Leader-Based Multi-Agent Coordination Through Hybrid Optimal Control

Staffan Björkenstam, Meng Ji, Magnus Egerstedt, and Clyde Martin

***Abstract*— The problem of optimally transferring a linear dynamical system between affine varieties arises in a number of applications such as path planning and robot coordination. In this paper, this problem, as well as generalizations to switched linear systems, is solved in the context of minimum energy control. In particular, we present a novel algorithm for obtaining the globally optimal solution through a combination of Hilbert space methods and dynamic programming. As a driving application, the problem of leader-based multi-agent coordination is considered, and we show how our proposed algorithm can be used for optimal coordination purposes.**

I. Introduction

This paper focuses on the problem of controlling a network of interacting mobile robots in a coordinated fashion. In particular, we consider heterogeneous networks, partitioned into leader and follower agents. The idea is that the followers execute a relatively simple, decentralized control program, designed with the explicit purpose of keeping the team together. At the same time, the leaders are assumed to have access to global information and their movements are to be defined in such a way as to take an overall performance objective into account.

By partitioning the network into followers and leaders, a balance is struck between information flow management and task completion. Such heterogeneous networks were first introduced in [1], [2], through a study of so-called anchor node networks. Following this, a number of issues concerned with *leader-follower networks* have been covered. For instance, controllability was discussed in [3], [4], and the problem of transferring the network between quasi-static equilibrium points was the topic of [5]. The problem of boundary value control was the concern in [6], and in this paper we focus on the issue of optimal control.

The particular example scenario under consideration in this paper is that of *repeated redeployment*, in which the overall mission is specified through a collection of waypoints. These waypoints are moreover defined as pairs of interpolation times and subformations, characterizing the desired positions of a subset of the agents at the particular interpolation times. The interpretation here is that additional degrees of control freedom is obtained from the fact that the remaining agents are unconstrained at the interpolation times.

If we let the local interactions, defining the followers' motions, be linear, the resulting problem is that of optimal transfer of a linear system between multiple affine varieties. In fact, this problem will be solved using a combination of Hilbert's projection theorem and dynamic programming, and the outline of this paper is as follows: In Section II, we formulate the problem under consideration as well as give an overview over the graph-based coordination control mechanisms needed for its formulation. For the sake of readability, we defer the derivation of the general solution to Sections IV and V, and give the solution to the formation control problem in Section III. Subsequently, Section IV recalls the problem of driving a linear system between boundary points as a minimum norm problem in Hilbert spaces, and a dynamic programming algorithm is given in Section V for stitching together multiple such solutions in order to move between multiple affine varieties in an optimal fashion.

II. Leader-Based Multi-Agent Coordination

Consider N mobile robots, each of which is given by a point in $\mathbb{R}^n$. We will assume that the dynamics associated with each agent is given by $\dot{x}_i = u_i,\ i = 1, \ldots, N$, which means that, along each dimension, the dynamics can be decoupled. Hence, we can, without loss of generality, consider each dimension independently. In other words, let $x_i \in \mathbb{R}$, $i = 1, \ldots, N$, be the position of the ith agent, and $x = (x_1, x_2, \ldots, x_N)^T$ be the aggregated state vector. A widely adopted distributed control strategy for such systems is the so-called *consensus equation*

$$\dot{x}_i = -\sum_{j \in N(i)} (x_i - x_j), \tag{1}$$

where $j \in N(i)$ means that there is a connection (i.e. a communication link) between agents i and j.

Throughout this paper we will assume that the network topology is static, i.e. that $N(i)$ does not vary over time. In fact, the consensus equation in Equation 1 has been thoroughly studied for static as well as dynamic networks. A representative sample of some of the highlights in this area of research can be found in [7], [8], [9], [10], [11], [12], [13], [14], [15].

Now, algebraic graph theory (see for example [16]) provides us with the tools for analyzing such control strategies: A graph $\mathcal{G} = (V, E)$ consists of a set of nodes $V = \{v_1, v_2, \ldots, v_N\}$, which corresponds to the different agents, and a set of edges $E \subset V \times V$, which relates to a set of unordered pairs of agents. A connection exists between

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agent i and j if and only if $(v_i, v_j) = (v_j, v_i) \in E$, and the interpretation here is that $(v_i, v_j) \in E$ if and only if agents i and j have established a communication link between them.

Furthermore the decentralized control law in Equation 1 can be written as

$$\dot{x} = -\mathcal{L}(\mathcal{G})x, \quad (2)$$

where $\mathcal{L}(\mathcal{G})$ is the graph Laplacian for the graph $\mathcal{G}$, given by $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$, where $D(\mathcal{G})$ is the degree matrix, and $A(\mathcal{G})$ is the adjacency matrix associated with $\mathcal{G}$.

The leader-follower structure of the heterogeneous network is obtained by partitioning the nodes (agents) into leaders and followers respectively. We will assume that this partitioning is done by assuming that the first $N_f < N$ robots are followers and the remaining $N_l = N - N_f$ robots are leaders, i.e. $x = (x_f^T \ x_l^T)^T$, where $x_f \in \mathbb{R}^{N_f}$ are the followers' positions and $x_l \in \mathbb{R}^{N_l}$ are the leaders' positions.

The graph Laplacian can then be partitioned as

$$\mathcal{L}(\mathcal{G}) = \begin{pmatrix} \mathcal{L}_f & l_{fl} \\ l_{fl}^T & \mathcal{L}_l \end{pmatrix},$$

where $\mathcal{L}_f \in \mathbb{R}^{N_f \times N_f}$, $\mathcal{L}_l \in \mathbb{R}^{N_l \times N_l}$, and $l_{fl} \in \mathbb{R}^{N_f \times N_l}$. Assuming that we can control the velocities of the leader agents directly, we thus get the following dynamics

$$\dot{x} = \begin{pmatrix} -\mathcal{L}_f & -l_{fl} \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} u, \quad (3)$$

or $\dot{x} = Ax + Bu$.

What we want to do is drive this system through a collection of waypoints defined as pairs of interpolation times and corresponding desired positions for particular subsets of the agents. This task can be described through a collection of specific affine varieties (defined at the interpolation times) as

$$G_i x(T_i) = d_i, \ \ i = 0, \ldots, q, \quad (4)$$

where the q is the total number of waypoints, the T_i's are the interpolation times, and where, for all i, G_i has full rank, $rank(G_i) \leq n$, and $d_i \in \mathbb{R}^{rank(G_i)}$. (For example, in three dimensions $\{x : G_i x = d_i\}$ represents a plane if $rank(G_i) = 1$, a line if $rank(G_i) = 2$, or a point if $rank(G_i) = 3$.)

We want to achieve this repeated transfer between affine varieties while minimizing the control energy expended, i.e. while minimizing the quadratic cost functional

$$J(u) = \int_{T_0}^{T_q} u^T(t)u(t)dt. \quad (5)$$

Note that we, under the problem formulation in this paper, allow for edges to appear and/or disappear at the interpolation times as long as the graph stays connected. Such changes correspond to changes in the dynamics (different A and B matrices), which calls for a hybrid, optimal control approach, which will be the consideration of Sections IV and Section V. However, before we go ahead and solve the general affine variety transfer problem, we will present the solution to an example multi-agent problem, which is the topic of the next section.

III. Solution for an Example Formation

As an example, consider the situation in which the planar agents interact through a network topology encoded through the graph $\mathcal{G}$, given in Figure 1.

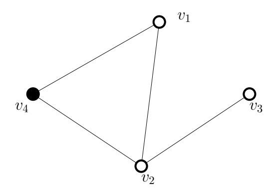

Fig. 1. A multi-agent network is shown, where agent v_4 is the sole leader.

The graph Laplacian for this system is

$$\mathcal{L}(\mathcal{G}) = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}.$$

If we now let $x = (x_1 \ x_2 \ x_3 \ x_4)^T$ be the agent positions in the x direction and $y = (y_1 \ y_2 \ y_3 \ y_4)^T$ be the agent positions in the y direction, we obtain the following completely controllable dynamical systems along the two dimensions

$$\dot{x} = -\mathcal{L}x + e_4 u, \ \ \dot{y} = -\mathcal{L}y + e_4 v,$$

where e_4 is the unit vector with a 1 in the fourth position, and where u, v are the scalar control inputs.

Now, the particular repeated redeployment task that we consider is as follows: Given initial positions for all the agents $x(T_0) = x_0$ and $y(T_0) = y_0$, we want to drive the system in such a way that the followers interpolate specific positions at specific times, i.e. $x_f(T_i) = x_{fi}$ and $y_f(T_i) = y_{fi}$, $i = 1, \ldots, q$. Since the leader position is unconstrained, we obtain a problem involving affine varieties, and since both the dynamics and the affine varieties are decoupled along the two dimension, we can solve the problem along each dimension independently. In fact, in Figure 2, the optimal solution is given for the minimum energy problem in which all the four agents start "close to" $(x_i, y_i)^T \approx (0, 50)^T, \ i = 1, \ldots, 4$, at time $t = 0$. The leader then moves the followers close to $(x_i, y_i)^T \approx (0, 0)^T, \ i = 1, 2, 3$, at $t = 5$ in an optimal fashion, and finally drives them to $(x_i, y_i) \approx (50, 0)^T, \ i = 1, 2, 3$, at time $t = 10$. Figure 2 shows this optimal coordinated maneuver.

IV. Optimal Transfer Between Affine Varieties

Even though the main goal of the following sections is to derive an algorithm for transferring the state of a dynamical system between affine varieties, we will start by recalling the solution to the point to point transfer problem.

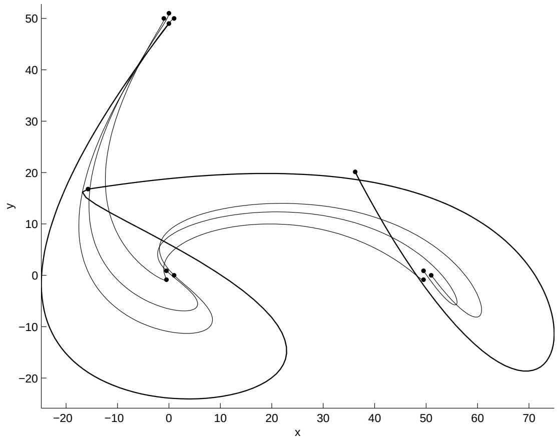

Fig. 2. Starting at a formation close to $(0\ 50)^T$ at time $t = 0$ the leader (thick curve) maneuvers the followers to the new positions close to $(0\ 0)^T$ at $t = 5$ and close to $(50\ 0)^T$ at $t = 10$. This is done while expending the smallest possible control energy.

A. Optimal Point to Point Transfer

The point to point transfer problem involves driving a linear system of differential equations between given boundary states, i.e.

$$\begin{cases} \dot{x} = Ax + Bu \\ x(T_0) = x_0,\ x(T_1) = x_1, \end{cases}$$

where $u \in \mathbb{R}^m$ is the control signal, $x \in \mathbb{R}^n$ the state vector, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$.

The point to point transfer should be done in such a way that a cost functional is minimized with respect to the control signal. The cost functional that we choose to study is

$$J(u) = \int_{T_0}^{T_1} u^T(t)u(t)dt,$$

which can be interpreted as the energy of the control signal.

The two point boundary value problem can be rewritten as the following constraint on u:

$$x_1 - e^{A(T_1-T_0)}x_0 = \int_{T_0}^{T_1} e^{A(T_1-s)}Bu(s)ds.$$

Although the point to point transfer problem can be solved in a number of ways, we will view it as a minimum-norm problem in an infinite dimensional Hilbert-space (e.g. [17]). This approach is not new (for example, see [18] and [17]), but we choose to include this construction in order to highlight and derive tools that will be of use in later sections.

Let H be the Hilbert space of m dimensional vectors of square-integrable input functions on $[T_0, T_1]$, with inner product

$$\langle u, v\rangle_H = \int_{T_0}^{T_1} u^T(t)v(t)dt.$$

Define $L : H \to \mathbb{R}^n$ as the linear mapping

$$Lu = \int_{T_0}^{T_1} e^{A(T_1-s)}Bu(s)ds$$

and let $d = x_1 - e^{A(T_1-T_0)}x_0$. We can then formulate the problem as

$$\min_u \|u\|_H^2,$$

under the condition that

$$Lu = d.$$

Observe that this problem only has a solution if $d \in Im(L)$. Let us assume this is in fact the case. Then $Lu = d$ is an affine variety, i.e. a translated linear subspace in H, and the Projection theorem guarantees the existence of a unique minimizer u^*. (For an accessible treatment of this subject, see [17].) Furthermore, we also know that $u^* \perp ker(L)$.

Let $L^* : \mathbb{R}^n \to H$ be the adjoint of L, i.e. $\langle a, Lu\rangle_{\mathbb{R}^n} = \langle L^*a, u\rangle_H$ for any $a \in \mathbb{R}^n$ and any $u \in H$. We then have the following fundamental relation between L^* and L $Im(L^*) = (ker(L))^\perp$. And, since $u^* \in (ker(L))^\perp$, we have that $u^* \in Im(L^*)$, i.e. $u^* = L^*\lambda$ for some $\lambda \in \mathbb{R}^n$. In other words

$$\begin{cases} u^* = L^*\lambda \\ Lu^* = d \end{cases} \quad \Leftrightarrow \quad \begin{cases} u^* = L^*\lambda \\ LL^*\lambda = d \end{cases}$$

By assuming that the system is completely controllable, and using the well-known expressions for L^* and M, we finally get

$$u^*(t) = B^T e^{A^T(T_1-t)}M^{-1}(x_1 - e^{A(T_1-T_0)}x_0)$$

and

$$J(u^*) = (x_1 - e^{A(T_1-T_0)}x_0)^T M^{-1}(x_1 - e^{A(T_1-T_0)}x_0).$$

V. Optimal Transfer Between Multiple Affine Varieties

The goal is to drive the system in such a way that the solution lies on specific affine varieties at given times, as illustrated in Figure 3. The dynamics of the system may differ between the affine varieties, as seen in Figure 4.

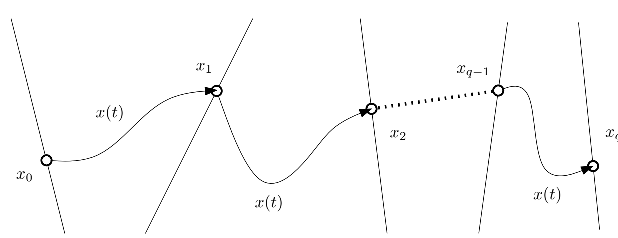

Fig. 3. Transfer between q+1 affine varieties.

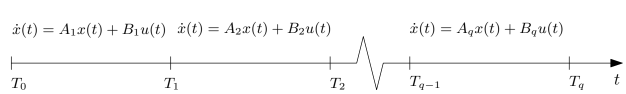

Fig. 4. System of q+1 affine varieties together with a switched linear system.

Again, as in the point to point transfer, we want to minimize the energy of the control signal. And, if we denote the affine variety at time T_i by $S_i = \{x \in \mathbb{R}^n : G_i x = d_i\}$, we get

$$\min_u J(u)$$

subject to

$$\begin{cases} \dot{x} = A_i x + B_i u \text{ for } t \in (T_{i-1}, T_i),\ i = 1, \dots, q \\ x(T_i) \in S_i,\ i = 0, \dots, q. \end{cases}$$

A. Dynamic Programming

Now, we will use the solution to the optimal point to point transfer problem to formulate a dynamical programming problem from which we can solve for the optimal intersection points on the affine varieties.

Dynamic programming divides the problem into stages with a decision required at each stage. Every stage has a state associated with it. In our case the stages are represented by the times we are supposed to be on an affine variety, and the state is the state vector at this time. From the point to point problem, we know that given the state at a stage there is a unique path of minimum cost that takes the system to a specific point at the succeeding stage. So the decision to be made at each stage is where on the affine variety at the succeeding stage we want to end up, given the state we are in at the current stage.

Let $c_i(a, b)$ be the cost of going from a state a in stage $i-1$ to a state b in stage i, and let $f_i(a)$ be the minimum cost of going to the affine variety at the final stage, via the intermediate affine varieties, when starting at state a in stage i, as illustrated in Figure 5. Let $x_i = x(T_i)$ for $0 \le i \le q$, we then have the following Bellman recursion:

$$f_{i-1}(x_{i-1}) = \min_{x_i \in S_i} (c_i(x_{i-1}, x_i) + f_i(x_i)),\ 0 < i \le q.$$

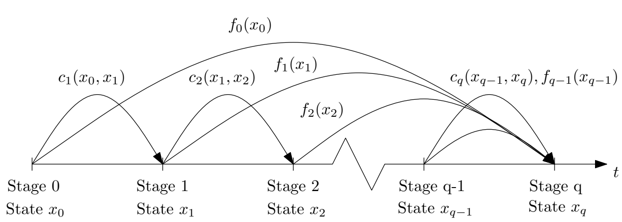

Fig. 5. Transfer costs in the dynamic programming algorithm.

Our problem can then be reformulated as

$$\min_{x_0 \in S_0} (f_0(x_0)).$$

Using the cost from the point to point problem gives us

$$c_i(a, b) = (b - e^{A_i \Delta T_i} a)^T M_i^{-1} (b - e^{-A_i \Delta T_i} a),$$

where $\Delta T_i = T_i - T_{i-1}$ and M_i is the controllability Grammian associated with the pair (A_i, B_i).

We will assume that the systems are completely controllable. Thus M_i is a symmetric, positive definite matrix and so is the inverse M_i^{-1}. Let us denote M_i^{-1} by Q_i. Then

$$c_i(a, b) = (b - e^{A_i \Delta T_i} a)^T Q_i (b - e^{A_i \Delta T_i} a) = \|b - e^{A_i \Delta T_i} a\|^2_{Q_i}.$$

B. Main Result

Since stage q is the final stage $f_q(a) = 0,\ \forall a \in \mathbb{R}^n$. With this as the starting condition we can work our way backwards using the Bellman recursion until we get an expression for $f_0(x_0)$. We then minimize $f_0(x_0)$ to get the minimizer x_0^*. By this time we will know the relation between x_0^* and the remaining optimal points on the affine varieties. Once all the optimal points on the affine varieties have been determined, all we need to do is to compute a sequence of point to point transfers:

$$\begin{aligned} f_{q-1}(x_{q-1}) &= \min_{x_q \in S_q} (c_{q-1}(x_{q-1}, x_q) + f_q(x_q)) \\ &= \min_{x_q \in S_q} \|x_q - e^{A_q \Delta T_q} x_{q-1}\|^2_{Q_q} \end{aligned}$$

Now, define the finite dimensional Hilbert space $\mathbf{H}_q = \mathbb{R}^n$ with inner product

$$\langle x, y \rangle_{\mathbf{H}_q} = \langle x, y \rangle_{Q_q} = x^T Q_q y$$

for $x, y \in \mathbf{H}_q$. Since S_q defines an affine variety

$$V_q^{d_q} = \{x \in \mathbf{H}_q : G_q x = d_q\}$$

in $\mathbf{H}_q$ what we want is to find $x_q \in V_q^{d_q}$ closest to $p_q = e^{A_q \Delta T_q} x_{q-1} \in \mathbf{H}_q$, i.e. to solve

$$\min_{x_q \in V_q^{d_q}} \|x_q - p_q\|^2_{H_q}.$$

According to Hilbert's projection theorem [17] this problem has a unique optimal solution given by

$$x_q^* = V_q^{d_q} \cap (V_q^{0\perp} + p_q),$$

as seen in Figure 6, where $V_q^{0\perp} = \{x : \langle x, y \rangle_{\mathbf{H}_q} = x^T Q_q y = 0\ \forall\ y \in V_q^0\}$ is the orthogonal complement of the linear subspace $V_q^0 = \{x \in \mathbf{H}_q : G_q x = 0\}$. And, since $Im(G_q^T) = Ker(G_q)^\perp$, we have

$$V_q^{0\perp} = \{x \in \mathbf{H}_q : \exists \lambda \in \mathbb{R}^{rank(G_q)} \text{ s.t. } Q_q x = G_q^T \lambda\}.$$

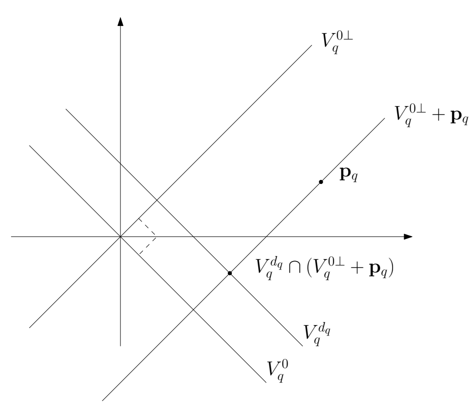

Fig. 6. The process of finding the point in $V_q^{d_q}$ closest to $\mathbf{p}_q \in \mathbf{H}_q$.

Now, since the optimal solution is given by $x_q^* = V_q^{d_q} \cap (V_q^{0\perp} + p_q)$, this results in the following linear system of equations for the optimal point

$$\begin{cases} G_q x_q^* = d_q \\ Q_q(x_q^* - p_q) = Q_q(x_q^* - e^{A_q \Delta T_q} x_{q-1}) = G_q^T \lambda_q \end{cases}$$

or, written in matrix form Written in matrix form

$$P_q \begin{pmatrix} x_q^* \\ \lambda_q \end{pmatrix} = \begin{pmatrix} Q_q p_q \\ d_q \end{pmatrix},$$

where

$$P_q = \begin{pmatrix} Q_q & -G_q^T \\ G_q & 0 \end{pmatrix}.$$

Since the system has a unique solution according to the projection theorem, P_q^{-1} exists. Denote the upper left $n \times n$ matrix of P_q^{-1} by $(P_q^{-1})_{11}$ and the upper right $n \times rank(G_q)$ matrix of P_q^{-1} by $(P_q^{-1})_{12}$.

This gives us

$$x_q^* = H_q x_{q-1} - h_q,$$

where

$$H_q = (P_q^{-1})_{11} Q_q e^{A_q \Delta T_q}$$

and

$$h_q = -(P_q^{-1})_{12} d_q$$

Claim: x_k^* can be written as an affine function of x_{k-1} i.e. $x_k^* = H_k x_{k-1} - h_k$, $0 < k \leq q$.

We have already seen that this is true for $k = q$. Let us have a look at an arbitrary k, $0 < k < q$.

Assume that the claim holds for all i, $k < i \leq q$.

$$f_{k-1}(x_{k-1}) = \min_{x_k \in S_k} (c_k(x_{k-1}, x_k) + f_k(x_k)),$$

which can be rewritten as

$$\min_{x_k \in S_k} (\|x_k - p_k\|_{Q_k}^2 + \sum_{j=k+1}^{q} \|F_k^{(j-k)} x_k - p_k^{(j-k)}\|_{Q_j}^2),$$

where

$$\begin{aligned} &F_k^{(1)} = H_{k+1} - e^{A_{k+1} \Delta T_{k+1}} \\ &F_k^{(j)} = F_{k+1}^{(j-1)} H_{k+1},\ j = 2 \ldots q-k \\ &p_k = e^{A_k \Delta T_k} x_{k-1} \\ &p_k^{(1)} = h_{k+1} \\ &p_k^{(j)} = F_{k+1}^{(j-1)} h_{k+1} + p_{k+1}^{(j-1)},\ j = 2 \ldots q-k. \end{aligned}$$

Define the finite dimensional Hilbert space $\mathbf{H}_k = \mathbb{R}^n \oplus \ldots \oplus \mathbb{R}^n$ with inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{H}_k} = \langle x, y \rangle_{Q_k} + \sum_{j=k+1}^{q} \langle x^{(j-k)}, y^{(j-k)} \rangle_{Q_j}$$

for $\mathbf{x} = (x, x^{(1)}, \ldots, x^{(q-k)}),\ \mathbf{y} = (y, y^{(1)}, \ldots, y^{(q-k)}) \in \mathbf{H}_k$. Define the affine variety $V_k^{d_k}$ as

$$\{\mathbf{x} \in \mathbf{H}_k : G_k x = d_k,\ F_k^{(1)} x^{(1)} = x, \ldots,\ F_k^{(q-k)} x^{(q-k)} = x\}.$$

Then we can write

$$f_{k-1}(x_{k-1}) = \min_{\mathbf{x}_k \in V_k^{d_k}} (\|\mathbf{x}_k - \mathbf{p}_k\|_{\mathbf{H}_k}^2)$$

So the problem is to find $\mathbf{x}_k = (x_k, x_k^{(1)}, \ldots, x_k^{(q-k)}) \in V_k^{d_k}$ closest to $\mathbf{p}_k = (p_k, p_k^{(1)}, \ldots, p_k^{(q-k)}) \in \mathbf{H}_q$. Again, the unique optimal solution is given by $\mathbf{x}_k^* = V_k^{d_k} \cap (V_k^{0\perp} + \mathbf{p}_k)$, where $V_k^{0\perp}$ is the orthogonal complement to the linear subspace V_k^0 given by

$$\{\mathbf{x} \in \mathbf{H}_k : G_k x = 0,\ F_k^{(1)} x^{(1)} = x, \ldots,\ F_k^{(q-k)} x^{(q-k)} = x\},$$

with $V_k^{0\perp}$ given by

$$\begin{aligned} &\{\mathbf{x} \in \mathbf{H}_k : \exists \lambda \in \mathbb{R}^{rank(G_k)} \\ &\quad \text{s.t. } Q_k x + \textstyle\sum_{j=k+1}^{q} F_k^{(j-k)T} Q_j x^{(j-k)} = G_k^T \lambda\}. \end{aligned}$$

The optimal point, $\mathbf{x}_k^* = V_k^{d_k} \cap (V_k^{0\perp} + \mathbf{p}_k)$, is then given by

$$\begin{cases} G_k x_k^* = d_k \\ F_k^{(1)} x_k^* = x_k^{*(1)} \\ \vdots \\ F_k^{(q-k)} x_k^* = x_k^{*(q-k)} \\ Q_k(x_k^* - p_k) + \\ \quad \sum_{j=k+1}^{q} F_k^{(j-k)T} Q_j (x_k^{*(j-k)} - p_k^{(j-k)}) = G_k^T \lambda_k \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} G_k x_k^* = d_k \\ Q_k(x_k^* - p_k) + \\ \quad \sum_{j=k+1}^{q} F_k^{(j-k)T} Q_j (F_k^{(j-k)} x_k^* - p_k^{(j-k)}) = G_k^T \lambda_k \end{cases}$$

Writing this in matrix form we have

$$P_k \begin{pmatrix} x_k^* \\ \lambda_k \end{pmatrix} = \begin{pmatrix} Q_k p_k + \sum_{j=k+1}^{q} F_k^{(j-k)T} Q_j p_k^{(j-k)} \\ d_k \end{pmatrix},$$

with

$$P_k = \begin{pmatrix} Q_k + \sum_{j=k+1}^{q} F_k^{(j-k)T} Q_j F_k^{(j-k)} & -G_k^T \\ G_k & 0 \end{pmatrix}.$$

Let $(P_k^{-1})_{11}$ be the upper left $n \times n$ matrix of P_k^{-1} and $(P_k^{-1})_{12}$ the upper right $n \times rank(G_k)$ matrix of P_k^{-1}.

We then have

$$x_k^* = H_k x_{k-1} - h_k$$

where

$$H_k = (P_k^{-1})_{11} Q_k e^{A_k \Delta T_k}$$

and

$$h_k = -(P_k^{-1})_{12} d_k - (P_k^{-1})_{11} \sum_{j=k+1}^{q} (F_k^{(j-k)T} Q_j p_k^{(j-k)})$$

So, if the claim holds for all i such that $k < i \leq q$ the claim holds for k. Since the claim holds for $k = q$, by induction, the claim holds for all k, $0 < k \leq q$.

Now we know how to compute $f_0(x_0)$ so all that remains is to find the minimizing x_0.

$$\min_{x_0 \in S_0} (f_0(x_0)) = \min_{x_0 \in S_0} (\sum_{j=1}^{q} \|F_0^{(j)} x_0 - p_0^{(j)}\|_{Q_j}^2),$$

where

$$\begin{array}{l} F_0^{(1)} = H_1 - e^{A_1 \Delta T_1} \\ F_0^{(j)} = F_1^{(j-1)} H_1, j = 2 \dots q \\ p_0^{(1)} = h_1 \\ p_0^{(j)} = F_1^{(j-1)} h_1 + p_1^{(j-1)},\ j = 2 \dots q \end{array}$$

Define the finite dimensional Hilbert space $\mathbf{H}_0 = \mathbb{R}^n \oplus \ldots \oplus \mathbb{R}^n$ with inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbf{H}_0} = \sum_{j=1}^{q} \langle x^{(j)}, y^{(j)} \rangle_{Q_j}$$

for $\mathbf{x} = (x^{(1)}, \ldots, x^{(q)}),\ \mathbf{y} = (y^{(1)}, \ldots, y^{(q)}) \in \mathbf{H}_0$. Define the affine variety

$$\begin{array}{l} V_0^{d_0} = \{\mathbf{x} \in \mathbf{H}_0 : G_0 x = d_0,\ F_0^{(1)} x^{(1)} = x, \ldots \\ \quad F_0^{(q-k)} x^{(q)} = x,\ \text{for some } x \in \mathbb{R}^n\}. \end{array}$$

We can then write

$$\min_{x_0 \in S_0} (f_0(x_0)) = \min_{\mathbf{x}_0 \in V_0^{d_0}} (\|\mathbf{x}_0 - \mathbf{p}_0\|_{\mathbf{H}_0}^2),$$

i.e. find $\mathbf{x}_0 = (x_0^{(1)}, \ldots, x_0^{(q)}) \in V_0^{d_0}$ closest to $\mathbf{p}_0 = (p_0^{(1)}, \ldots, p_0^{(q)}) \in \mathbf{H}_0$. Again, there exists a unique optimal solution $\mathbf{x}_0^* \in \mathbf{H}_0$ but x_0^* may not be uniquely defined by $\mathbf{x}_0^*$. In fact x_0^* is uniquely defined if and only if $Ker(G_0) \cap Ker(F_0^{(1)}) \cap \ldots \cap Ker(F_0^{(q)})) = \{0\}$.

Using the definition of orthogonality in $\mathbf{H}_0$ and the fact that $Im(G_0^T) = Ker(G_0)^\perp$, we get

$$\begin{array}{l} V_0^{0\perp} = \{\mathbf{x} \in \mathbf{H}_0 : \exists \lambda \in \mathbb{R}^{rank(G_0)} \\ \quad \text{s.t. } \sum_{j=1}^{q} F_0^{(j)T} Q_j x^{(j)} = G_0^T \lambda\}. \end{array}$$

The optimal point, $\mathbf{x}_0^* = V_0^{d_0} \cap (V_0^{0\perp} + \mathbf{p}_0)$, is then given by

$$\left\{ \begin{array}{l} G_0 x_0^* = d_0 \\ F_0^{(1)} x_0^* = x_0^{*(1)} \\ \vdots \\ F_0^{(q)} x_0^* = x_k^{*(q)} \\ \sum_{j=1}^{q} F_0^{(j)T} Q_j (x_0^{*(j)} - p_0^{(j)}) = G_0^T \lambda_0 \end{array} \right.$$

$$\Updownarrow$$

$$\left\{ \begin{array}{l} G_0 x_0^* = d_0 \\ \sum_{j=1}^{q} F_0^{(j)T} Q_j (F_0^{(j)} x_0^* - p_0^{(j)}) = G_0^T \lambda_0 \end{array} \right.$$

Writing this in matrix form we get

$$\left(\begin{array}{cc} \sum_{j=1}^{q} F_0^{(j)T} Q_j F_0^{(j)} & -G_0^T \\ G_0 & 0 \end{array} \right) \left(\begin{array}{c} x_0^* \\ \lambda_0 \end{array} \right) =$$
$$= \left(\begin{array}{c} \sum_{j=1}^{q} F_0^{(j)T} Q_j p_0^{(j)} \\ d_0 \end{array} \right)$$

As mentioned earlier the solution to this system may not be unique. However if we only need to find one solution we can use the Moore-Penrose inverse to get the solution of minimum norm.

VI. Conclusions

To summarize: We have obtained an algorithm that solves the problem of optimal transfer between multiple affine varieties. This algorithm is based on dynamic programming and minimum norm optimization in nested Hilbert spaces. And, the multi-agent coordination problem considered in Section II is solved in Figure 2 using this algorithm for computing the optimal leader maneuver.

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