A Separation Signal for Heterogeneous Networks

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***Abstract*— Organizing a large-scale, heterogeneous network of agents into clusters based on the agents' class is a useful preprocessing step for cooperative tasks, where agents with the same capabilities need to be in the same location to cooperatively solve a task. In this paper, we investigate whether it is possible to apply an exogenous control signal to a heterogeneous network of agents, such that these agents form clusters of the same class of agents. We demonstrate that if each agent belongs to one of M "weight" classes and executes a weighted, forced agreement protocol, then it is possible to apply an external input signal that separates the network into M clusters corresponding to the M classes of agents.**

I. Introduction

In the wake of natural or man-made disasters, we want to deploy robots and sensors with different capabilities that are capable of cooperatively solving tasks of a search and rescue mission. The tasks that need to be solved by these heterogeneous networks often involve, but are not limited to, surveillance ([1]), localization ([2]), or transport ([3]). There exist several different ways agents with different capabilities can cooperate as discussed in [4], but in particular, we are interested in the situation when a task requires multiple agents with the same capability to cooperate. For example, we may want all robots with grippers to work together to move a large object, while separately, but simultaneously, all robots with chemical sensor work together to detect chemical leaks. However, when it comes to deploying these systems, it is often not feasible or too expensive to deliberately deploy devices (as discussed in [5]) with the same capabilities together in a specific location. For example, the disaster areas are often not accessible and the only method to deploy the network is to spread the agents over a region from the air.

In fact, these large-scale, heterogeneous multi-agent networks are typically deployed in a random fashion over some region to keep the deployment cost small, as is described by [6]. The first step after deployment is for the agents to aggregate in a common location. Provided that the network is connected, a rendezvous protocol can quickly aggregate the network in a common location (as was shown, for example, in [7]). Once aggregated, we want to separate the network into clusters, each corresponding to a collection of all agents with a particular capability. This particular way of organizing the network would allow us to then assign each cluster of agents to solve some task of the search and rescue mission cooperatively.

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In this paper, we focus on separating the network into separate connected components or "clusters" that correspond to all agents of a particular class (where this class may correspond to a unique capability) being co-located, but completely separated from agents belonging to another class. This problem has not been widely studied in robotics aside from [8], but it is well known that in physical processes involving granular mixtures, granules of different sizes segregate under external perturbations ([9], [10], and [11]). The authors of [8] were inspired by this phenomenon and modeled robots as particles of different sizes to achieve separation based on differences in size. Separation was mainly achieved by having each robot broadcast its size locally and repulse from other robots based on their reported sizes.

We were also inspired by the physical phenomenon of achieving separation by perturbing a granular mixture, but we have taken a different approach. We let the agents execute a weighted, forced agreement protocol (see, for example, [12]), where the agents are attracted to each other, rather than repulsed. Agents within the same class have the same weight, while agents from different classes have different weights. Different weights in the dynamics are an analog for the agents having different sizes. We will show that we can apply an exogenous control signal to this network that will completely separate agents from each other if they are in different classes, while agents from the same class remain together.

The application of a single, external input signal to a large system is not novel and is known as broadcast or ensemble control. This type of control mechanism has been used to achieve positional consensus (demonstrated in [13]), ensure collision avoidance (shown in [14]), align quantum spins (discussed in [15]), and achieve geometric formations (described in [16]). The key is that these multi-agent systems are heterogeneous, meaning that agents with different characteristics will react differently to the same input signal. This difference in response to an input signal is exploited to achieve a global behavior for the network. Our contribution is to show that there exists an external input signal that can separate a heterogeneous network of agents into cluster of agents of the same type.

As a first step, we derive this external signal for separating two classes of agents and then show that it is also possible to separate three classes of agents, while assuming that the initial position of all agents is the same and that the separation happens along a single dimension. Next, we generalize to separating M classes of agents. Once we have made this generalization, we will continue to generalize our results to show that this network separation principle can be applied to agents moving in multiple dimensions and to the case when the agents start separating while co-located in a small ball, i.e. not coincident as previously assumed. Last, we conclude with some experimental verification in the form of simulations.

II. A Separating Signal

Suppose that a network of agents is comprised of M types of agents, belonging to one of the classes in $\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_M\}$. The problem we are interested in is how to separate these M classes of agents from each other without separating agents from the *same* class. As an initial way of approaching this problem, let us assume that the agents in the different class somehow carry different weights, such that they respond differently to an external signal. In addition, assume that these agents are all running a local, forced agreement protocol, in the sense that

$$\dot{x}_i = \gamma_{\pi(i)} \left(\sum_{j \in N(i)} (x_j - x_i) + u \right), \tag{1}$$

where x_i is the position of agent i, $\gamma_{\pi(i)}$ is a scalar weight, $N(i)$ is the set of neighbors that agent i has in the network, and u is an exogenous input signal. The set of neighbors is defined by the condition $\|x_j - x_i\| \leq \Delta$, i.e. $j \in N(i)$ and $i \in N(j)$ if agents i and j are close enough to each other. It is important to note that a neighborhood, $N(i)$, is not a function of class, since an agent is not aware of its neighbor's class. The key object in (1) is the class membership function $\pi : \mathcal{N} \to \mathcal{C}$, where $\mathcal{N} = \{1, \ldots, N\}$ is the set of all agents, i.e the function π maps agent i into one of the M classes with weights $\gamma_1, \gamma_2, \ldots, \gamma_M$. The notion that these classes weigh differently is encoded in the following two properties:

1) Each class has a unique weight; otherwise, any two classes with the same weight can be merged into a single class.
2) The weights can be ordered in ascending order,

$$0 < \gamma_1 < \gamma_2 < \ldots < \gamma_M,$$

by simply relabeling $\mathcal{C}$ if necessary.

What we aim to do is come up with a separating signal u that ensures that the agents are separated from each other, as is illustrated in Fig. 1. This signal will be broadcast simultaneously to all agents. As a first step towards deriving such a separating signal, let us first assume that a) the agents are all scalar, i.e., that $x_i \in \mathbb{R}$, and b) that all agents start at the same position, i.e., $x_i(0) = x_j(0), \forall (i,j)$. We will first consider the two class case, then the three class case, and finally, we will generalize to M classes. The two class case will provide us with a way to separate two classes, while the three class case will provide us with a strategy for separating more than two classes by using the results from the two class case. It is then possible for us to use the results from these two cases to generalize to M classes.

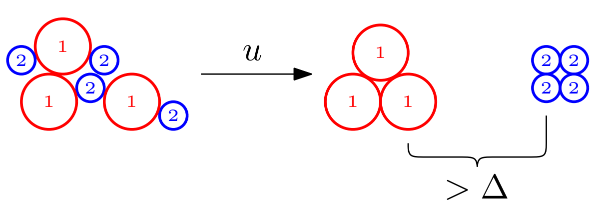


Fig. 1. Separation of Δ between two classes using an external signal u applied to all agents simultaneously.

A. Two Classes

Under the two assumption, $x_i(0) = x_j(0), \forall (i,j)$ and $x_i \in \mathbb{R}$, we will derive a constant, scalar separating signal that is guaranteed to achieve a desired separation of greater than Δ between the two classes. In fact, assume that there are N_1 agents of $\mathcal{C}_1$ and N_2 agents of $\mathcal{C}_2$. Since all agents within a class start at the same position and execute the same dynamics, they will always stay together under any input signal u. As such, if we let χ_i be the position of any member of $\mathcal{C}_i$, $i = 1, 2$, we can let $d_{12} = \chi_2 - \chi_1$ denote the distance separating the two classes. And, the system dynamics for the two classes become

$$\dot{\chi}_1 = \gamma_1 (N_2 d_{12} + u), \quad \dot{\chi}_2 = \gamma_2 (-N_1 d_{12} + u)$$

or

$$\dot{d}_{12} = \dot{\chi}_2 - \dot{\chi}_1 = (\gamma_2 - \gamma_1) u - (\gamma_2 N_1 + \gamma_1 N_2) d_{12}. \tag{2}$$

Now, assume that the agents are no longer connected if an inter-agent distance is greater than Δ, i.e. if $i \in \mathcal{C}_1, j \in \mathcal{C}_2$, then $j \notin N(i), i \notin N(j)$ when $|x_j - x_i| > \Delta$.

Theorem 1: u is a scalar, constant separating signal if

$$u > \frac{(\gamma_2 N_1 + \gamma_1 N_2)\Delta}{\gamma_2 - \gamma_1}, \tag{3}$$

which ensures that $\dot{d}_{12} > 0$ when $d_{12} \in [0, \Delta]$, i.e., guarantees that all agents belonging to different classes are completely separated by a distance greater than Δ.

Proof: The proof follows directly from applying $\dot{d}_{12} > 0$ to (2) and solving for u to get

$$u > \frac{(\gamma_2 N_1 + \gamma_1 N_2) d_{12}}{\gamma_2 - \gamma_1}.$$

$u > 0$ is needed to start the separation process when $d_{12} = 0$. Since d_{12} increases monotonically on the interval $[0, \Delta]$, it is sufficient to suppose that $d_{12} = \Delta$ and require that

$$u > \frac{(\gamma_2 N_1 + \gamma_1 N_2)\Delta}{\gamma_2 - \gamma_1}$$

is applied on the interval.

Then, a scalar, constant separating signal u is

$$u = \frac{(\gamma_2 N_1 + \gamma_1 N_2)(\Delta + \epsilon)}{\gamma_2 - \gamma_1}, \tag{4}$$

where $\epsilon > 0$. ■

Moreover, we can explicitly compute the duration T_s for which u needs to be applied to ensure that no two agents from different classes are connected.

Corollary 1: To separate the two classes, u needs to be applied for a duration of

$$T_s = \frac{\ln\left(\frac{\Delta+\epsilon}{\epsilon}\right)}{\gamma_1 N_2 + \gamma_2 N_1}, \tag{5}$$

where $\epsilon > 0$.

Proof: The distance separating the two classes at time t is

$$\begin{aligned} d_{12}(t) &= \int_0^t e^{-(\gamma_2 N_1+\gamma_1 N_2)(t-s)}(\gamma_2 N_1 + \gamma_1 N_2)(\Delta+\epsilon)ds \\ &= (\Delta+\epsilon)\left(1 - e^{-(\gamma_2 N_1+\gamma_1 N_2)t}\right), \end{aligned} \tag{6}$$

where $d_{12}(0) = 0$ is the initial condition. At time T_s, we know that $d_{12}(T_s) = \Delta$, so we can solve (6) in terms of T_s to get (5). Since $\dot{d}_{12}(T_s) > 0$, the two classes will be separated by a distance greater than Δ after T_s time. ■

B. Three Classes

The next step is to find a separating signal u, which can completely separate agents belonging to three classes, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$. We will use this result as a stepping stone for generalizing our results to M classes. With the addition of $\mathcal{C}_3$, we define d_{23} to be distance between agents in $\mathcal{C}_2$ and $\mathcal{C}_3$ and d_{13} to be the distance between agents in $\mathcal{C}_1$ and $\mathcal{C}_3$. Let $d_{13} = d_{12} + d_{23}$, such that $d_{13} > d_{12}$ and $d_{13} > d_{23}$ during the separation process, i.e. $t > 0$. Recall that the initial conditions are $d_{13}(0) = d_{12}(0) = d_{23}(0) = 0$.

The system dynamics for the three classes before separation are

$$\begin{aligned} \dot{\chi}_1 &= \gamma_1(N_2 d_{12} + N_3 d_{13} + u), \\ \dot{\chi}_2 &= \gamma_2(-N_1 d_{12} + N_3 d_{23} + u), \\ \dot{\chi}_3 &= \gamma_3(-N_1 d_{13} - N_2 d_{23} + u). \end{aligned} \tag{7}$$

Instead of finding a single separating signal u, we are going to be strategic and find a series of separating signals $u_M, \ldots, u_2$ that separate the classes by peeling off classes in descending order of their weights. For example, in the three class problem, u_3 will separate $\mathcal{C}_3$ from $\mathcal{C}_2$ and $\mathcal{C}_1$, and u_2 will separate $\mathcal{C}_2$ from $\mathcal{C}_1$[1].

First, we want to find u_3 that separates $\mathcal{C}_3$ from $\mathcal{C}_2$. u_3 will also separate $\mathcal{C}_3$ from $\mathcal{C}_1$, since when $d_{23} > \Delta$, then $d_{13} > \Delta$ must be true, because $d_{13} > d_{23}$ by definition. As before, we want $\dot{d}_{23} > 0$ when $d_{23} = \Delta$:

$$\begin{aligned} \dot{d}_{23} &= \dot{\chi}_3 - \dot{\chi}_2 > 0 \\ &= \gamma_3(-d_{13}N_1 - d_{23}N_2 + u_3) \\ &\quad - \gamma_2(-d_{12}N_1 + d_{23}N_3 + u_3) > 0 \end{aligned} \tag{8}$$

Suppose we find a u_3 that satisfies this inequality and achieves a separation of $\mathcal{C}_3$ from $\mathcal{C}_1$ and $\mathcal{C}_2$. The dynamics are now sightly different:

$$\begin{aligned} \dot{\chi}_1 &= \gamma_1(N_2 d_{12} + u_2), \\ \dot{\chi}_2 &= \gamma_2(-N_1 d_{12} + u_2), \\ \dot{\chi}_3 &= \gamma_3 u_2. \end{aligned} \tag{9}$$

[1]When we state that we want to separate $\mathcal{C}_i$ from $\mathcal{C}_j$, what we really mean is that we want to separate the agents in $\mathcal{C}_i$ from the agents in $\mathcal{C}_j$.

Since we now want $\dot{d}_{12} \geq 0$ when $d_{12} = \Delta$, the separating signal u_2 that separates $\mathcal{C}_1$ and $\mathcal{C}_2$ is exactly (4).

It is important to note that this strategy seems to ignore the change in dynamics that occurs when $\mathcal{C}_1$ and $\mathcal{C}_3$ separate before $\mathcal{C}_2$ and $\mathcal{C}_3$ have separated. Similarly, depending on the choice of parameters γ_1, γ_2, and γ_3, as well as, N_1, N_2, and N_3, $\mathcal{C}_1$ and $\mathcal{C}_2$ may have separated before $\mathcal{C}_2$ and $\mathcal{C}_3$ separate. This scenario renders applying u_2 unnecessary. Therefore, this strategy will not be optimal; however, we will show that this strategy will still successfully separate the three class. Our motivation is to define a *simple* strategy, which will guarantee the separation of the different classes of agents independent of the parameters (with respect to the strategy only, not with respect to how the input signals are defined).

Theorem 2: u_3 is a scalar, constant separating signal if

$$u_3 > \frac{(\gamma_3(N_1+N_2) + \gamma_2 N_3)\Delta}{\gamma_3 - \gamma_2}, \tag{10}$$

which ensures that $\dot{d}_{23} > 0$ when $d_{23} \in [0, \Delta]$, i.e. guarantees that all agents belonging to $\mathcal{C}_3$ are separated from all agents in $\mathcal{C}_1$ and $\mathcal{C}_2$. Once this separation has occurred, the scalar, constant separating signal u_2 equal to (4) can be applied to separate agents belonging to $\mathcal{C}_2$ from agents belonging to $\mathcal{C}_1$.

Proof: We want to find u_3 that guarantees $\dot{d}_{23} > 0$ when $d_{23} = \Delta$. First we solve for u_3 in (8).

$$u_3 > \frac{(\gamma_3 N_2 + \gamma_2 N_3)d_{23} + (\gamma_3 d_{13} - \gamma_2 d_{12})N_1}{\gamma_3 - \gamma_2}$$

We select a sufficiently large u_3 by applying the fact that $d_{13} > d_{23}$ and $\gamma_2 d_{12} > 0$:

$$\begin{aligned} u_3 &> \frac{(\gamma_3(N_1+N_2) + \gamma_2 N_3)d_{13}}{\gamma_3 - \gamma_2} \\ &= \frac{(\gamma_3(N_1+N_2) + \gamma_2 N_3)(\Delta+\epsilon)}{\gamma_3 - \gamma_2}, \end{aligned}$$

where $\epsilon > 0$.

This is not an airtight upper bound, since the contribution from agents in $\mathcal{C}_1$ will be zero sometime before $d_{23} = \Delta$ and separation between $\mathcal{C}_2$ and $\mathcal{C}_3$ is achieved. However, it still guarantees that $\dot{d}_{23} > 0$ when $d_{23} \in [0, \Delta]$.

Once $\mathcal{C}_3$ is separated from $\mathcal{C}_1$ and $\mathcal{C}_2$, we are justified in using u_2 as defined by (4) to separate $\mathcal{C}_1$ and $\mathcal{C}_2$ if and only if the inequality $\dot{d}_{23} \geq 0$ still holds when applying u_2. Otherwise, we cannot guarantee that $\mathcal{C}_2$ and $\mathcal{C}_3$ remain separated. After separation,

$$\dot{d}_{23} = \gamma_3 u_2 - \gamma_2(-d_{12}N_1 + u_2) \geq 0,$$

so we need to plug in u_2 and make sure the inequality holds.

$$u_2 = \frac{(\gamma_2 N_1 + \gamma_1 N_2)(\Delta+\epsilon)}{\gamma_2 - \gamma_1} \geq -\frac{\gamma_2 d_{12} N_1}{\gamma_3 - \gamma_2}$$

We can directly see that this inequality will hold, because a) $\gamma_3 > \gamma_2 > 0$, and b) $\gamma_2 d_{12} N_1 > 0$. Therefore, if we use u_3 to separate $\mathcal{C}_3$ from $\mathcal{C}_1$ and $\mathcal{C}_2$ and then u_2 to separate $\mathcal{C}_2$ from $\mathcal{C}_1$, we are able to completely separate all three classes from each other. ■

Moreover, we can compute a duration for applying u_3, $T_{s,3}$, that is sufficient to separate $\mathcal{C}_3$ from the other two classes and a duration of applying u_2, $T_{s,2}$, that is sufficient to separate $\mathcal{C}_2$ from $\mathcal{C}_1$. These durations will be relaxed upper bounds, because of the following two assumptions:

1) We assume that the dynamics in (7) are unchanged on the interval $[0, T_{s,3}]$, which is not accurate, since $\mathcal{C}_1$ separates from $\mathcal{C}_3$ before $\mathcal{C}_2$, and $\mathcal{C}_1$ may even separate from $\mathcal{C}_2$ before then depending on the weights and the sizes of the classes.
2) We assume that $d_{12}(T_{s,3}) = 0$, which is not accurate, since $d_{12} > 0$ is guaranteed by the fact that $\gamma_1 \neq \gamma_2$.

Under these assumptions, we can compute a simple schedule for applying u_3 and u_2 to achieve separation of the three classes.

Corollary 2: To separate the three classes, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, we will apply the signal u_3 for a duration of $T_{s,3}$ and at time $T_{s,3}$, we will apply the signal u_2 for a duration of $T_{s,2}$, where $T_{s,3}$ and $T_{s,2}$ are defined as,

$$\begin{aligned} T_{s,3} &= \frac{\ln\left(\frac{\Delta+\epsilon}{\epsilon}\right)}{\gamma_3(N_1+N_2)+\gamma_2 N_3} \\ T_{s,2} &= \frac{\ln\left(\frac{\Delta+\epsilon}{\epsilon}\right)}{\gamma_1 N_2+\gamma_2 N_1}, \end{aligned} \tag{11}$$

and $\epsilon > 0$.

Proof: The proof follows the same process as Corollary 1 by deriving $d_{23}(t)$ on the interval $t \in [0, T_{s,3}]$ and solving $d_{23}(T_{s,3}) = \Delta$ for $T_{s,3}$. $d_{23}(t)$ is derived from (7) assuming that these dynamics are unchanged on the interval $[0, T_{s,3}]$. $d_{12}(t)$ is derived from (9) on the interval $t \in (T_{s,3}, T_{s,2}]$. We solve $d_{12}(T_{s,2}) = \Delta$ for $T_{s,2}$ and assume that $d_{12}(T_{s,3}) = 0$ to find the time to separate $\mathcal{C}_1$ and $\mathcal{C}_2$ as in Corollary 1. ■

This simple schedule is not optimal. We will hold u_3 for longer than is needed to separate $\mathcal{C}_3$ from $\mathcal{C}_2$, because we assume that the dynamics do not change on the interval $[0, T_{s,3}]$. Similarly, assuming that $d_{12}(T_{s,3}) = 0$ is pessimistic, since $d_{12} > 0$ is guaranteed by the fact that $\gamma_1 \neq \gamma_2$. The consequence is that we will hold u_2 for longer than is needed to separate $\mathcal{C}_2$ from $\mathcal{C}_1$. We could be more exact in computing these durations, since the parameters are known, and thus, we know how the system evolves. However, we are motivated to find a strategy that is *simple* to implement, in the sense that this strategy is independent of the choice in parameters (weights and sizes of the classes) that determine the order of separation[2]. Despite the lack of optimality, this strategy of applying u_3 for $T_{s,3}$ and then applying u_2 for $T_{s,2}$ will separate the three classes successfully.

C. M Classes

The next step is to find a similar strategy to completely separate agents belonging to M classes, $\mathcal{C}_1, \ldots, \mathcal{C}_M$. We define d_{ij} to be the distance between agents in $\mathcal{C}_i$ and $\mathcal{C}_j$. If three classes, $\mathcal{C}_i$, $\mathcal{C}_j$, and $\mathcal{C}_k$, are ordered such that $\gamma_i < \gamma_j < \gamma_k$, then by our previous construction, $d_{ik} = d_{ij} + d_{jk}$, such that $d_{ik} > d_{ij}$ and $d_{ik} > d_{jk}$ (with the exception of the initial conditions, where $d_{ik}(0) = d_{ij}(0) = d_{jk}(0) = 0$).

The system dynamics for the M classes before any separation are

$$\dot{\chi}_i = \gamma_i \left(\sum_{k=1}^{M} N_k d_{ik} + u \right),$$

where $d_{ii} = 0$ and $d_{ik} = -d_{ki}$.

Instead of finding a single separating signal u, we are going to again be strategic and find a series of separating signals, $u_M, \ldots, u_2$, that separate the classes by peeling off classes in descending order of their weights.

First, we want to find u_M that separates $\mathcal{C}_M$ from $\mathcal{C}_{M-1}$. u_M will also separate $\mathcal{C}_M$ from $\mathcal{C}_{M-2}, \ldots, \mathcal{C}_1$. Next, we want to find u_{M-1} that separates $\mathcal{C}_{M-1}$ from $\mathcal{C}_{M-2}$, and so on until we have separated all of the M classes from each other.

As was the case before, we will assume that the dynamics are unchanged while we separate two classes, even though $\mathcal{C}_{k-2}$ would separate from $\mathcal{C}_k$ before $\mathcal{C}_k$ and $\mathcal{C}_{k-1}$ have separated, and $\mathcal{C}_{k-1}$ may even separate from $\mathcal{C}_{k-2}$ before separation between $\mathcal{C}_k$ and $\mathcal{C}_{k-1}$ has been achieved. As we have shown before, u_k will still be an input signal that separates $\mathcal{C}_k$ and $\mathcal{C}_{k-1}, \ldots, \mathcal{C}_1$ successfully.

Theorem 3: $u_M, \ldots, u_2$ is a series of scalar, constant separating signals that separate M classes completely, i.e. $d_{ij} > \Delta, \forall i \neq j$, and $i, j \in \{1, \ldots, M\}$, if

$$u_k > \frac{\left(\gamma_k \sum_{j=1}^{k-1} N_j + \gamma_{k-1} N_k\right)\Delta}{\gamma_k - \gamma_{k-1}}, \tag{12}$$

where $k \in \{2, \ldots, M\}$.

Proof: We want to find u_k that separates agents in $\mathcal{C}_k$ from the agents in $\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}$. Assuming that when u_k is applied, $\mathcal{C}_{k+1}, \ldots, \mathcal{C}_M$ are already separated from $\mathcal{C}_1, \ldots, \mathcal{C}_k$, then we know from the developments in the previous sections that u_k is of the form (12). However, we have to make sure that applying u_k does not result in the merging of any of the already separated classes $\mathcal{C}_{k+1}, \ldots, \mathcal{C}_M$. Therefore, the following inequality must be satisfied,

$$u_k \geq -\frac{\gamma_k \sum_{j=1}^{k-1} d_{jk} N_j}{\gamma_{k+1} - \gamma_k},$$

for all $k = 2, \ldots, (M-1)$ such that $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$ do not to merge, as well as, that $u_k \geq 0$ such that none of the separated classes $\mathcal{C}_{k+1}, \ldots, \mathcal{C}_M$ merge. Both of these inequalities are satisfied by the fact that u_k is always positive by inspection of (12). ■

Moreover, we can again compute a duration of applying u_M, $T_{s,M}$, that is sufficient to separate $\mathcal{C}_M$ from the other classes, a duration of applying u_{M-1}, $T_{s,M-1}$, that is sufficient to separate $\mathcal{C}_{M-1}$ from the other classes, and so on. As before, we apply the assumption that the dynamics do not change while a class $\mathcal{C}_k$ is separated from $\mathcal{C}_{k-1}$ and the other classes, and that when we start separating $\mathcal{C}_k$ from $\mathcal{C}_{k-1}$, that distance separating $\mathcal{C}_k$ from $\mathcal{C}_{k-1}, \ldots, \mathcal{C}_1$ is zero. These

[2] Independence from the parameters does not imply that u or T_s do not depend on the parameters, but rather that we can apply u for T_s time to separate two specific classes without worrying about how exactly these two classes separate from the other classes.

assumptions, as before, lead to conservative upper bounds on the durations that the input signals are applied to separate the classes.

Corollary 3: To separate the M classes, $\mathcal{C}_1, \ldots, \mathcal{C}_M$, we will apply the signal u_M for a duration of $T_{s,M}$, and at time $T_{s,M}$, we will apply the signal u_{M-1} for a duration of $T_{s,M-1}$, and so on, where u_k is held for a duration of $T_{s,k}$,

$$T_{s,k} = \frac{\ln\left(\frac{\Delta+\epsilon}{\epsilon}\right)}{\gamma_k \sum_{j=1}^{k-1} N_j + \gamma_{k-1} N_k}, \tag{13}$$

where $k = M, \ldots, 2$ and $\epsilon > 0$. Let $T_{s,M+1} = 0$.

Proof: Suppose that we pick the separating signal such that

$$u_k = \frac{\left(\gamma_k \sum_{j=1}^{k-1} N_j + \gamma_{k-1} N_k\right)(\Delta+\epsilon)}{\gamma_k - \gamma_{k-1}}.$$

The proof follows directly from the proof of Corollary 1 by deriving $d_{(k-1)k}(t)$ on the interval $t \in [T_{s,k+1}, T_{s,k}]$ and solving $d_{(k-1)k}(T_{s,k}) = \Delta$ for $T_{s,k}$, as if $d_{(k-1)k}(T_{s,k+1}) = 0$. ∎

This strategy is not optimal for the same reasons as before. Assuming that $d_{(k-1)k}(T_{s,k+1}) = 0$ is pessimistic, since $d_{(k-1)k} > 0$ is guaranteed by the fact that $\gamma_{k-1} \neq \gamma_k$. Similarly, assuming that the dynamics do not change when peeling a class away from the rest of the classes is not accurate. The consequence is that we will hold u_k for longer than is needed to separate $\mathcal{C}_k$ from $\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}$, $\forall k = 2, \ldots, M$. However, complete separation of all M classes is achieved under this strategy.

D. Other Generalizations

In the previous sections we assumed that $x_i \in \mathbb{R}$; however, the exact same arguments apply to non-scalar agents ($x_i \in \mathbb{R}^n, n \geq 2$) under the same dynamics. The only difference is that we still insist on a constant u_k, where the separation condition becomes

$$\|u_k\| > \frac{(\gamma_k \sum_{j=1}^{k-1} N_j + \gamma_{k-1} N_k)\Delta}{\gamma_k - \gamma_{k-1}}.$$

This condition follows from the fact that the dynamics are decoupled along all dimensions and that the magnitude of u_k is independent of its direction in $\mathbb{R}^n$.

We want to be able to remove our assumption about the initial conditions, i.e., that $x_i(0) = x_j(0),\ \forall(i,j)$. These initial condition could be achieved by simply running the unforced version of the dynamics, which we know will asymptotically drive all agents to a common location (as long as the network stays connected). However for practical purposes, it may be too long to wait for all agents to converge to exactly the same location, so what we will do is see how the argument needs to change when we insist that $\|x_i(0) - x_j(0)\| \leq 2\delta,\ \forall(i,j)$ for a given, small $\delta > 0$.

To show that separation is possible between two classes, $\mathcal{C}_1$ and $\mathcal{C}_2$, we need to show that u is a constant, separating signal that completely separates all pairs of agents $(i,j), i \in \mathcal{C}_1, j \in \mathcal{C}_2$. In fact, we will show that if we assume that $\Delta > 4\delta$, $x_i \in \mathbb{R}$, and pick a u that separates the centroids of the two classes by $\Delta + 2\delta$, then

1) While the network is completely connected, the centroids are separating, i.e. $\dot{\bar{x}}_2 - \dot{\bar{x}}_1 > 0$, and agents are moving towards the centroid of their class.
2) Once some of the agents from the two classes start separating, $\dot{\bar{x}}_2 - \dot{\bar{x}}_1 > 0$ holds and the closest pair $(i,j), i \in \mathcal{C}_1, j \in \mathcal{C}_2$ is separating.

Lemma 1: If the agents are initially co-located in a δ-ball, i.e. $|x_i(0) - x_j(0)| \leq 2\delta, \forall(i,j)$ and $\Delta > 4\delta$, then an input signal

$$u > \frac{(\gamma_1 N_2 + \gamma_2 N - 1)(\Delta + 2\delta)}{\gamma_2 - \gamma_1},$$

will ensure that $\dot{\bar{x}}_2 - \dot{\bar{x}}_1 > 0$ on the interval $(\bar{x}_2 - \bar{x}_1) \in [-2\delta, \Delta + 2\delta]$.

Proof: Suppose $\bar{x}_k(t)$ is the centroid of the positions of the agents in class $\mathcal{C}_k$, i.e.

$$\bar{x}_k(t) = \frac{1}{N_k} \sum_{j \in \mathcal{C}_k} x_j(t),$$

then, the first step is to find the derivative of the two centroids, $\dot{\bar{x}}_1$ and $\dot{\bar{x}}_2$. We can rewrite Equation (1) for an agent $x_i \in \mathcal{C}_1$ as,

$$\begin{aligned}\dot{x}_i &= \gamma_1 \left(\sum_{j \in N(1)} (x_j - x_i) + \sum_{j \in N(2)} (x_j - x_i) + u \right), \\ &= \gamma_1 \left(N_1(\bar{x}_1 - x_i) + N_2(\bar{x}_2 - x_i) + u \right),\end{aligned}$$

under the assumption that all agents of $\mathcal{C}_1$ and $\mathcal{C}_2$ are connected. This assumption is certainly true while all agents are inside the δ-ball and before the distance between any two agents from different classes exceeds Δ. Then,

$$\begin{aligned}\dot{\bar{x}}_1 &= \frac{1}{N_1} \sum_{j \in \mathcal{C}_1} \dot{x}_j \\ &= \frac{\gamma_1}{N_1} \sum_{j \in \mathcal{C}_1} \left(N_1(\bar{x}_1 - x_j) + N_2(\bar{x}_2 - x_j) + u \right) \\ &= \gamma_1 \left(N_2(\bar{x}_2 - \bar{x}_1) + u \right)\end{aligned}$$

Following the same procedure, we can compute $\dot{\bar{x}}_2$,

$$\dot{\bar{x}}_2 = \gamma_2 \left(N_1(\bar{x}_1 - \bar{x}_2) + u \right),$$

and in turn we can compute,

$$\begin{aligned}\dot{\bar{x}}_2 - \dot{\bar{x}}_1 &= \gamma_2 \left(N_1(\bar{x}_1 - \bar{x}_2) + u \right) - \gamma_1 \left(N_2(\bar{x}_2 - \bar{x}_1) + u \right) \\ &= -(\gamma_2 N_1 + \gamma_1 N_2)(\bar{x}_2 - \bar{x}_1) + (\gamma_2 - \gamma_1)u\end{aligned}$$

Without an external input, $u = 0$, the distance between the centroid decays to zero asymptotically; however, if we were to apply

$$u = \frac{(\gamma_2 N_1 + \gamma_1 N_2)(\Delta + 2\delta + \epsilon)}{\gamma_2 - \gamma_1}, \tag{14}$$

where $\epsilon > 0$, then we ensure that the distance between the centroids is always increasing.

One of the assumptions we made is that all agents of $\mathcal{C}_1$ and $\mathcal{C}_2$ are connected during the separation process;

however, we know that not all agents of $\mathcal{C}_1$ will separate from all agents of $\mathcal{C}_2$ simultaneously. In fact, the dynamics will change as agents start to separate, but we will show that u will still ensure complete separation of the two classes.

Suppose that an agent, $x_i \in \mathcal{C}_1$, starts to separate from some of the agents in $\mathcal{C}_2$, and therefore, this agent's dynamics change to

$$\dot{x}_i = \gamma_1 \left(\sum_{j \in N(1)} (x_j - x_i) + \sum_{j \in \tilde{N}(2,i)} (x_j - x_i) + u \right),$$

where $\tilde{N}(2,i)$ is the set of $\tilde{N}_{2,i}$ agents from class $\mathcal{C}_2$ that are still connected to agent x_i from class $\mathcal{C}_1$. As a consequence, the dynamics of the centroid of class $\mathcal{C}_1$, now denoted $\tilde{\bar{x}}_1$, are also changed to

$$\begin{aligned}\dot{\tilde{\bar{x}}}_1 &= \frac{\gamma_1}{N_1} \sum_{j \in \mathcal{C}_1} \left(\sum_{k \in \tilde{N}(2,j)} (x_k - x_j) + u \right) \\ &\leq \gamma_1 \left(N_2(\bar{x}_2 - \bar{x}_1) + u \right),\end{aligned}$$

and similarly,

$$\begin{aligned}\dot{\tilde{\bar{x}}}_2 &= \frac{\gamma_2}{N_2} \sum_{j \in \mathcal{C}_2} \left(\sum_{k \in \tilde{N}(1,j)} (x_k - x_j) + u \right) \\ &\geq \gamma_2 \left(N_1(\bar{x}_1 - \bar{x}_2) + u \right).\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{\tilde{\bar{x}}}_2 - \dot{\tilde{\bar{x}}}_1 &\geq \gamma_2 \left(N_1(\bar{x}_1 - \bar{x}_2) + u \right) - \gamma_1 \left(N_2(\bar{x}_2 - \bar{x}_1) + u \right) \\ &= -(\gamma_2 N_1 + \gamma_1 N_2)(\bar{x}_2 - \bar{x}_1) + (\gamma_2 - \gamma_1)u.\end{aligned}$$

If we apply u as defined in Equation (14), then $\dot{\tilde{\bar{x}}}_2 - \dot{\tilde{\bar{x}}}_1 > 0$ continues to hold even if some of the pairs $(i,j), i \in \mathcal{C}_1, j \in \mathcal{C}_2$, have separated. ■

Lemma 2: If the agents are initially co-located in a δ-ball, i.e. $|x_i(0) - x_j(0)| \leq 2\delta, \forall (i,j)$, $\Delta > 4\delta$, and while $(\bar{x}_2 - \bar{x}_1) \in [-2\delta, 2\delta]$, the network is completely connected and each agent is moving towards the centroid of their class.

Proof: We want to be able to show that while the networks is still completely connected (which is certainly true while $|\bar{x}_2 - \bar{x}_1| \leq 2\delta$, because $\Delta > 4\delta$), agent $i \in \mathcal{C}_1$ is moving towards $\bar{x}_1$ and agent $j \in \mathcal{C}_2$ is moving towards $\bar{x}_2$:

$$\begin{aligned}\dot{\bar{x}}_1 - \dot{x}_i &= \gamma_1 \left(N_2(\bar{x}_2 - \bar{x}_1) + u \right) \\ &\quad - \gamma_1 \left(N_1(\bar{x}_1 - x_i) + N_2(\bar{x}_2 - x_i) + u \right) \\ &= -\gamma_1 (N_1 + N_2)(\bar{x}_1 - x_i),\end{aligned}$$

and similarly,

$$\dot{\bar{x}}_2 - \dot{x}_j = -\gamma_2 (N_1 + N_2)(\bar{x}_2 - x_j).$$

These equations show that each agent is moving towards the centroid of their class. The result is that if we have separated the centroids by 2δ, we can be sure that all agents $j \in \mathcal{C}_2$ are to the right of all agents $i \in \mathcal{C}_1$, i.e. $x_j > x_i, \ \forall (i,j)$. ■

Unfortunately, once some of the agents start to separate, we can no longer guarantee that agents are moving towards the centroid of their class. However, we can show that the closest pair $(i,j), i \in \mathcal{C}_1, j \in \mathcal{C}_2$ continues to separate.

Lemma 3: Once some of the agents from the two different classes have started to separate, the closest pair $(i,j), i \in \mathcal{C}_1, j \in \mathcal{C}_2$ is separating if we continue to apply

$$u > \frac{(\gamma_1 N_2 + \gamma_2 N - 1)(\Delta + 2\delta)}{\gamma_2 - \gamma_1},$$

i.e. $\dot{x}_j - \dot{x}_i > 0$.

Proof: Suppose $\tilde{N}_{2,i}$ is the number of agents of $\mathcal{C}_2$ that agent i can detect, then $\tilde{\bar{x}}_{2,i}$ is the centroid of those agents. Similarly, $\tilde{N}_{1,j}$ is the number of agents of $\mathcal{C}_1$ that agent j can detect, and $\tilde{\bar{x}}_{1,j}$ is the centroid of those agents. Recall that we have separated the two classes in such a way that $\bar{x}_2 > \bar{x}_1$, $\bar{x}_2 > \tilde{\bar{x}}_{2,i}$, $\bar{x}_1 < \tilde{\bar{x}}_{1,j}$, and $x_j > x_i$. If $(i,j), i \in \mathcal{C}_1, j \in \mathcal{C}_2$ is the closest pair, then $\bar{x}_1 < x_i$ and $\bar{x}_2 > x_j$. We will use these inequalities to show that $\dot{x}_j - \dot{x}_i > 0$:

$$\begin{aligned}\dot{x}_j - \dot{x}_i &= \gamma_2 (\tilde{N}_{1,j}(\tilde{\bar{x}}_{1,j} - x_j) + N_2(\bar{x}_2 - x_j) + u) \\ &\quad - \gamma_1 (N_1(\bar{x}_1 - x_i) + \tilde{N}_{2,i}(\tilde{\bar{x}}_{2,i} - x_i) + u) \\ &= (\gamma_2 - \gamma_1)u + \gamma_2 N_2(\bar{x}_2 - x_j) - \gamma_1 N_1(\bar{x}_1 - x_i) \\ &\quad + \gamma_2 \tilde{N}_{1,j}(\tilde{\bar{x}}_{1,j} - x_j) - \gamma_1 \tilde{N}_{2,i}(\tilde{\bar{x}}_{2,i} - x_i) \\ &> (\gamma_2 - \gamma_1)u + \gamma_2 N_2(\bar{x}_2 - x_j) - \gamma_1 N_1(\bar{x}_1 - x_i) \\ &\quad + \gamma_2 N_1(\bar{x}_1 - x_j) - \gamma_1 N_2(\bar{x}_2 - x_i) \\ &= (\gamma_2 - \gamma_1)(u + N_1\bar{x}_1 + N_2\bar{x}_2) \\ &\quad - \gamma_2(N_1 + N_2)x_j + \gamma_1(N_1 + N_2)x_i \\ &> (\gamma_2 - \gamma_1)(u + N_1\bar{x}_1 + N_2\bar{x}_2) \\ &\quad - \gamma_2(N_1 + N_2)\bar{x}_2 + \gamma_1(N_1 + N_2)\bar{x}_1 \\ &= (\gamma_2 - \gamma_1)u - (\gamma_2 N_1 + \gamma_1 N_2)(\bar{x}_2 - \bar{x}_1) \\ &= \dot{\bar{x}}_2 - \dot{\bar{x}}_1 > 0\end{aligned}$$

■

Since the centroids and the closest pair $(i,j), i \in \mathcal{C}_1, j \in \mathcal{C}_2$ are separating under u from Equation (14), we know that the two classes continue to separate even as some of the agents in each class have already separated. However, we have no guarantee that the centroids are not separating significantly faster than the closest pair, such that when the centroids are separated by $\Delta + 2\delta$, the closest pair is separated by distance less than Δ. Therefore, there may exist a permutation of γ_1, γ_2, N_1, and N_2, for which u is not sufficient to separate the two classes. Despite this possibility, we can make the following conjecture based on these lemmas and our simulations:

Conjecture 1: If the agents are initially co-located in a δ-ball, i.e. $\|x_i(0) - x_j(0)\| \leq \delta, \ \forall (i,j)$, then it is possible to completely separate two classes of agents, $\mathcal{C}_1$ and $\mathcal{C}_2$, by separating the centroids of the two classes by a distance greater than $\Delta + 2\delta$. u is a separating signal if

$$u > \frac{(\gamma_1 N_2 + \gamma_2 N_1)(\Delta + 2\delta)}{\gamma_2 - \gamma_1}, \tag{15}$$

where $\delta > 0$ and $\Delta > 4\delta$.

III. SIMULATIONS

We want to verify numerically in simulation whether our results hold, and demonstrate the effect of varying the parameters γ_k and N_k for each of the M classes. First, let us consider the two class case, where we are interested in separating $\mathcal{C}_1$ from $\mathcal{C}_2$. Figure 2 demonstrates a successful separation using the control signal

$$u = \frac{(\gamma_2 N_1 + \gamma_2 N_2)(\Delta + \epsilon)}{\gamma_2 - \gamma_1},$$

for some $\epsilon > 0$. The separation distance Δ is indicated by the black dashed line. The distance between the two classes logarithmically approaches Δ until separation occurs, after which the distance that separates the two classes increases quickly. If it were the case that u was not sufficient to separate the two classes, we would see that the distance between $\mathcal{C}_1$ and $\mathcal{C}_2$ in the plots would stay under the black dashed line.

Figure 2a illustrates the effect of varying N_1 and N_2, while Fig. 2b illustrates the effect of varying γ_1 and γ_2. In all cases, the distance between $\mathcal{C}_1$ and $\mathcal{C}_2$ eventually exceeds the separation distance Δ (the black dashed line in the figures).

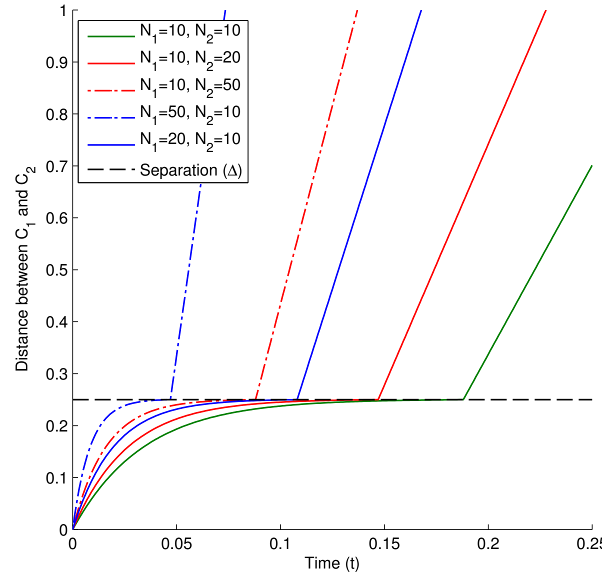
We can also demonstrate that our choice of $u_M, \ldots, u_2$ applied to the case when we want to separates six classes, $\mathcal{C}_1, \ldots, \mathcal{C}_6$ is also successful. Again, a failure to separate a pair of classes (i.e., u_k is not sufficient for separation) would have been indicated by one of the separation distances (lines in the plot) staying under the black dashed line. Figure 3 illustrates that we can successfully separate the six classes from each other. In both cases, all classes have the same number of agents, i.e. $N_1 = \ldots = N_6 = 10$. Figure 3a specifically considers the case when the inter-class difference in γ increases, i.e. $\gamma_2 - \gamma_1 < \gamma_3 - \gamma_2 < \ldots < \gamma_6 - \gamma_5$. In this scenario, $\mathcal{C}_6$ and $\mathcal{C}_5, \ldots, \mathcal{C}_1$ completely separate first, then $\mathcal{C}_5$ and $\mathcal{C}_4, \ldots, \mathcal{C}_1$ separate completely, and so on. Figure 3b illustrates the simple schedule used to separate the classes. In this scenario, u_6, u_5, and u_4 are sufficient to actually separate all M classes.

Last, we want to demonstrate that if the agents move in $\mathbf{R}^2$ and do not start in the same location, namely they are co-located in some δ-ball, then we can still achieve separation using the control signal

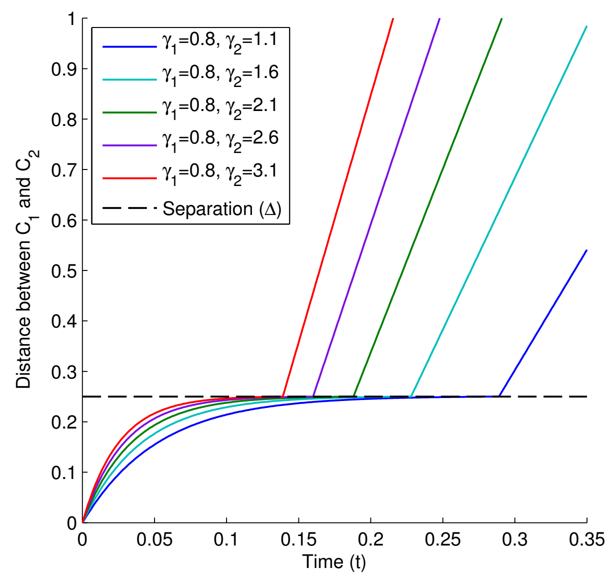
$$u = \frac{(\gamma_2 N_1 + \gamma_2 N_2)(\Delta + 2\delta + \epsilon)}{\gamma_2 - \gamma_1},$$

for some $\delta, \epsilon > 0$.

Figure 4 illustrates the case where 100 agents in $\mathcal{C}_1$ separate from 75 agents in $\mathcal{C}_2$. The separation distance when any two agents disconnect is $\Delta = 0.4$, and all agents start from a location within a δ-ball, where $\delta = 0.1$. The centroids of the two classes are separated by a distance greater than $\Delta + 2\delta$ when the simulation ends. The minimum separation between two agents of each class is $\Delta_{\min}$, and since $\Delta_{\min} > \Delta$, the two classes are completely separated. Figure 4a illustrates the case when $x_i \in \mathbb{R}$, while Fig. 4b illustrates the case when $x_i \in \mathbb{R}^2$. The signal u separates the two classes in both cases.



(a) Distance separating $\mathcal{C}_1$ and $\mathcal{C}_2$, when $\gamma_1 = 0.8, \gamma_2 = 2.1$ and N_1, N_2 are varied.



(b) Distance separating $\mathcal{C}_1$ and $\mathcal{C}_2$, when $N_1 = 10, N_2 = 10$ and γ_1, γ_2 are varied.

Fig. 2. Successful separation of $\mathcal{C}_1$ and $\mathcal{C}_2$ for a variety of parameters.

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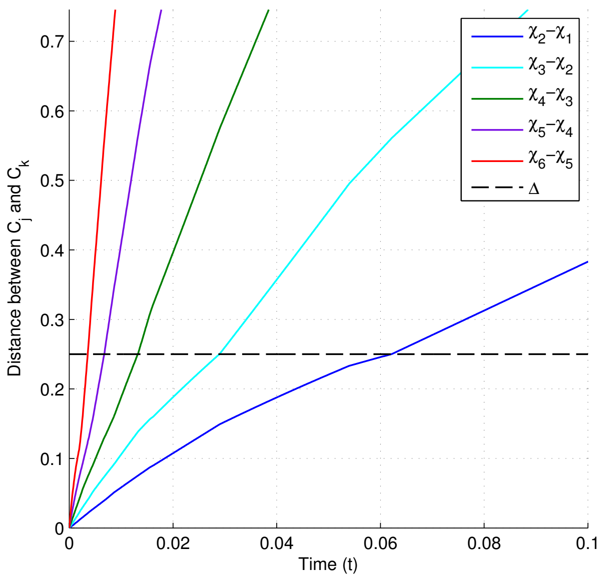
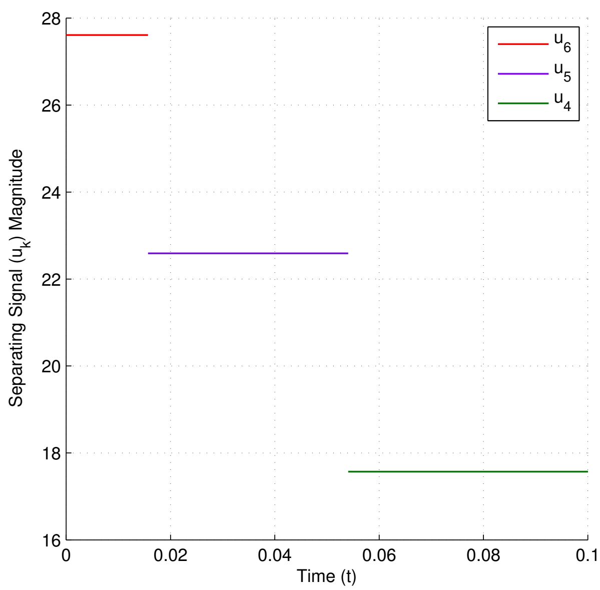
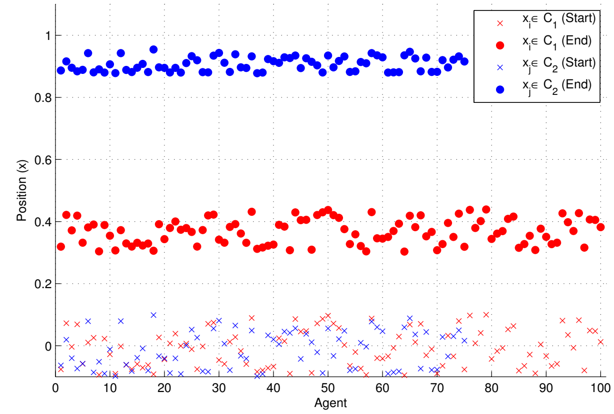


(a) Distances separating $\mathcal{C}_1, \ldots, \mathcal{C}_6$, when $\gamma_1 = 0.2, \gamma_2 = 0.4, \gamma_3 = 0.8, \gamma_4 = 1.6, \gamma_5 = 3.2, \gamma_6 = 6.4$, $N_1 = \ldots = N_6 = 10$

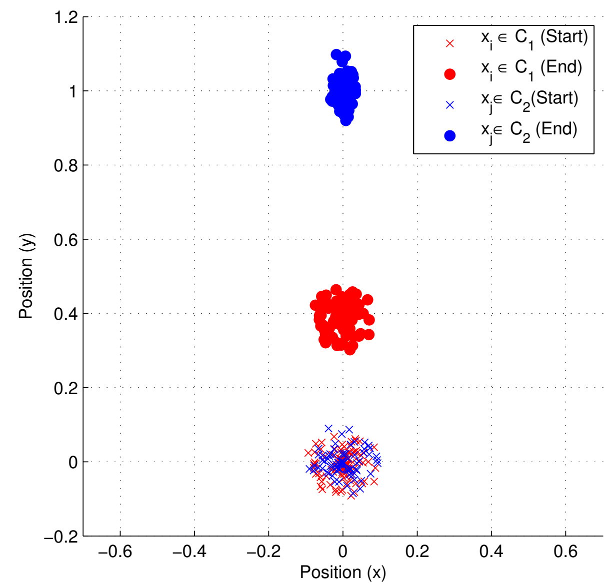


(b) Each u_k is applied for a duration of $T_{s,k}$, where $k = \{4, 5, 6\}$ on the interval $t \in (0, 0.1)$.

Fig. 3. Successful separation of $\mathcal{C}_1, \ldots, \mathcal{C}_6$ with a simple schedule.



(a) Separating $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathbb{R}$, $\Delta = 0.4, \delta = 0.1$. $\Delta_{\min} = 0.446$.



(b) Separating $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathbb{R}^2$, $\Delta = 0.4, \delta = 0.1$. $\Delta_{\min} = 0.457$.

Fig. 4. Successful separation of $\mathcal{C}_1$ and $\mathcal{C}_2$ when agents start in a δ-ball, $N_1 = 100, N_2 = 75, \gamma_1 = 0.2, \gamma_2 = 0.7$.

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