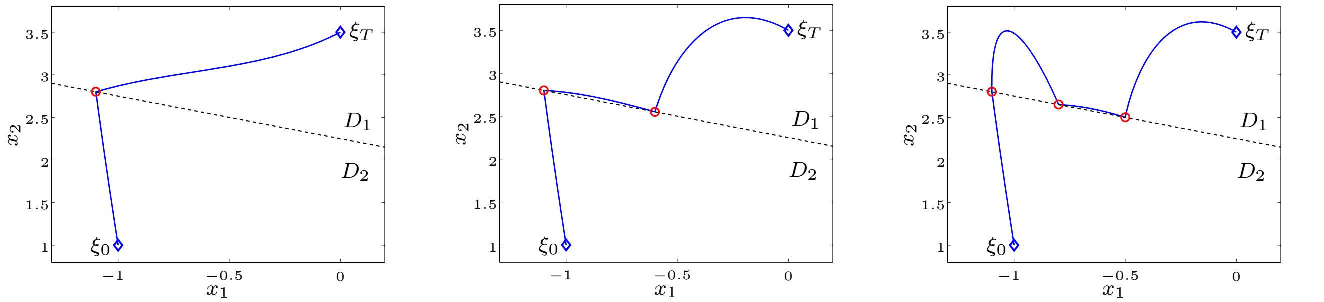
The final time is $T = 1.8$, with the maximum number of switches being given by $N = 3$. The particular cost function (7) under consideration is the control energy of the control signal $\ell(x(t), u(t)) = u(t)^2$. The numerical solution is obtained by discretizing the time interval $[0, T]$ into 18 equally spaced temporal steps and by discretizing the switching manifold $m_{(1,2)}: \; (1 \;\; 2)\, x = 4.5$ into 40 equally spaced spatial steps over the interval $x_1 \in [-2, 2]$.

Figure 5 shows the intermediate results, i.e., the optimal solutions for a given fixed number of switches $M \in \{1, 2, 3\}$, with their corresponding costs $V^M(\xi_0, q_2, \xi_T, q_1, T)$. The phenomenon of "bouncing back" at the switching manifold can be observed in the cases $M = 2$ and $M = 3$, Figure 5.2 and Figure 5.3, respectively. Using (IV), these results lead to the final optimal solution with the associated cost $W^3(\xi_0, q_2, \xi_T, q_1, T)$. The optimal solution is obtained when $M = 1$ with the corresponding optimal cost being $W^3(\xi_0, q_2, \xi_T, q_1, T) = V^1(\xi_0, q_2, \xi_T, q_1, T) \approx 3.865$. However, it is straightforward to construct examples for which the optimal solution does in fact involve multiple switches, as shown in [11].

Example 2: As depicted in Figure 2, the state space $X \subset \mathbb{R}^2$ is divided by two parallel lines $m_{(1,2)}: \; (-1 \;\; 1)\, x = -0.5$ and $m_{(2,3)}: \; (-1 \;\; 1)\, x = -2$ resulting in the three regions $D_1 = \{x \mid -0.5 < (-1 \;\; 1)\, x\}$, $D_2 = \{x \mid -2 < (-1 \;\; 1)\, x < -0.5\}$, and $D_3 = \{x \mid (-1 \;\; 1)\, x < -2\}$. The system is driven between the points $\xi_0 = (-0.5 \;\; 0)^T \in D_1$ and $\xi_T = (-1.1 \;\; -4)^T \in D_3$ with an unconstrained input $u(t) \in \mathbb{R}$ under the system dynamics

$$\dot{x} = \begin{cases} \begin{pmatrix} -1 & 2 \\ -5 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, & x \in D_1 \\ \begin{pmatrix} 0.5 & 1 \\ -0.05 & -0.25 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, & x \in D_2 \\ \begin{pmatrix} 0 & 0.25 \\ -3 & -0.1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, & x \in D_3. \end{cases}$$

With $T = 6.7$ and $N = 2$, according to the automaton in Figure 2, the only possible string of discrete control inputs is $w = e_{12} e_{23}$. Using the same cost function as in



5.1: $M=1\,,\ V^1(\xi_0,q_2,\xi_T,q_1,T)\approx 3.865$ 5.2: $M=2\,,\ V^2(\xi_0,q_2,\xi_T,q_1,T)\approx 3.866$ 5.3: $M=3\,,\ V^3(\xi_0,q_2,\xi_T,q_1,T)\approx 3.867$

Fig. 5. Optimal Trajectories of Example 1 for a Given Fixed Number of Switches $M \in \{1,2,3\}$ – The solid line shows the optimal trajectory, the dashed line presents the switching manifold, and the circles depict the switching points.

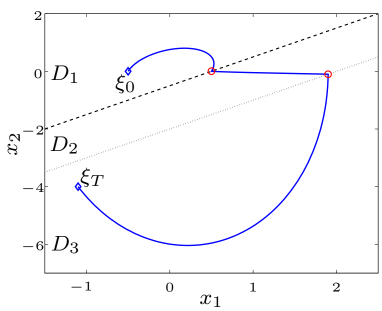


Fig. 6. Optimal Trajectory of Example 2 – The solid line shows the optimal trajectory, the dashed and dotted lines present the switching manifolds $m_{(1,2)}$ and $m_{(2,3)}$, respectively, and the circles depict the switching points.

Example 1, the numerical solution is obtained by discretizing the time interval $[0,T]$ into 67 equally spaced temporal steps and by discretizing the switching manifold $m_{(1,2)}$ and $m_{(2,3)}$ into 40 equally spaced spatial steps over the interval $x_1 \in [-1.5,\, 2.5]$. The resulting optimal solution is given in Figure 6 and $W^2(\xi_0,q_1,\xi_T,q_3,T)\approx 0.039\,$.

VI. Conclusions

This paper presented a Hybrid Bellman Equation for hybrid systems with regional dynamics. This equation thus provided a characterization of global optimality in a hybrid setting, in which the control variable consisted not only of the continuous control signal, but also of a decision variable dictating what regions the system should switch between. A number of examples were also presented that illustrate the use of the proposed method.

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