Automatica 47 (2011) 1035–1040

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Containment in leader–follower networks with switching communication topologies[☆]

G. Notarstefano[a,*], M. Egerstedt[b], M. Haque[b]

[a] Department of Engineering, University of Lecce, Via per Monteroni, 73100 Lecce, Italy

[b] School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA

ARTICLE INFO

Article history:
Available online 5 March 2011

Keywords:
Multi-agent coordination
Leader–follower networks
Containment

ABSTRACT

We study bipartite, first-order networks where the nodes take on leader or follower roles. Specifically, we let the leaders' positions be static and assume that leaders and followers communicate via an undirected switching graph topology. This assumption is inspired by the swarming behavior of silkworm moths, where female moths intermittently release pheromones to be detected by the males. The main result presented here states that if the followers execute the linear agreement protocol, they will converge to the convex hull spanned by the leaders' positions as long as the time-varying undirected graph defining the communication among all agents is jointly connected. The novelty of this research is that we use LaSalle's Invariance Principle for switched systems, and additionally, the result is shown to hold for arbitrary state dimensions.

1. Introduction

The research on multi-agent robotics and decentralized, networked control has drawn significant inspiration from interaction rules in social animals and insects (Couzin & Franks, 2003; Gazi & Passino, 2004; Grünbaum, Viscido & Parrish, 2004). In particular, the widely used nearest-neighbor-based interaction rules, used for example for formation control (e.g. Ji, Muhammad, and Egerstedt (2006) and Olfati-Saber (2006)), consensus (e.g. Jadbabaie, Lin, and Morse (2003) and Ren and Beard (2004)), and coverage control (Cortes, Martinez, & Bullo, 2006; McNew, Klavins, & Egerstedt, 2007), have direct biological counterparts, as pointed out in Couzin and Franks (2003). In this paper, we follow this line of inquiry by seeing if we can understand how leader–follower systems behave if: (i) the leaders are only intermittently visible to the followers, and (ii) the agents interact over a switching network topology. This model is inspired by a particular swarming phenomenon observed in the silkworm moth *Bombyx Mori*.

Silkworm moths are known to swarm in tight geometrical configurations, such as vertical cylindrical structures. This is caused by the females' intermittent releasing of a pheromone – *bombykol* – to attract male moths, and by the males' mutual attraction to determine each other's gender through visual inspection. These two phenomena in essence make the females act as attractors to the males, but the intermittent nature of the release and of the individuals' interactions produces an inherently switched system. Moreover, the spatial distribution of the females imply that the males are attracted to a general area rather than to a particular point, which is what is believed to cause their characteristic swarming geometry (see, e.g., Hummel and Miller (1984), Pasteels and Deneubourg (1987), Thornhill and Alcock (1983) and Wheeler (1923)).

Based on this discussion, in this paper we investigate a first-order network model in which stationary leaders (the female moths) and moving followers (the males) are only intermittently visible to each other. This corresponds to applying a switched control input of varying dimension (since the number of communicating agents may be changing) to the system. Our main result is that, asymptotically, the followers will end up in the convex hull spanned by *all* the leaders' positions. For the case in which the leaders are always visible and no edges appear or disappear between followers, this is already known Ferrari-Trecate, Egerstedt, Buffa, and Ji (2006). Along a similar line of inquiry, rendezvous in switching directed networks with at most one leader has been studied in Moreau (2004).

☆ The research leading to these results has received funding from the European Community's Seventh Framework Programme (FP7/2007-2013) under grant agreement no. 224428, CHAT, and no. 231378, CO3AUV, (G. Notarstefano) and from the US Office of Naval Research's HUNT MURI project (M. Egerstedt and M. Haque). The material in this paper was partially presented at the 48th IEEE Conf. on Decision and Control, December 16–18, 2009, Shanghai, China. This paper was recommended for publication in revised form by Associate Editor Xiaobo Tan under the direction of Editor Miroslav Krstic. Preliminary versions of this work with partial results were presented in Notarstefano, Egerstedt, and Haque (2009) and Haque, Egerstedt, and Martin (2008).

* Corresponding author.
E-mail addresses: giuseppe.notarstefano@unile.it (G. Notarstefano), magnus@ece.gatech.edu (M. Egerstedt), musad.haque@ece.gatech.edu (M. Haque).

0005-1098/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2011.01.077

The main contribution of this paper is a containment result for switched networks with intermittently visible, static leaders. Using tools from hybrid stability theory, namely a hybrid version of LaSalle's Invariance Principle Mancilla-Aguilar and García (2006), we show that the convex hull of leaders' and followers' positions shrinks to the convex hull of the leaders' fixed positions, which is proven to be the largest invariant set for the followers' positions. Preliminary results were provided in Haque et al. (2008), where it was shown (under stronger assumptions) that the followers end up in a larger ellipsoidal set that contains the convex hull of the leaders' positions, and in Notarstefano et al. (2009), where containment under fixed interaction topologies was studied. A similar question to the one under consideration here was pursued in Cao and Ren (2009), where the containment problem was studied for systems with scalar dynamics; whereas in this paper, the result is proven for arbitrary state dimensions. Furthermore, the LaSalle-based approach used here is different from Cao and Ren (2009), which has the advantage of being directly applicable to non-scalar systems.

The outline of the paper is as follows: We next establish some of the basic notation that will be used in the paper. We then, in Section 2, recall the switched version of LaSalle's Invariance Principle, followed by a discussion of the underlying network model in Section 3 and the static case, in Section 4. The main result for switched systems is given in Section 5, followed by a simulation study in Section 6.

Notation. We let $\mathbb{N}$ and $\mathbb{R}_{\geq 0}$ denote the natural numbers and the nonnegative real numbers, respectively. Given the sets M, M_1 and M_2 such that $M \subset M_1 \times M_2$, we denote by $\pi_1(M)$ (respectively $\pi_2(M)$) the projection of M on M_1 (respectively M_2), i.e. $\pi_1(M_1 \times M_2) = M_1$ and $\pi_2(M_1 \times M_2) = M_2$. We denote by $\mathbf{1}_d$, $d \in \mathbb{N}$, the vector of dimension d with all entries equal to 1 (e.g. $\mathbf{1}_2 = [1\ 1]^T$). Given a vector $v \in \mathbb{R}^d$, $d \in \mathbb{N}$, and a set $M \subset \mathbb{R}^d$, we denote dist (v, M) the distance between v and M, that is, dist $(v, M) = \inf_{w \in M} \|v - w\|_2$, where $\|\cdot\|_2$ is the two norm.

2. LaSalle's Invariance Principle for switched systems

In this section, we recall LaSalle's Invariance Principle for switched systems proved in Mancilla-Aguilar and García (2006) that will be useful to prove our main result. For the sake of clarity, we will not use the most general assumptions used in the paper, but we will impose stronger assumptions that are verified by our problem formulation.

Given a parameterized family of locally Lipschitz vector fields $\{f_\gamma : \mathbb{R}^n \to \mathbb{R}^n \mid \gamma \in \Gamma\}$, where Γ is a finite index set, we consider the switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \tag{1}$$

where $\sigma : \mathbb{R}_{\geq 0} \to \Gamma$ is a piecewise constant (continuous from the right) switching signal.

Let $\mathscr{S}$ be the set of all switching signals. A pair $(x(\cdot), \sigma(\cdot))$ is a *trajectory* of (1) if and only if $\sigma(\cdot) \in \mathscr{S}$ and $x : [0, T) \to \mathbb{R}^n$, $0 < T \leq +\infty$, is a piecewise differential solution to $\dot{x}(t) = f_{\sigma(t)}(x(t))$, $t \in [0, T)$. Note that T is, in general, a function of $x(0)$ and $\sigma(\cdot)$ so that we should write $T(x(0), \sigma(\cdot))$.

In the following we will consider switching signals that have positive average dwell-time, i.e. signals for which the number of discontinuities in any open interval is bounded above by the length of the interval normalized by an "average dwell-time" plus a "chatter bound".

More formally, we say that a switching signal $\sigma(\cdot)$ has an average dwell-time $\tau_D > 0$ and a chatter bound $N_0 \in \mathbb{N}$ if the number of its switching times in any open interval $(\tau_1, \tau_2) \subset \mathbb{R}_{\geq 0}$ is bounded by $N_0 + (\tau_2 - \tau_1)/\tau_D$. We denote by $\mathscr{S}_a[\tau_D, N_0]$ the set of all switching signals with average dwell-time τ_D and chatter bound N_0, and by $\mathscr{T}_a[\tau_D, N_0]$ the subclass of all trajectories of (1) corresponding to some $\sigma(\cdot) \in \mathscr{S}_a[\tau_D, N_0]$. Also, we let

$$\mathscr{S}_a = \cup_{\tau_D > 0,\, N_0 \in \mathbb{N}} \mathscr{S}_a[\tau_D, N_0],$$

and consequently, we let $\mathscr{T}_a$ be the corresponding subclass of trajectories.

In order to deal with LaSalle's Invariance Principle it is useful, following Mancilla-Aguilar and García (2006), to introduce the following subclasses of trajectories.

Definition 2.1 (*Class of Trajectories $\mathscr{T}_V$*)**.** Let $V : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. $\mathscr{T}_V$ is the class of trajectories $(x(\cdot), \sigma(\cdot)) \in \mathscr{T}$ which verify the conditions:

(i) $x(t) \in \Omega$ for all $t \in [0, T)$;
(ii) for any pair of times $t, t' \in [0, T)$ such that $t \leq t'$ and $\sigma(t) = \sigma(t')$, then $V(x(t), \sigma(t)) \geq V(x(t'), \sigma(t'))$.

$\mathscr{T}_V^*$ is the subfamily of $(x(\cdot), \sigma(\cdot)) \in \mathscr{T}_V$ verifying $V(x(t), \sigma(t)) = V(x(t'), \sigma(t'))$ for $\sigma(t) = \sigma(t')$. □

Then, we introduce a suitable notion of a weakly invariant set.

Definition 2.2 (*Weakly Invariant Set*)**.** Given a family $\mathscr{T}'$ of trajectories of (1), a non-empty subset $M \subset \mathbb{R}^n \times \Gamma$ is said to be weakly invariant with respect to $\mathscr{T}'$ if, for each $(\xi, \gamma) \in M$, there is a trajectory $(x(\cdot), \sigma(\cdot)) \in \mathscr{T}'$ such that $x(0) = \xi$, $\sigma(0) = \gamma$ and $(x(t), \sigma(t)) \in M$ for all $t \in [0, T)$. □

We are now ready to state (a slightly modified version of) LaSalle's Invariance Principle proved in Mancilla-Aguilar and García (2006) (Theorem 2.4).

Theorem 2.1 (*LaSalle's IP for Switched Systems, (Mancilla-Aguilar & García, 2006)*)**.** *Let $V : \Omega \times \Gamma \to \mathbb{R}$, with Ω an open subset of $\mathbb{R}^n$, be continuous. Suppose that $(x(\cdot), \sigma(\cdot))$ is a trajectory belonging to $\mathscr{T}_V \cap \mathscr{T}_a[\tau_D, N_0]$ for some $\tau_D > 0$ and $N_0 \in \mathbb{N}$, such that for some compact subset $B \subset \Omega$, $x(t) \in B$ for all $t \geq 0$. Let $M \subset \mathbb{R}^n \times \Gamma$ be the largest weakly invariant set with respect to $\mathscr{T}_V^* \cap \mathscr{T}_a[\tau_D, N_0]$ contained in $\Omega \times \Gamma$. Then $x(t)$ converges to $\pi_1(M)$ as $t \to \infty$.* □

3. Network model

In this section, we introduce a mathematical model, based on the model in Haque et al. (2008), that describes the swarming behavior encountered among the silkworm moths. Informally, we consider a network with agents of two sorts: leaders (representing the female moths) and followers (representing the males). Leaders and followers are both described as first-order integrators, but they apply different control laws. In this paper we assume the leaders to be stationary, that is, their control input is identically zero. Also, we assume that they may be active or inactive, equivalently visible or invisible to the followers. The followers apply a Laplacian-based averaging control law. They communicate among themselves and with active leaders according to a switching undirected communication graph.

More formally, we consider a network of agents labeled by a set of identifiers $\{1, \ldots, n\}$, $n \in \mathbb{N}$, such that the labels $\{1, \ldots, n^f\}$, $n^f \in \mathbb{N}$, correspond to the followers and the remaining ones to the leaders. The agents live in state space $\mathbb{R}^d$, $d \in \mathbb{N}$, and obey first order, continuous time dynamics, that is, $\dot{x}_i = u_i$, for all $i \in \{1, \ldots, n\}$, where $x_i \in \mathbb{R}^d$ and $u_i \in \mathbb{R}^d$ are respectively the state and the input of agent i. In order to distinguish between follower and leader dynamics, we will use the notation x_i^f and x_j^l for the states of follower i and leader j, respectively. It is worth noting that the dynamics are decoupled; thus along each direction the dynamics is exactly the dynamics of a system with $d = 1$.

The agents communicate according to a switching undirected communication graph. Formally, we let $\{1,\ldots,n\}$ be the set of nodes of the graph and $\sigma:\mathbb{R}_{\geq 0}\to\Gamma:=\{0,1\}^{2^n}$ be a switching signal with positive average dwell-time, that is $\sigma(\cdot)\in\mathscr{T}_a[\tau_D,N_0]$ for some $\tau_D>0$ and $N_0\in\mathbb{N}$. The communication graph $G_{\sigma(\cdot)}=(\{1,\ldots,n\},E_{\sigma(\cdot)})$ is defined as follows. An edge $(i,j)\in\{1,\ldots,n\}\times\{1,\ldots,n\}$ belongs to $E_{\sigma(t)}$ if agents i and j communicate at time t. For any admissible $\sigma(\cdot)$, we assume that the graph $G_{\sigma(\cdot)}$ is jointly connected. That is, let $t_k, k\in\mathbb{N}$, denote the kth switching time of $\sigma(\cdot)$ greater than or equal to a given time $t_0\in\mathbb{R}_{\geq 0}$, we assume that, for any $t_0\in\mathbb{R}_{\geq 0}$, $\cup_{k\in\mathbb{N}}G_{\sigma(t_k)}$ is connected.

We let $N_i(t)$ be the set of neighbors of follower i. If the graph is fixed the set of neighbors does not depend on time, thus we will denote it by simply N_i.

The dynamics of the followers is given by

$$\dot{x}_i^f(t)=-\sum_{j\in N_i(t)}(x_i^f(t)-x_j(t)),\quad i\in\{1,\ldots,n^f\}.$$

It is useful to highlight the contribution of neighboring leaders and followers separately. Let $N_i^f(t)$ and $N_i^l(t)$ respectively be the set of followers and leaders communicating with follower i at instant t. Thus, the follower dynamics can be rewritten as

$$\dot{x}_i^f(t)=-\sum_{j\in N_i^f(t)}(x_i^f(t)-x_j^f(t))-\sum_{j\in N_i^l(t)}(x_i^f(t)-x_j^l(t)),$$

for $i\in\{1,\ldots,n^f\}$.

The leaders are stationary, that is, their dynamics is simply

$$\dot{x}_i^l(t)=0,\quad i\in\{n^f+1,\ldots,n\}.$$

Next, we introduce some compact notation to write the dynamics along each direction in $\mathbb{R}^d$. We recall that the dynamics along different directions are decoupled and coincide with the dynamics of the scalar system ($d=1$). In order to avoid the introduction of extra indices, we present such a compact form assuming that $d=1$, that is, $x_i\in\mathbb{R}$ for all $i\in\{1,\ldots,n\}$. It is well known that for the scalar case the state matrix of the linear system obtained by stacking all the agents' states is the graph Laplacian. Recall that, for the undirected graph $G_\gamma=(\{1,\ldots,n\},E_\gamma)$ the Laplacian matrix $L_\gamma:=(\ell_{i,j})_{n\times n}$ is defined as:

$$\ell_{i,j}:=\begin{cases}\deg(i) & \text{if } i=j\\ -1 & \text{if } i\neq j \text{ and } (i,j)\in E_\gamma\\ 0 & \text{otherwise,}\end{cases}$$

where $\deg(i)$ is the degree of node i (i.e., the number nodes sharing an edge with node i). Although leaders do not apply a "Laplacian control law" (as followers do), it is useful to consider the dynamics obtained as if all the agents (both leaders and followers) did. Indeed, the dynamics would be

$$\dot{x}(t)=-L_{\sigma(t)}x(t),$$

where $x(t)=[x_1(t),\ldots,x_n(t)]^T$ and $L_{\sigma(t)}$ is the Laplacian of the graph $G_{\sigma(t)}$ at instant t.

If we partition the Laplacian with respect to leaders and followers as

$$L_{\sigma(t)}=\begin{bmatrix}L_{\sigma(t)}^f & l_{\sigma(t)}^{fl}\\ l_{\sigma(t)}^{lf} & L_{\sigma(t)}^l\end{bmatrix}$$

the followers dynamics turns out to be

$$\dot{x}^f(t)=-L_{\sigma(t)}^f x^f(t)-l_{\sigma(t)}^{fl}x^l,\tag{2}$$

where $x^f(t)=[x_1^f(t),\ldots,x_{n^f}^f(t)]^T$ is the vector of follower positions at time t and $x^l=[x_{n^f+1}^l,\ldots,x_n^l]^T$ is the constant vector of leaders positions. It is worth noting that $L_{\sigma(t)}^f$ is not the follower's Laplacian, but depends on active leaders as well. However, it can be written in terms of the Laplacian of the follower's subgraph, $L_0^f(t)$, as

$$L_{\sigma(t)}^f=L_0^f(t)+D^l(t)$$

where $D^l(t)$ is a diagonal matrix whose entries are the degrees of the followers with respect to the active leaders only.

4. Problem statement and static case

Before stating and proving the main result, i.e. the followers communicating according to a switching graph end up in the convex hull spanned by the static and only intermittently visible leaders, we first need to investigate and recall what happens under static network topologies.

Next, we prove two lemmas that are useful to prove the convergence result for fixed topology (see for example Ji et al. (2006) for other versions of the proofs). The results of the next two lemmas deal with the dynamics along each direction. As in the previous section, rather than overloading the notation with extra indices, we just state the result for $d=1$.

Lemma 4.1. *If the graph is connected, then L^f is positive definite.* □

Proof. We know that L is positive semi-definite, $L\succeq 0$. In addition, if the graph is connected, we have that null $(L)=\operatorname{span}\{\mathbf{1}_n\}$. Since

$${x^f}^T L^f x^f=[{x^f}^T\ 0]L\begin{bmatrix}x^f\\0\end{bmatrix}$$

and $[{x^f}^T\ 0]^T\notin\operatorname{null}(L)$, we have that

$$[{x^f}^T\ 0]L\begin{bmatrix}x^f\\0\end{bmatrix}>0\quad\forall x^f\neq 0.\quad\square$$

This lemma allows us to state the following lemma (also available in Ji et al. (2006)).

Lemma 4.2. *Given fixed leader positions x^l, then*

$${x^f}_{eq}=-{L^f}^{-1}L^{fl}x^l\tag{3}$$

is a globally asymptotically stable equilibrium point.

Proof. The proof follows directly by the fact that L^f is invertible. □

We are now ready to recall the result from Ferrari-Trecate et al. (2006) (formulated in a slightly different way) stating that for a leader–follower network with fixed topology, the followers' positions will converge to the convex hull of the leaders' positions. We provide a different and simpler proof. We stress the fact that the result holds (and is proven) for arbitrary dimension $d\in\mathbb{N}$.

Lemma 4.3 (*Containment for a Static Topology*)**.** *Given a connected, static network topology with multiple static leaders, the followers will asymptotically end up in the convex hull, Ω_L, spanned by the leaders' positions, i.e.*

$$x_{i,\mathrm{eq}}^f\in\Omega_L,\quad i=1,\ldots,n^f.$$

Proof. As a result of Lemma 4.2, we have that if the leaders are stationary (located at x_i^l, $i\in\{1,\ldots,n\}$), the followers will asymptotically approach the equilibrium point ${x^f}_{\mathrm{eq}}$ whose component along each direction can be computed by using the scalar expression in (3).

Now, since ${x^f}_{\mathrm{eq}}$ is an equilibrium, we must have that

$$\dot{x}_{i,\mathrm{eq}}^f=0=-\sum_{j\in N_i}(x_{i,\mathrm{eq}}^f-x_{j,\mathrm{eq}})$$

for all follower agents. (Here we have used the notation that if agent j is a leader, $x_{\mathrm{j,\,eq}}$ is the static position of that leader.) This means that

$$x^f{}_{\mathrm{i,\,eq}} = \frac{1}{|N_i|}\sum_{j\in N_i} x_{j,\ \mathrm{eq}}.$$

In other words, the equilibrium point $x^f_{i,\mathrm{eq}}$ for follower agent i lies in the convex hull spanned by agent i's neighbors—may they be leaders or followers.

Now, if every follower ends up in the convex hull spanned by its neighbors' positions, and the only agents who do not need to satisfy this condition are the leaders, every follower will end up in the convex hull spanned by the leaders' positions. □

Remark 4.1. Two straightforward results follow from the previous lemma and are well known results in the consensus literature; see, e.g., Moreau (2004). First, the convex hull of the followers' positions is a decreasing function of time. Second, if there are no leaders, then the agents rendezvous at a common point. □

It should be pointed out again that the main result presented in this section is previously known. For the remainder of the paper, we extend Lemma 4.3 to hold also for a switching topology, which thus constitutes the main contribution in the paper.

5. Containment under switching topology

In this section, we prove the main result of the paper, i.e. in a leader–follower network with switching topology the followers asymptotically converge to the convex hull spanned by the all stationary leaders' positions.

Lemma 5.1 (*Boundedness of Followers' Trajectories*)**.** *Consider a leader–follower first-order network as in Section 3 (with stationary leaders and followers' dynamics as in (2)). Suppose that for any $\sigma(\cdot)\in\mathscr{S}_a[\tau_D,N_0]$, $\tau_D>0$ and $N_0\in\mathbb{N}$, the communication graph $G_{\sigma(\cdot)}$ is jointly connected. Then, for any $x^f(0)\in\mathbb{R}^{dn^f}$, there exists a compact set $B\subset\mathbb{R}^{dn^f}$ such that $x^f(t)\in B$ for all $t\geq 0$.*

Proof. Regardless of the connectivity, each follower executes

$$\dot{x}^f_i(t) = -\sum_{j\in N_i(t)}(x^f_i(t)-x_j(t)),\quad i\in\{1,\ldots,n^f\},$$

which means that at each time t, x^f_i moves towards the convex hull of the agents in its neighborhood set $N_i(t)$, which was shown, for example, in Ferrari-Trecate et al. (2006). We call this the convex-hull-seeking property.

Now, let $\Omega(t)$ be the convex hull spanned by all the leaders' (active as well as inactive at time t) and followers' positions. We will show that the volume of $\Omega(t)$ is uniformly non-increasing and thus that $\Omega(0)$ will serve as the compact set B in which the followers are uniformly confined. In fact, the only way an agent can increase the volume of $\Omega(t)$ is by being placed on the boundary of the convex hull, $\partial\Omega(t)$, and moving away from $\Omega(t)$, which is contradicted by the convex-hull-seeking property. As such, $|\Omega(t)|$ never increases, and $B=\Omega(0)$ above, which concludes the proof. □

We are now ready to state the main result.

Theorem 5.1 (*Containment for Switching Topology*)**.** *Consider a leader–follower first-order network as in Section 3 (with stationary leaders and follower dynamics as in (2)). Suppose that for any $\sigma(\cdot)\in\mathscr{S}_a[\tau_D,N_0]$, $\tau_D>0$ and $N_0\in\mathbb{N}$, the communication graph $G_{\sigma(\cdot)}$ is jointly connected. Let Ω_L be the convex hull spanned by all the leaders' positions. Then, each follower asymptotically converges to Ω_L. In other words, for any $\epsilon>0$, there exists $\bar{t}>0$ such that, for any $j\in\{1,\ldots,n^f\}$,*

$$\mathit{dist}\ (x^f_j(t),\Omega_L)<\epsilon$$

for all $t\geq\bar{t}$.

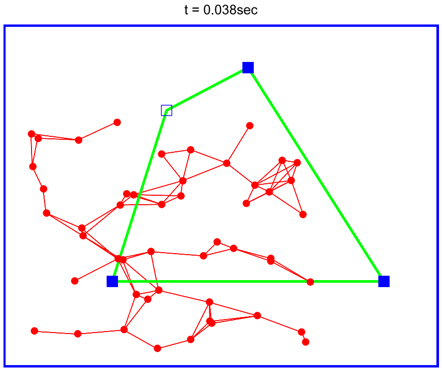
Proof. We prove the result by using the LaSalle's invariant principle stated in Theorem 2.1. First of all, observe that from Lemma 5.1 there exists a compact set $B\subset\mathbb{R}^{dn^f}$ such that for any $(x^f(\cdot),\sigma(\cdot))\in\mathscr{T}_a[\tau_D,N_0]$, $x(t)\in B$ for all $t\geq 0$. Next, let $V(x^f,\gamma)$ be the volume of the convex hull of the agents (leaders and followers) for any value of γ. Notice that, since the leaders are stationary, the volume is only a function of the followers positions, while the leaders' positions can be considered as fixed parameters. First, we show that V is non-increasing between two switching intervals for all followers' trajectories, i.e., we show that $\mathscr{T}_V=\mathscr{T}_a[\tau_D,N_0]$. Second, we prove that the set $M=(\Omega_L)^{n^f}\times\Gamma$ is the largest weakly invariant set for the family of trajectories $\mathscr{T}_V^*$, i.e., the subfamily of trajectories in $\mathscr{T}_V$ for which V is constant between two switching intervals.

To prove that V is non-increasing between two switching intervals, consider the agents (them being leaders or followers) inside the convex hull and the ones on it. Now, between two switching intervals each agent is connected to other agents (possibly none) via a connected component of the graph $G_{\sigma(t_k)}$. By the results proven in the previous section, we know that each follower has two possibilities to evolve. It will either evolve so as to shrink the convex hull of the agents in the same connected components of $G_{\sigma(t_k)}$ or remain stationary if it is not connected to anyone. Also, we know that leaders are stationary. Therefore, if an agent (leader or follower) is inside the convex hull it will remain inside or become part of the boundary because the convex hull is shrinking. If it is outside it will either remain stationary (it is a leader or a disconnected follower) or it will contribute to shrink the convex hull. Therefore the volume of the convex hull cannot increase.

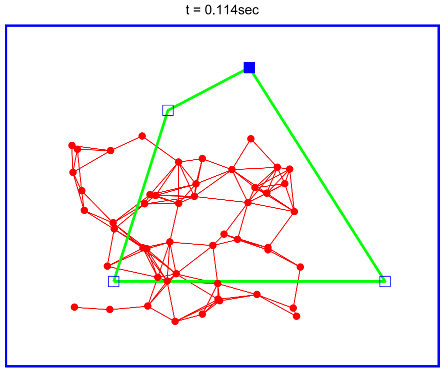
Now, we need to prove that M is the largest weakly invariant set for $\mathscr{T}_V^*$. Clearly, M is weakly invariant for $\mathscr{T}_V^*$. Indeed, take any $(\xi,\gamma)\in M$, with $\xi_i=x^f_i(0)\in\Omega_L$ for all $i\in\{1,\ldots,n^f\}$ and G_γ the initial (possibly disconnected) communication graph. Using the results for the static case, it follows easily that for $\sigma(\cdot)\equiv\gamma$ each follower trajectory remains in Ω_L so that $(x(t),\sigma(t))\in M$ for all $t\geq 0$ with $(x(\cdot),\sigma(\cdot))\in\mathscr{T}_V^*$. To prove that M is the largest weakly invariant set, let, by contradiction, $M'\supset M$ be a larger weakly invariant set. Since M' is weakly invariant with respect to $\mathscr{T}_V^*$, then for any $(x^f(0),\gamma)\in M'$ there exists a trajectory $(x^f(\cdot),\sigma(\cdot))$ such that the volume of $\pi_1(M')$ stays constant and $x^f(t)\in\pi_1(M')$ for all $t\geq 0$. Now, since $M'\supset M$ there exists $i\in\{1,\ldots,n^f\}$ such that $x^f_i(0)\notin\Omega_L$ and $x^f_i(0)$ on the boundary of $\pi_1(M')$. The contradiction follows by the joint connectivity of $G_{\sigma(\cdot)}$. Indeed, the only way for the volume of $\pi_1(M')$ to remain constant is that $x^f_i(t)=x^f_i(0)$ for all t. But, from the previous arguments we know that for this to happen agent i must be isolated (not connected to any other agent) for all t. This gives the contradiction and concludes the proof. □

Remark 5.1 (*Extensions of the Main Result*)**.** The convex hull of any subgroup of the leaders' positions is a subset of the entire convex hull. Thus, our result remains true under the milder assumption that each follower is connected to a leader. Also, if starting from a given time instant a subgroup of the leaders remains disconnected from the followers, then the followers will converge to the convex hull of the remaining leaders' positions. □

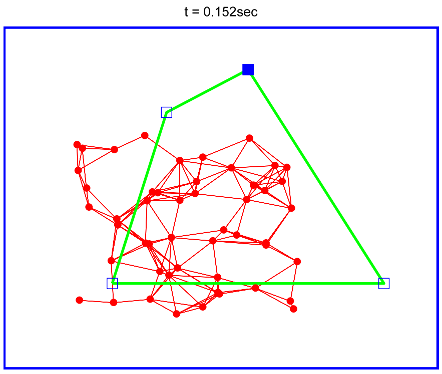
Remark 5.2 (*Average Dwell-Time Assumption*)**.** The assumption that the communication graph switches according to a signal with bounded average dwell-time is introduced for the sake of analytical treatment. How to provide bounds on dwell-time and



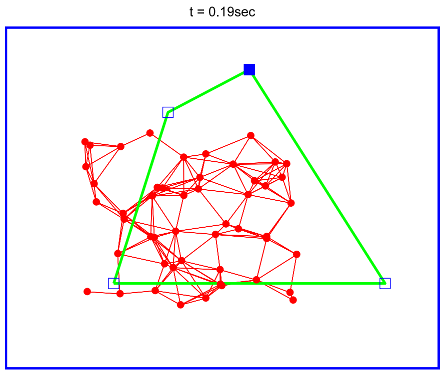
(a) Three leaders are visible to the followers.



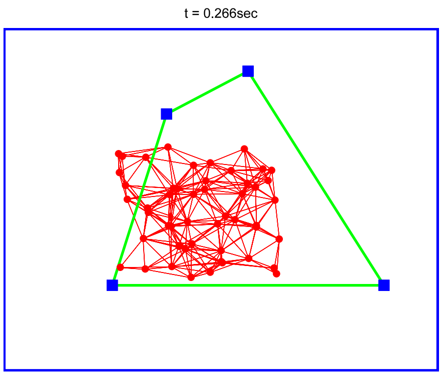
(b) One leader is visible to the followers.



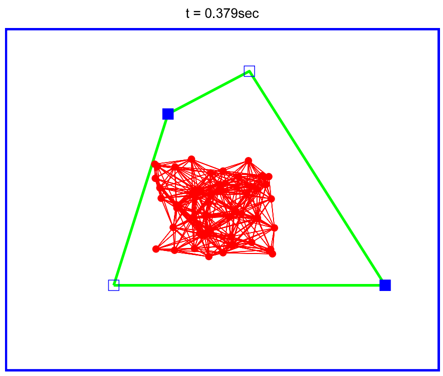
(c) One leader is visible to the followers.



(d) One leader is visible to the followers.



(e) Four leaders are visible to the followers.



(f) Two leaders are visible to the followers.

Fig. 1. Simulation of the followers agents (dots) converging to the convex hull spanned by the static leaders agents (squares). The convex hull is shown by the line segments connecting the leader agents, while lines between follower agents denote edges. Also, a leader agent visible to the follower agents in the network is depicted as a filled-in box..

chatter bound in real practice on the basis of biological data is an interesting issue. Notice, however, that, to prove our main result we do not need to know these bounds. □

6. Simulations

We simulate the leader–follower network using 50 follower agents (dots) and 4 leader agents (squares), as shown in Fig. 1. Leaders from this network that influence all the followers are selected at random. The simulation illustrates the fact that the followers in the network converge to locations inside the convex hull spanned by the static leader agents.

7. Conclusions

In this paper we studied leader–follower first-order networks. We showed that the subset of follower agents converge to the convex hull spanned by the positions of the stationary leader agents. This is the case even if leaders and followers communicate only intermittently. The main result in this paper relies on recent advances in the switched LaSalle's Invariance Principle, and it can help explain the swarming behaviors observed in the silkworm moth, where the male moths are attracted to the female moths that only intermittently release pheromones. Future directions of this work include the analysis of the convergence rate of the followers to the convex hull and more complex models for the system dynamics that allow us to capture other interesting phenomena observed in the silkworm moth.

References

Cao, Y., & Ren, W. (2009). Containment control with multiple stationary or dynamic leaders under a directed interaction graph. In *IEEE conference on decision and control*. China: Shanghai.

Cortes, J., Martinez, S., & Bullo, F. (2006). Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions. *IEEE Transactions on Automatic Control*, *51*(8), 1289–1298.

Couzin, I. D., & Franks, N. R. (2003). Self-organized lane formation and optimized traffic flow in army ants. *Proceedings of the Royal Society of London, Series B*, *270*, 139–146.

Ferrari-Trecate, G., Egerstedt, M., Buffa, A., & Ji, M. (2006). Laplacian sheep: a hybrid, stop–go policy for leader-based containment control. In *Hybrid systems: computation and control* (pp. 212–226). Santa Barbara, CA: Springer-Verlag.

Gazi, V., & Passino, K. M. (2004). A class of attraction/repulsion functions for stable swarm aggregations. *International Journal of Control*, *77*(18), 1567–1579.

Haque, M.A., Egerstedt, M., & Martin, C. (June 2008). First-order, networked control models of swarming silkworm moths. In *American control conference*. Seattle, WA.

Hummel, H. E., & Miller, T. A. (1984). *Techniques in pheromone research*. New York, NY: Springer-Verlag.

Jadbabaie, A., Lin, J., & Morse, A. S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transaction on Automatic Control*, *48*(6), 9881001.

Ji, M., Muhammad, A., & Egerstedt, M. (June 2006). Leader-based multi-agent coordination: controllability and optimal control. In *American control conference* (pp. 1358–1363). Minneapolis, MN.

Mancilla-Aguilar, J., & García, R. (2006). An extension of Lasalle's invariance principle for switched systems. *Systems and Control Letters*, *vol. 55*, 376–384.

McNew, J. M., Klavins, E., & Egerstedt, M. (2007). Solving coverage problems with embedded graph grammars. In *Hybrid systems: computation and control* (pp. 413–427). Pisa, Italy: Springer-Verlag.

Moreau, L. (Dec 2004). Stability of continuous-time distributed consensus algorithms. In *43rd IEEE conference on decision and control* (pp. 3998–4003) *vol. 4*.

Notarstefano, G., Egerstedt, M., & Haque, M. (2009). Rendezvous with multiple, intermittent leaders. In *IEEE conference on decision and control*. China: Shanghai.

Olfati-Saber, R. (2006). Flocking for multi-agent dynamic systems: algorithms and theory. *IEEE Transaction Automatic Control*, *51*(3), 401420.

Grünbaum, D., Viscido, S., & Parrish, J. K. (2004). Extracting interactive control algorithms from group dynamics of schooling fish. In V. Kumar, N. E. Leonard & A. S. Morse (Eds.), *Proc. of block island workshop on cooperative control*. Springer-Verlag.

Pasteels, J. M., & Deneubourg, J. L. (1987). *From individual to collective behavior in social insects*. Basel, Boston, MA: Birkhauser Verlag.

Ren, W., & Beard, R. W. (2004). Consensus of information under dynamically changing interaction topologies. In *Proceedings of the American control conference* (pp. 4939–4944), *Vol. 6*, June 30–July 2 2004.

Thornhill, R., & Alcock, J. (1983). *The evolution of insect mating systems*. Cambridge, MA: Harvard University Press.

Wheeler, W. M. (1923). *Social life among the insects*. New York, NY: Harcourt, Brace & World.

G. Notarstefano is "ricercatore" (Assistant Professor) at the University of Lecce (Italy) since February 2007. He received the Laurea degree "summa cum laude" in Electronics Engineering from the University of Pisa (Italy) in 2003 and the Ph.D. degree in "Automatica e Ricerca Operativa" (Automation and Operation Research) from the University of Padova (Italy) in April 2007. He was visiting scholar at the University of Stuttgart (October 2009), at the University of California Santa Barbara (April 2009 and March–September 2005) and at the University of Colorado Boulder (April–May 2006). His research interests include distributed optimization and motion coordination in multi-agent networks, trajectory optimization and tracking of nonlinear systems, and dynamics, control and trajectory optimization of new generation aerial vehicles.

M. Egerstedt was born in Stockholm, Sweden, and is a Professor in the School of Electrical and Computer Engineering at the Georgia Institute of Technology, where he has been on the faculty since 2001. He received the M.S. degree in Engineering Physics and the Ph.D. degree in Applied Mathematics from the Royal Institute of Technology in 1996 and 2000, respectively. He also received a B.A. degree in Philosophy from Stockholm University in 1996. Dr. Egerstedt's research interests include hybrid and networked control, with applications in motion planning, control, and coordination of mobile robots, and he serves as Editor for Electronic Publications for the IEEE Control Systems Society and Associate Editor for the IEEE Transactions on Automatic Control. Magnus Egerstedt is the director of the Georgia Robotics and Intelligent Systems Laboratory (GRITS Lab), is a Senior Member of the IEEE, received the ECE/GT Outstanding Junior Faculty Member Award in 2005, and the CAREER award from the US National Science Foundation in 2003.

M. Haque was born in Houston, Texas. He received a Bachelor's degree in Electrical Engineering from The University of Texas in 2006, where he also performed research in high-power semiconductor lasers. At present, Musad is a Ph.D. candidate at the Georgia Institute of Technology and works as a graduate research assistant at the Georgia Robotics and Intelligent Systems Lab. Musad's research interests are broad, but have a general theme centered on the social behavior of biological systems and multi-agent robotics.