Multi-Modal Control Using Adaptive Motion Description Languages ⋆

Tejas R. Mehta [a], Magnus Egerstedt [b].

[a] *School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*

[b] *School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA*

Abstract

In this paper, we address the problem of adaptively enriching motion description languages for multi-modal control by systematically enlarging the set of available modes. This problem is formulated as an optimal control problem, where new modes are designed by combining recurring mode string fragments into smooth, new "meta-modes" and variational arguments are employed to derive optimality conditions for this construction.

Key words: Hybrid Systems, Optimal Control, Motion Description Languages

1 Introduction

A *multi-modal system* is a system that switches between different modes of operation. These switches can be time-driven, or occur in response to changes in the environment (i.e. event-driven). The dynamics during each mode are typically given by differential equations. Since these systems combine continuous dynamics with discrete switching dynamics, they belong to the wider class of *hybrid systems*.

In the case of controlled switching, the original system dynamics do not naturally change between different modes. Rather, a multi-modal control strategy is employed to break up the control task into a sequence of simpler tasks. The main idea is to design different controllers with respect to a particular control task, operating point, or system configuration, and then combine these controllers using supervisory logic to obtain the desired overall behavior. A prime example of this paradigm is *behavior-based robotics*, in which the overall behavior is obtained through a combination of simpler behaviors (or controllers) such as avoiding obstacles and approaching landmarks [1,5].

In this paper, we define a mode as consiting of a feedback control law together with a condition for its termination (referred to as an interrupt condition), as is the case within the Motion Description Language (MDL) framework [4,10]. Using this representation for the mode, the control task involves designing a set of feedback laws and interrupt conditions pertinent to the control task, and then sequencing them in order to achieve the desired, overall behavior.

Assume that we are given a set of feedback laws (denoted by K) and interrupts (denoted by Ξ), a control program (mode string) is a concatenation of pairs $(\kappa, \xi) \in K \times \Xi$. Such control programs can be thought of as having an information theoretic content [9] in that we can let the *complexity* of the control program be given by the number of bits required to encode it. This is given by the description length: $|\bar{\sigma}| \log_2(card(K \times \Xi))$, where $|\bar{\sigma}|$ denotes the length of the control program $\bar{\sigma}$, and $card(\cdot)$ denotes the cardinality. Moreover, one can think of *expressiveness* as a measure of what overall behaviors are producible from the set modes, characterized by the reachable set. (In fact, the reachable set has been thoroughly studied and there is an abundance of literature pertaining to its estimation, e.g. [3,17].) One can thus ask the question of whether or not it is worth adding new modes to the mode set, i.e. try to weigh the increase in complexity against the potential increase in expressiveness and the overall performance. In this paper, we extend our preliminary work along these lines on *motion alphabet augmentation* [13,15]. In particular, previous methods neglected the interrupt conditions and focused solely on

⋆ Corresponding author T. R. Mehta. Tel. (404) 385-4077. Fax (404) 385-1746.

Email addresses: tmehta@ece.gatech.edu (Tejas R. Mehta), magnus@ece.gatech.edu (Magnus Egerstedt).

designing new feedback laws. These feedback laws were designed to replace individual occurrences of recurring mode fragments with the hope that it would also be useful as a replacement of other occurrences of the mode fragments, although no such guarantees were provided. In this paper, we remedy this shortcoming by optimizing over the entire mode string instead of a single occurrence of the mode fragment. Moreover, we also directly tackle the issue of how to design the interrupt conditions in the event-driven setting.

2 Problem Description

Formally, we define a mode σ as a pair (κ, ξ), where $\kappa : X \to U$ corresponds to a particular feedback law, the interrupt $\xi : X \to \{0, 1\}$ encodes conditions for its termination, and where X and U denote the state and input space respectively. (Time-driven interrupts can be facilitated by including time in the state space.) Given a finite set of feedback mappings K and interrupts Ξ, we let $\Sigma = K \times \Xi$ denote the set of all modes, or control-interrupt pairs, and by $\Sigma^\star$ we understand the set of all finite length strings over Σ.

Assume that we are given a mode string $\bar{\sigma} \in \Sigma^\star$ for solving a particular control task. What we address in this paper is the question: *What new modes, if any, should be added in order to improve the performance?* By this, we understand not only a reduction in performance cost, but also a reduction in the specification complexity of the control program. Moreover, the new modes should be added in a structured manner so that this procedure can be automated.

In [13,15], we showed that it may be beneficial to replace recurring mode string fragments, if they exist, by a smooth, new "meta-mode." For example, consider the mode sequence $\bar{\sigma} = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_4 \in \Sigma^\star$, where we observe that $\sigma_1\sigma_2$ is a recurring mode string fragment. To see how augmenting the mode set in this manner can reduce the complexity, assume that $\sigma_1, \ldots, \sigma_r$ occurs (possibly repeatedly) in a control program. If p occurrences of $\sigma_1, \ldots, \sigma_r$ are replaced by a single "meta-mode" σ_{1r} in the original control program $(\bar{\sigma}_{old})$, then the complexity of new mode string $(\bar{\sigma}_{new})$ is now $(|\bar{\sigma}_{old}| - p(r - 1)) \log_2(card(\Sigma_{old})+1) < |\bar{\sigma}_{old}| \log_2(card(\Sigma_{old}))$ as long as $p \geq 1$ and $r > 1$. Thus, the complexity would be reduced if such a σ_{1r} could be found. So the main problem under consideration here is the construction of such a σ_{1r}. The construction of this new mode σ_{1r} involves constructing a new feedback law κ_{1r} and an interrupt ξ_{1r}.

More specifically, we will construct the feedback law as a linear combination of *basis functions* g_i's, where g_i is given by some combination of the previous feedback laws, i.e. $g_i = \zeta(\kappa_1, \ldots, \kappa_r)$. In this paper, we do not concern ourselves with the particular choice of $\zeta(\cdot)$ exactly, but refer the reader to [15] for a discussion of this topic. Once these basis functions have been selected, the construction of the new feedback law can be cast in terms of finding appropriate (possibly state-dependent) weights of the basis functions in the linear combination that best approximates the trajectory produced by the mode string fragment we are trying to replace.

In fact, given the trajectory $x(t) \in \mathbb{R}^n$, obtained using the mode string fragment $\sigma_1\sigma_2 \cdots \sigma_r$ that we are trying to replace, the main idea behind this work is to introduce an approximation function $z(t) \in \mathbb{R}^n$. This function is supposed to track $x(t)$ and is defined through

$$\dot{z} = f(z, \kappa_{1r}) \text{ until } \xi_{1r}(z) = 1, \tag{1}$$

with $z(0) = x(0)$. Hence, the problem is reduced to finding the feedback mapping κ_{1r} that best approximates $x(t)$. In [13], the feedback law κ_{1r} was constrained to be a static, linear combination of the basis functions g_i's, which reduces the problem to finding appropriate coefficients in order to minimize a particular cost function that is designed to capture how well the original trajectory is being approximated.

In this paper we generalize this idea by proposing to let the feedback law κ_{1r} be more general, using membership functions, as proposed in [14],

$$\kappa_{1r}(z(t)) = \sum_{i=1}^{N} \mu_i(z(t), \alpha_i) g_i(z(t)), \tag{2}$$

where $\mu_i : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$. In (2), the weight of each basis function g_i is determined by a weighing function μ_i, which is parameterized by vector $\alpha_i \in \mathbb{R}^k$. We refer to these weighing functions as *membership functions* as they closely resemble membership functions in fuzzy-logic control [8,16]. Here the control vector is the concatenation of the shaping vectors α_is for each of the N membership functions, and hence $\vec{\alpha} = [\alpha_1, \ldots, \alpha_N]^T \in \mathbb{R}^{Nk}$.

But, finding the feedback laws is not enough. We also need to find the appropriate interrupt functions. And, in this paper we will assume that the interrupts are parameterized by some control vector. In particular, we let the interrupt ξ_i be shaped by a control parameter $\beta_i \in \mathbb{R}^k$, e.g. $\xi_i : \mathbb{R}^n \times \mathbb{R}^k \to \{0, 1\}$. Now we can design ξ_{1r} by adjusting β_r, hence the optimization problem involves an additional control vector $\beta \in \mathbb{R}^k$.

In light of the preceding discussion, in this paper we will propose a systematic construction for finding the new control laws and interrupt conditions. Before detailing this construction, we will first examine a simple example to make the key concepts more concrete. Suppose we have a mode string $\bar{\sigma} = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_4$, and note that $\sigma_1\sigma_2$ is a recurring mode string in $\bar{\sigma}$. The new mode

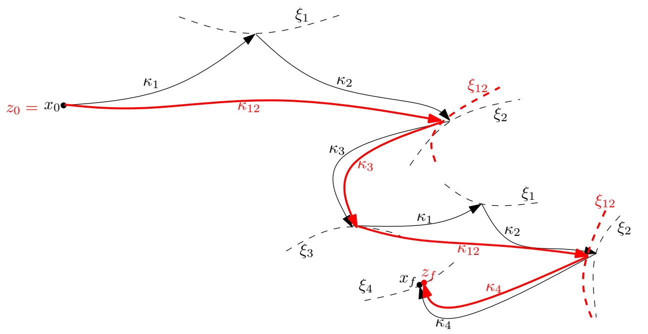

Fig. 1. Depicted is the original trajectory $x(t)$ and the approximation trajectory z(t).

would be constructed using an approximation trajectory $z(t)$ as shown in Figure 1. In particular, the approximation trajectory is given by

$$\dot{z} = \begin{cases} f(z, \kappa_{12}(z)) & \text{until } \xi_{12}(z) = 1 \\ f(z, \kappa_3(z)) & \text{until } \xi_3(z) = 1 \\ f(z, \kappa_{12}(z)) & \text{until } \xi_{12}(z) = 1 \\ f(z, \kappa_4(z)) & \text{until } \xi_4(z) = 1, \end{cases} \tag{3}$$

with $z(0) = x(0)$. In equation (3) above, κ_{12} is shaped by the control vector $\vec{\alpha}$ as shown in equation (2) and ξ_{12} is ξ_2 reshaped by the control vector β_{12}.

3 A Variational Problem

To summarize: What we need to do is simultaneously shape the new, meta-mode as well as the interrupt condition for its termination, as a function of the state of the system. In fact, without the interrupts, a number of approaches to solving similar problems have been proposed, including [6,7,12], and a similar derivation is also given in [11].

For this, it will be advantageous to introduce an identifier $p(i)$, taking values in a finite set, denoting the mode of operation during the time interval $[\tau_{i-1}, \tau_i)$, where we will use τ to denote the time instances at which the system switches between different modes of operation.

Using this terminology, the approximation trajectory $z(t)$ is given by

$$\dot{z}(t) = \begin{cases} f(z, \kappa_{p(1)}(z)) & \text{until } \xi_{p(1)}(z, \beta_{p(1)}) = 1 \\ \quad\vdots & \quad\vdots \\ f(z, \kappa_{p(M)}(z)) & \text{until } \xi_{p(M)}(z, \beta_{p(M)}) = 1, \end{cases} \tag{4}$$

with $z(0) = x(0)$. Observe that $f(z, \kappa_{p(i)}(z))$ is a function of z and the control vector $\vec{\alpha}$ when $p(i) = 1r$ and just a function of z otherwise. Thus, for ease of notation, we introduce a new indexing function $\tilde{f}_i(z, \vec{\alpha})$ defined as

$$\tilde{f}_i(z, \vec{\alpha}) = \begin{cases} f(z, \sum_{j=1}^{N} \mu_j(z, \alpha_j) g_j(z)) & \text{if } p(i) = 1r \\ f(z, \kappa_{p(i)}) & \text{otherwise.} \end{cases} \tag{5}$$

Similarly, since we are only reshaping ξ_{1r}, we can treat $\beta_{p(i)}$ as a fixed constant when $p(i) \neq 1r$. Hence our control parameter for the interrupt shaping is $\vec{\beta} = \beta_{1r} \in \mathbb{R}^k$. Again, for ease of notation, we introduce $\tilde{\xi}_i(z, \vec{\beta})$ defined as follows:

$$\tilde{\xi}_i(z, \vec{\beta}) = \begin{cases} \xi_{1r}(z, \vec{\beta}) & \text{if } p(i) = 1r \\ \xi_{p(i)}(z, \beta_{p(i)}) & \text{otherwise.} \end{cases} \tag{6}$$

Moreover, we assume that the "first" switching instant $\tau_0 = 0$ is fixed, and the other switching instants are given by the interrupts as

$$\tau_i = \{t > \tau_{i-1} \mid \xi_{p(i)}(z(t), \beta_{p(i)}) = 1\}, \tag{7}$$

for $i = 1, \dots, M$.

Now our problem involves finding the control vectors $\vec{\alpha}$ and $\vec{\beta}$ such that the cost

$$J = \int_{\tau_0}^{\tau_M} L(x(t), z(t))dt + \psi(x_f, z(\tau_M)) \tag{8}$$

is minimized, where L and ψ are continuously differentiable in their second argument. We also assume that the basis functions g_i's and membership functions μ_i's are continuously differentiable. In addition to these assumptions, we must make one more assumption to ensure that the approximation function z does not approach any of the switching surfaces tangentially. Namely, we assume that

$$\frac{\partial \tilde{\xi}_i}{\partial z} \tilde{f}_i \neq 0$$

for $i = 1, \dots, M$. In other words, we assume that the Lie derivative of $\tilde{\xi}_i$ with respect to z along the flow $\tilde{f}_i$ does not equal 0.

Defining the Hamiltonian as

$$H_i(x, z, \lambda_i, \vec{\alpha}_i) = L(x, z) + \lambda_i \tilde{f}_i(z, \vec{\alpha}), \tag{9}$$

the augmented (but unaltered from an evaluation point of view) unperturbed cost is given by

$$\tilde{J}_0 = \sum_{i=1}^{M} \int_{\tau_{i-1}}^{\tau_i} \Big[H_i(x, z, \lambda_i, \vec{\alpha}) - \lambda_i \dot{z} \Big] dt + \\ + \sum_{i=1}^{M} \nu\big(\tilde{\xi}_i(z(\tau_i), \vec{\beta})\big) + \psi(x_f, z(\tau_M)). \tag{10}$$

Note here that the continuous co-state $\lambda(t)$ is corresponds to the constraints of the continuous dynamics, while the co-state ν corresponds to the discrete switching dynamics. Now we perturb (10) in such a way that $\vec{\alpha} \to \vec{\alpha} + \epsilon\vec{\gamma}_{l_r}$, where $\vec{\gamma}_{l_r} = [0, \ldots, \gamma_{l_r}, \ldots, 0]^T$ (note the $(kl+r)^{th}$ entry is γ_{l_r} and all other entries are 0's, e.g. we are perturbing the r^{th} entry of shaping vector α_l) and $\vec{\beta} = \vec{\beta} + \epsilon\vec{\delta}$, and $\epsilon << 1$, then $z \to z + \epsilon\eta$ is the resulting variation in $z(t)$ and $\tau_i \to \tau_i + \epsilon\theta_i$ for $i = 1, \ldots, M$ is the resulting variation in the switching times.

It can now be shown that the first order variation in $\tilde{J}$ is

$$\begin{aligned}
\delta\tilde{J} &= \lim_{\epsilon\to 0} \frac{\tilde{J}_\epsilon - \tilde{J}_0}{\epsilon} \\
&= \sum_{i=1}^{M} \int_{\tau_{i-1}+\epsilon\theta_{i-1}}^{\tau_i} \Big[\frac{\partial H_i}{\partial z}\eta + \frac{\partial H_i}{\partial \alpha_{l_r}}\gamma_{l_r} - \lambda_i \dot{\eta}\Big] dt + \\
&\quad + \sum_{i=1}^{M-1} \theta_i \Big[\lambda_{i+1}(\tau_i+)\big(f_i(\tau_i-) - f_{i+1}(\tau_i+)\big)\Big] + \\
&\quad\quad + \theta_M L(x(\tau_M), z(\tau_M)) + \\
&\quad + \sum_{i=1}^{M} \nu_i \Big[\frac{\partial\tilde{\xi}_i}{\partial z}\tilde{f}_i\theta_i + \frac{\partial\tilde{\xi}_i}{\partial z}\eta + \frac{\partial\tilde{\xi}_i}{\partial\vec{\beta}}\delta\Big]_{t=\tau_i} + \\
&\quad + \Big[\frac{\partial\psi}{\partial z}\tilde{f}_{M+1}\theta_M + \frac{\partial\psi}{\partial z}\eta\Big]_{t=\tau_M},
\end{aligned} \tag{11}$$

where $\tilde{J}_\epsilon$ denotes the perturbed cost.

The integral terms in (11), denoted by $\delta\chi$, can be further reduced by integrating $\lambda_i\dot{\eta}$ by parts to obtain

$$\begin{aligned}
\delta\chi &= \sum_{i=1}^{M} \int_{\tau_{i-1}+\epsilon\theta_{i-1}}^{\tau_i} \Big[\frac{\partial H_i}{\partial z} + \dot{\lambda}_i\Big]\eta dt + \\
&+ \sum_{i=1}^{M} \gamma_{l_r} \int_{\tau_{i-1}+\epsilon\theta_{i-1}}^{\tau_i} \frac{\partial H}{\partial \alpha_{l_r}} dt - \sum_{i=1}^{M} \Big[\lambda_i \eta\Big]_{\tau_{i-1}+\epsilon\theta_{i-1}}^{\tau_i}
\end{aligned} \tag{12}$$

Recall that $\theta_0 = 0$ since $\tau_0 = 0$ is fixed, and note that $\eta(0) = 0$ since $z(0) = x(0) = x_0$. Using the fact that $\eta(t)$ is continuous, (12) is reduced to

$$\begin{aligned}
\delta\chi &= \sum_{i=1}^{M} \int_{\tau_{i-1}+\epsilon\theta_{i-1}}^{\tau_i} \Big[\frac{\partial H_i}{\partial z} + \dot{\lambda}_i\Big]\eta dt + \\
&\quad + \sum_{i=1}^{M} \gamma_{l_r} \int_{\tau_{i-1}+\epsilon\theta_{i-1}}^{\tau_i} \frac{\partial \tilde{f}_i}{\partial \alpha_{l_r}} dt - \\
&\quad - \sum_{i=1}^{M-1} \big[\lambda_i(\tau_i-) - \lambda_{i+1}(\tau_i+)\big]\eta(\tau_i) - \\
&\quad - \lambda_M(\tau_M-)\eta(\tau_M-).
\end{aligned} \tag{13}$$

Substituting $\delta\chi$ back into $\delta\tilde{J}$, we want to select the co-states λ_is and ν_is so that we avoid having to calculate the variations η and θ_is. For the evolution of $\lambda(t)$, we get the expected (standard) result:

$$\dot{\lambda}_i = -\frac{\partial H_i}{\partial z} = -\frac{\partial L}{\partial z} - \lambda_i \frac{\partial \tilde{f}_i}{\partial z}. \tag{14}$$

The boundary conditions are, however, very different. It turns out that the co-state $\lambda(t)$ is discontinuous at the switching instants. To see this, lets first look at the variation $\eta(\tau_M-)$:

$$\begin{aligned}
&\eta(\tau_M-)\Big[-\lambda_M(\tau_M-) + \nu_M \frac{\partial\tilde{\xi}_M}{\partial z}(\tau_M-) + \\
&\quad\quad + \frac{\partial\psi}{\partial z}(\tau_M-)\Big] \equiv 0 \\
&\Longrightarrow \lambda_M(\tau_M) = \frac{\partial\psi}{\partial z}(\tau_M-) + \nu_M \frac{\partial\tilde{\xi}}{\partial z}(\tau_M-).
\end{aligned} \tag{15}$$

Similarly looking at the variation $\eta(\tau_i-)$ for $i = 1, \ldots, M-1$, the boundary conditions at the switching instants τ_i's are

$$\lambda_i(\tau_i-) = \lambda_{i+1}(\tau_i+) + \nu_i \frac{\partial\tilde{\xi}_i}{\partial z}(\tau_i-), \tag{16}$$

for $i = 1, \ldots, M-1$.

Using Equations (14),(15), and (16), $\lambda(t)$ can be solved by integrating backwards in time. However, the boundary conditions at τ_i depend on the costate ν_i. To see how we should select ν_i, let's first examine the variation θ_i, for $i = 1, \ldots, M-1$:

$$\begin{aligned}
&\theta_i\Big[\lambda_{i+1}(\tau_i+)\big(f_i(\tau_i-) - f_{i+1}(\tau_i+)\big) + \nu_i \frac{\partial\tilde{\xi}_i}{\partial z}\tilde{f}_i(\tau_i-)\Big] \equiv 0 \\
&\Longrightarrow \nu_i = -\frac{\lambda_{i+1}(\tau_i+)\big(f_i(\tau_i-) - f_{i+1}(\tau_i+)\big)}{L_{\tilde{f}_i}\frac{\partial\tilde{\xi}_i}{\partial z}(\tau_i-)},
\end{aligned} \tag{17}$$

where $L_{\tilde{f}_i}\frac{\partial\tilde{\xi}_i}{\partial z} = \frac{\partial\tilde{\xi}_i}{\partial z}\tilde{f}_i$ denotes the Lie derivative of $\tilde{\xi}_i$ with respect to z along the flow $\tilde{f}_i$. Similarly looking at θ_M, we select ν_M as

$$\nu_M = -\Big[\frac{L(x_f, z) + L_{\tilde{f}_M}\frac{\partial\psi}{\partial z}}{L_{\tilde{f}_M}\frac{\partial\tilde{\xi}_M}{\partial z}}\Big]_{t=\tau_M-}. \tag{18}$$

Note that all of the expressions derived above are well defined as long as $L_{\tilde{f}_i}\frac{\partial\tilde{\xi}_i}{\partial z}(\tau_i) \neq 0$ for $i = 1, \ldots, M$. Recall that this condition simply means that the trajectory of z does not approach the interrupt (or switching) surface tangentially.

With this choice of the co-states, (11) is reduced to

$$\delta\tilde{J} = \sum_{i=1}^{M} \gamma_{l_r} \int_{\tau_{i-1}}^{\tau_i} \frac{\partial \tilde{f}_i}{\partial \alpha_{l_r}} dt + \sum_{i=1}^{M} \nu_i \frac{\partial\tilde{\xi}_i}{\partial\vec{\beta}}\vec{\delta}. \tag{19}$$

Since the α_{l_r}s and $\vec{\beta}$ are independent, the necessary conditions are optimality (e.g. $\delta J = 0$) are

$$\frac{\partial J}{\partial \vec{\beta}} = \sum_{i=1}^{M} \nu_i \frac{\partial \tilde{\xi}_i}{\partial \vec{\beta}}(\tau_i-) \equiv 0, \text{ and} \tag{20}$$

$$\frac{\partial J}{\partial \alpha_{l_r}} = \sum_{i=1}^{M} \int_{\tau_{i-1}}^{\tau_i} \lambda_i \frac{\partial \tilde{f}_i}{\partial \alpha_{l_r}}\, dt \equiv 0, \quad \text{for } l = 1, \ldots, N, \text{ and } r = 1, \ldots, k. \tag{21}$$

Relating this back to our original problem formulation, the partial derivative $\frac{\partial \tilde{f}_i}{\partial z}$ is

$$\frac{\partial \tilde{f}_i}{\partial z} = \begin{cases} \frac{\partial f}{\partial z} + \frac{\partial f}{\partial u}\Big[\sum_{j=1}^{N}\big[g_j \frac{\partial \mu_j}{\partial z} + \mu_j \frac{\partial g_j}{\partial z}\big]\Big] & \text{if } p(i) = 1r \\ \frac{\partial f}{\partial z} & \text{otherwise.} \end{cases} \tag{22}$$

The partial derivative of $\tilde{f}_i$ with respect to the shaping vector is

$$\frac{\partial \tilde{f}_i}{\partial \alpha_{l_r}} = \frac{\partial f}{\partial u}\frac{\partial \mu_l}{\partial \alpha_{l_r}} g_l \tag{23}$$

if p(i) = 1r, and 0 otherwise. Also, the partial derivative of $\tilde{\xi}$ are

$$\frac{\partial \tilde{\xi}_i}{\partial z} = \frac{\partial \xi_{p(i)}}{\partial z}, \tag{24}$$

$$\frac{\partial \tilde{\xi}_i}{\partial \vec{\beta}} = \frac{\partial \xi_{12}}{\partial \beta} \text{ if } p(i) = 1r, \text{ and} \tag{25}$$

$$\frac{\partial \tilde{\xi}_i}{\partial \vec{\beta}} = 0 \text{ if } p(i) \neq 1r. \tag{26}$$

Hence, the necessary conditions for can be further reduced to

$$\frac{\partial J}{\partial \vec{\beta}} = \sum_{\{i \,\mid\, p(i)=1r\}} \nu_i \frac{\partial \xi_{1r}}{\partial \beta}(\tau_i-) \equiv 0, \text{ and} \tag{27}$$

$$\frac{\partial J}{\partial \alpha_{l_r}} = \sum_{\{i \,\mid\, p(i)=1r\}} \int_{\tau_{i-1}}^{\tau_i} \lambda_i \frac{\partial f}{\partial u}\frac{\partial \mu_l}{\partial \alpha_{l_r}} g_l\, dt \equiv 0, \quad \text{for } l = 1, \ldots, N, \text{ and } r = 1, \ldots, k. \tag{28}$$

These results are summarized in a theorem below:

Theorem 1 *Given a function $x(t) \in \mathbb{R}^n$ and a set of continuously differentiable functions $g_i : \mathbb{R}^n \to \mathbb{R}^m$ and $\mu_i : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ for $i = 1, 2, \ldots, N$, with $z(t) \in \mathbb{R}^n$ given by (4) and (2), an extremum to the cost function*

$$J = \int_0^T L(x(t), z(t))dt + \psi(x_f, z(T))$$

is attained when the control vectors $\vec{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T \in \mathbb{R}^{Nk}$ and $\vec{\beta} =\in \mathbb{R}^k$ are chosen as follows:
Euler-Lagrange Equations:

$$\dot{\lambda}(t) = -\frac{\partial H_i}{\partial z} = -\frac{\partial L}{\partial z}(x, z) - \lambda(t)\frac{\partial \tilde{f}_i}{\partial z}(z, \alpha_i), \quad \textit{when } t \in (\tau_{i-1}, \tau_i),$$

$$\nu_M = -\Big[\frac{L(x_f, z) + L_{\tilde{f}_M}\frac{\partial \psi}{\partial z}}{L_{\tilde{f}_M}\frac{\partial \tilde{\xi}_M}{\partial z}}\Big]_{t=\tau_M-},$$

$$\nu_i = \frac{\lambda_{i+1}(\tau_i+)\big(f_i(\tau_i-) - f_{i+1}(\tau_i+)\big)}{L_{\tilde{f}_i}\frac{\partial \tilde{\xi}_i}{\partial z}(\tau_i-)}, \quad \textit{for } i = 1, \ldots, M-1,$$

Boundary Conditions:

$$\lambda_M(\tau_M) = \frac{\partial \psi}{\partial z}(\tau_M-) + \nu_M \frac{\partial \tilde{\xi}}{\partial z}(\tau_M-),$$

$$\lambda_i(\tau_i-) = \lambda_{i+1}(\tau_i+) + \nu_i \frac{\partial \tilde{\xi}_i}{\partial z}(\tau_i-), \quad \textit{for } i = 1, \ldots, M-1,$$

Optimality Conditions:

$$\frac{\partial J}{\partial \vec{\beta}} = \sum_{\{i \,\mid\, p(i)=1r\}} \nu_i \frac{\partial \xi_{1r}}{\partial \beta}(\tau_i-) \equiv 0, \textit{ and}$$

$$\frac{\partial J}{\partial \alpha_{l_r}} = \sum_{\{i \,\mid\, p(i)=1r\}} \int_{\tau_{i-1}}^{\tau_i} \lambda_i \frac{\partial f}{\partial u}\frac{\partial \mu_l}{\partial \alpha_{l_r}} g_l\, dt \equiv 0, \quad \textit{for } l = 1, \ldots, N, \textit{ and } r = 1, \ldots, k.$$

4 Numerics

In the previous sections, we derived the necessary conditions for the extremum of the performance index J. Particulary, the control parameters included the shaping vectors α_i parameterizing membership function μ_i and the switching times τ_i, defined through the control vector β for parameterizing the interrupt ξ_i. In this section, we present a numerical algorithm that utilizes these optimality conditions to converge to a stationary solution. This algorithm employs a gradient descent method, in which, the control parameters are updated in the negative gradient direction until a stationary solution has been reached.

- Initialize with a guess of the control variables $\vec{\alpha}_i^{(0)}$ and $\vec{\tau}^{(0)}$, and let $p := 0$.
- `while` $p < 2$ `or` $|J^{(p)} - J^{(p-1)}| > \epsilon$
  - Compute the approximation function $z(t)$ and cost $J^{(p)}$ forward in time from τ_0 to $\tau_M^{(p)}$.
  - Compute the co-state $\lambda(t)$ backward in time from $\tau_M^{(p)}$ to τ_0.
  - Compute the gradients $\nabla J(\vec{\tau}^{(p)})$, and $\nabla J(\vec{\alpha}_i^{(p)})$.
  - Update the control variables as follow :

$$\vec{\tau}^{(p+1)} := \vec{\tau}^{(p)} - \gamma^{(p)} \nabla J(\vec{\tau}^{(p)}),$$
$$\vec{\alpha}^{(p+1)} := \vec{\alpha}^{(p)} - \gamma^{(p)} \nabla J(\vec{\alpha}^{(p)}).$$

  - $p := p + 1$.
- `end while`

In the algorithm above, the control vectors are $\vec{\tau} = [\tau_1, \ldots, \tau_M]^T$ and $\vec{\alpha} = [\alpha_1, \ldots, \alpha_M]^T$. The gradients $\nabla J(\vec{\tau}) = [\frac{\partial J}{\partial \tau_1}, \ldots, \frac{\partial J}{\partial \tau_M}]^T$ and $\nabla J(\vec{\alpha}) = [\frac{\partial J}{\partial \alpha_1}, \ldots, \frac{\partial J}{\partial \alpha_N}]^T$.

Note that the choice of the step-size $\gamma^{(p)}$ can be critical for the method to converge. An efficient method among others is the use of Armijo's algorithm presented in [2]. Because of the non-convex nature of the cost function J, this gradient descent algorithm will only converge to a local minimum. Hence the attainment of a "good" local minimum can be quite dependent on the choice of a "good" initial guess for the control variables. The association of such a local method with heuristic strategies in order to find a global minimum is not investigated here.

5 Conclusions

When humans acquire new motor skills, they are typically obtained from a combination of previously established skills. This observation constituted the starting point for this work that addressed the problem of adaptively augmenting the motion alphabet to improve the overall performance of the system. New modes are systematically constructed to replace recurring mode fragments, thus resulting in a possible reduction in the complexity of the control programs while increasing the expressiveness of the motion alphabet. The construction of the new modes involved designing a feedback control law, which was designed as a combination of the already established control laws, and the design of an interrupt. We explicitly addressed the design of the interrupt for both time-driven and event-driven systems, and designed the new interrupt by incrementally adapting a previously established interrupt. In particular, the mode augmentation problem was casted as an optimal control problem and solved using variational arguments.

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