On the LQ-Based Optimization Techniques for for Impulsive Hybrid Control Systems

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***Abstract*— This paper deals with a quadratic optimization problem for linear impulsive hybrid systems. We study a class of LQ-type impulsive hybrid optimal control problems (OCPs) and consider the application of the hybrid Maximum Principle (MP). Our aim is to investigate the natural relationship between the Pontryagin-type MP and the Bellman Dynamic Programming (DP) approach. As next we develop the "hybrid" Riccati formalism and discuss some related computational aspects.**

I. Introduction

This paper addresses optimization problems for a class of linear impulsive hybrid systems (LIHSs). It is well-known that the ability to operate a hybrid control system in an optimal way remains a challenging theoretical task (see e.g., [18], [13], [16], [17], [12], [3], [9], [10], [1], [2], [4], [5], [6], [7]). For a classical closed-loop based OCP, one of the main tools toward the construction of optimal trajectories is the celebrated Bellman DP method. For a conventional OCP the DP approach is equivalent to the techniques based on the classic Pontryagin MP (see e.g., [8], [11]). On the other hand, the above-mentioned optimization techniques are not sufficiently advanced to the LQ-type OCPs governed by impulsive hybrid dynamical systems. The class of hybrid models considered in this contribution involves systems driven by continuous control inputs where switching is accompanied by a jump in the state. A similar class has been considered in [5], [6]. The aim of this contribution is to study a possible relationship between the DP and MP in the case of a LQ-optimal impulsive hybrid OCPs. Moreover, we deduce the corresponding Riccati-formalism (similarly to the conventional LQ-theory), propose the constructive numerical schemes based on this Riccati approach and illustrate our computational approach by way of some examples.

The remainder of our paper is organized as follows. Section 2 contains some preliminary facts, basic concepts and the problem formulation. Section 3 is devoted to an equivalent representation of the LIHSs and to the Riccati-formalism for the LQ problems in the hybrid setting. Section 4 contains a simple numeric example. Moreover, we also discuss shortly some general numerical aspects in the context of impulsive hybrid OCPs. Section 5 summarizes the paper.

II. Optimization of Linear Impulsive Hybrid Systems

Let us start introducing a variant of general concept of a LIHS used in this paper (see e.g., [6], [7] for details).

Definition 1: An LIHS is a 8-tuple

$$\chi = \{Q, X, U, A, B, \mathscr{U}, \Theta, S\}$$

where

- Q is a finite set of locations;
- $X = \{X_q\},\ q \in Q$, where $X_q \subseteq \mathbb{R}^n$, is a family of the state spaces;
- $U \subseteq \mathbb{R}^m$ is a set of admissible control input values;
- $A = \{A_q(\cdot)\},\ B = \{B_q(\cdot)\},\ q \in Q$ are families of continuously matrix-functions

$$A_q : \mathbb{R} \to \mathbb{R}^{n\times n},\ B_q : \mathbb{R} \to \mathbb{R}^{n\times m};$$

- $\mathscr{U} := \{u(\cdot) \in \mathbb{L}^\infty_m(0,t_f) : u(t) \in U \text{ a. e. on } [0,t_f]\}$ is a set of all admissible control functions;
- $\Theta = \{\Theta_q\},\ q \in Q$ is a collection of maximal constant amplitudes (state jumps);
- S is a subset of Ξ, where

$$\Xi := \{(q,x,q',x') : q,q' \in Q, x \in X_q, x' \in X_{q'}\}.$$

In the following, we assume that the control set U is compact and convex. Moreover, we suppose that smooth functions $m_{q,q'} : \mathbb{R}^n \to \mathbb{R},\ q,q' \in Q$,

$$m_{q,q'}(x) = b^T_{q,q'}x + c_{q,q'}$$

are given such that the hyperplanes

$$M_{q,q'} := \{x \in \mathbb{R}^n : m_{q,q'}(x) = 0\}$$

are pairwise disjoint. Here $b_{q,q'} \in \mathbb{R}^n$ and $c_{q,q'} \in \mathbb{R}$ for every $q,q' \in Q$. The given hyperplanes $M_{q,q'}$ represents the affine switching sets at which a switch from location q to location q' can take place. In our paper we consider LIHSs with $r \in \mathbb{N}$ switchings. The switching times are given by the following sequence: $\{t_i\}$, $i = 1,\dots,r$, where

$$0 = t_0 < t_1 < \dots < t_{r-1} < t_r = t_f.$$

Note that the above sequence of switching times $\{t_i\}$ is not defined a priori. A hybrid control system remains in location $q_i \in Q$ for all $t \in [t_{i-1}, t_i), i = 1,\dots,r$. Let us also note that the pair $(q, x(t)), q \in Q, x \in \mathbb{R}^n$, represents the hybrid state

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at time t. Moreover, every switching time $t_i,\ i=1,...,r-1$ depends on the state of the given LHS.

We now introduce some standard spaces, namely, the space $\mathbb{C}_0^\infty\left(0,t_f\right)$ of all $\mathbb{C}^\infty$ functions that vanish outside a compact subset of $\left(0,t_f\right)$ and the space $D'\left(0,t_f\right)$ of generalized functions (the Schwartz distributions). It is well-known that $D'\left(0,t_f\right)$ can be considered as a space of linear, sequentially continuous functionals with respect to the convergence on the space $\mathbb{C}_0^\infty\left(0,t_f\right)$. We are ready to introduce the notion of a linear hybrid trajectory to the LIHS under consideration (see e.g., [6]).

Definition 2: The hybrid trajectory of a LIHS is a triple $X=(x(\cdot),\{q_i\},\tau)$, where $x(\cdot)\in D'\left(0,t_f\right)$ is a discontinuous trajectory, $\{q_i\}, i=q,\ldots,r$ is a finite sequence of locations and τ is the corresponding sequence of switching times such that for each $i=1,\ldots,r$ there exist $u(\cdot)\in\mathscr{U}$ such that:

- $x(0)=x_0\notin\bigcup\limits_{q,q'\in Q}M_{q,q'}$ and $x_i(\cdot)=x(\cdot)|_{(t_{i-1},t_i)}$ is an absolutely continuous function on (t_{i-1},t_i);
- $x_i(t_i)\in M_{q_i,q_{i+1}}$ for i=1,…,r-1;
- $\dot{x}_i(t)=A_i(t)x_i(t)+B_i(t)u_i(t)+\theta_{q_i}\delta(t-t_i)$ for almost all $t\in[t_{i-1},t_i]$, where δ is the Dirac function and $\left\|\theta_{q_i}\right\|\leq\Theta_{q_i}$ and $u_i(\cdot)$ is a restriction of the chosen control function $u(\cdot)$ on the time interval $[t_{i-1},t_i]$.

The derivative $\dot{x}_i(\cdot)$ in Definition 2 is considered as a weak derivative of the generalized function $x_i(\cdot)$ defined on the full interval $[t_{i-1},t_i]$. It is also evident that a function $x(\cdot)$ from definition 2 consists of absolutely continuous parts defined on the open interval (t_{i-1},t_i) and involves jumps of magnitude θ_{q_i} at the switching times t_i. The global evolution equation of the given LIHS can also be represented as follows.

$$\begin{aligned}\dot{x}(t)=&\sum_{i=1}^{r}\beta_{[t_{i-1},t_i)}(t)\left[A_i(t)x_i(t)+B_i(t)u_i(t)\right]+\\&\sum_{i=1}^{r}\theta_{q_i}\delta(t-t_i)\ \text{a.e. on}\ \left[0,t_f\right],\quad x(0)=x_0\end{aligned}\tag{1}$$

where $\beta_{[t_{i-1},t_i)}(\cdot)$ is the characteristic function of the intervals $[t_{i-1},t_i),\ i=1,\ldots,r$. Since every switching time t_i, where $i=1,...,r-1$, depends on the trajectory $x(\cdot)$ of a LHS, every characteristic function $\beta_{[t_{i-1},t_i)}(\cdot)$ is also a function of the state vector.

Note that the initial value problem (1) is also considered in the sense of weak derivatives on the space $D'\left(0,t_f\right)$ and for each $u(\cdot)\in\mathscr{U}$ and all $\left\|\theta_{q_i}\right\|\leq\Theta_{q_i},\ i=1,\ldots,r$, the initial value problem (1) has a unique solution in $D'\left(0,t_f\right)$. Let $S_f:\mathbb{R}\to\mathbb{R}^{n\times n}$, $S_q:\mathbb{R}\to\mathbb{R}^{n\times n}$ and $R_q:\mathbb{R}\to\mathbb{R}^{m\times m}$, for all locations $q\in Q$. Assume that S_f is symmetric and positive semidefinite, and that for every $t\in\left[0,t_f\right]$ and every $q\in Q$, $S_q(t)$ is also a symmetric and positive semidefinite matrix. Moreover, let $R_q(t)$ be a symmetric and positive definite for every $t\in\left[0,t_f\right]$ and every $q\in Q$. We also assume that the given matrix-functions $S_q(\cdot)$, $R_q(\cdot)$ are continuous. Given an LIHS we consider the following LQ-type problem:

$$\begin{aligned}&\text{minimize } J(u(\cdot),\theta,x(\cdot)):=\frac{1}{2}\left(x_r^T\left(t_f\right)S_f x_r\left(t_f\right)\right)+\\&\sum_{i=1}^{r}\frac{1}{2}\int_{t_{i-1}}^{t_i}\left(x_i^T(t)S_{q_i}(t)x_i(t)+u_i^T(t)R_{q_i}(t)u_i(t)\right)\mathrm{d}t\\&\text{over all admissible trajectories } X \text{ of the LIHS}\end{aligned}\tag{2}$$

Evidently (2) is a problem of minimizing the quadratic Bolza cost functional J over all trajectories of the given linear hybrid system. Note that we study the impulsive hybrid OCP (1) in the absence of possible target and state constraints. Throughout this paper we assume that the LQ problem (2) has an optimal solution

$$\left(u^{opt}(\cdot),\theta^{opt},X^{opt}(\cdot)\right)\in\mathscr{C}:=\mathscr{U}\times\Theta^r\times\mathbb{R}^{n\times r},$$

where $\theta^{opt}:=\left(\theta_{q_1}^{opt}\ldots\theta_{q_r}^{opt}\right)$ is a matrix representing the optimal jumps. The optimal control problem 2 is an optimization problem formulated on the space $\mathscr{C}$ which involves the space of generalized functions $D'\left(0,t_f\right)$. Our aim is to simplify the initial problem 2. Considering the following auxiliary initial value problem

$$\begin{aligned}\dot{y}(t)=&\sum_{i=1}^{r}\beta_{[t_{i-1},t_i)}(t)\left[A_i(t)\left(y_i(t)+\theta_{q_i}\eta(t-t_i)\right)+\right.\\&\left.B_i(t)u_i(t)\right]\ \text{a.e. on}\ \left[0,t_f\right],\quad y(0)=x_0,\end{aligned}\tag{3}$$

where $i=1,\ldots,r$ and $\eta(\cdot)$ is a Heaviside step-function, we could formulate an auxiliary hybrid OCP governed by a hybrid system with autonomous location transitions (without jumps in the continuous state, see [4]). Evidently, the initial value problem (3) has a unique absolutely continuous solution for each $u(\cdot)\in\mathscr{U}$. We could consider $y(\cdot)$ as an element of the Sobolev space $x(\cdot)\in\mathbb{W}_n^{1,\infty}\left(0,t_f\right)$, i.e, the space of absolutely continuous functions with essentially bounded derivatives. We have the following equivalence result.

Theorem 1: Under the assumptions of Section II, the (unique) solution $x(\cdot)\in D'\left(0,t_f\right)$ of the initial value problem (1) can be represented in the following form:

$$x(t)=y(t)+\sum_{i=1}^{r}\theta_{q_i}\eta(t-t_i),$$

where $y(\cdot)\in\mathbb{W}_n^{1,\infty}\left(0,t_f\right)$ is a (unique) solution to the auxiliary initial value problem (3)

We refer to [6] for the proof of this result. Using Theorem 1 we could study an auxiliary hybrid system with autonomous location transitions (see [4]). Recall the corresponding concept.

Definition 3: A hybrid control system with autonomous location transitions (HSAL) is the following 7-tuple

$$\zeta=\{Q,X,U,A,B,\mathscr{U},S^a\}$$

where $Q,X,U,\mathscr{U},A,B$ are from Definition 1 and

$$S^a\subset\Xi^a:=\left\{(q,y,q',y'),\ q,q'\in Q,y\in X_q,y'\in X_{q'}\right\}.$$

Moreover a hybrid trajectory of ζ is a triple

$$Y = \left(y(\cdot), \{q_i\}^a, \tau^a\right),$$

where $y(\cdot): [0,T] \to \mathbb{R}^n$, and for each $i = 1,\dots,r$ there exist $u(\cdot) \in \mathscr{U}$ such that:

- $y(0) = y_0$ and $y_i(\cdot) = y(\cdot)|_{(t_{i-1},t_i)}$ is an absolutely continuous function on (t_{i-1},t_i) continuously prolongable to $[t_{i-1},t_i]$, $i = 1,\dots,r$;
- $\dot{y}_i(t) = A_i(t)y_i(t) + B_i(t)u_i(t)$ for almost all $t \in [t_{i-1},t_i]$, $i = 1,\dots,r$

The switching manifolds $M_{q_i,q_{i+1}}$ are now characterized by the following equations: $m_{q,q'}(y_i(t) + \theta_{q_i}) = 0$, $i = 1,\dots,r$.

Now we could formulate the auxiliary OCP using the above transformation and the initial value problem (3)

$$\begin{aligned}&\text{minimize } \frac{1}{2}\Big[\left(y_r(t_f)+\theta_{q_r}\right)^T S_f \left(y_r(t_f)+\theta_{q_r}\right) + \\ &\sum_{i=1}^{r}\int_{t_{i-1}}^{t_i}\Big((y_i(t)+\theta_{q_i}\eta(t-t_i))^T S_{q_i}(t)(y_i(t)+\theta_{q_i}\eta(t-t_i)) + \\ &u_i^T(t)R_{q_i}(t)u_i(t)\Big)\,\mathrm{d}t\Big]\end{aligned} \tag{4}$$

over all admissible trajectories Y of the HSAL.

It is necessary to stress that an optimal solution $(v^{opt}(\cdot), Y^{opt}(\cdot))$, where $v := (u,\theta)$, of the auxiliary OCP (4) defines the corresponding optimal solution $(u^{opt}(\cdot), \theta^{opt}, X^{opt}(\cdot))$ for the problem (2). We formulate this relationship more precisely.

Theorem 2: Suppose that problems (2) and (4) have both optimal solutions. Under the assumptions of this Section, every optimal solution $(v^{opt}(\cdot), Y^{opt})$ of problem (4) defines the corresponding optimal solution $(v^{opt}(\cdot), X^{opt}(\cdot))$ for problem (2), where

$$x^{opt}(t) = y^{opt}(t) + \sum_{i=1}^{r}\theta_{q_i}^{opt}\eta(t - t_i^{opt}).$$

The proof of this result follows from the following fact: the transformation from Theorem 1 is a bijective transformation (see [6] for theoretical details). Let us now formulate the hybrid MP presented in [6] for the case of LQ hybrid OCP (4).

Theorem 3: Let the matrices A, B be continuous and the optimal control problem (4) be regular. Then there exist a function $\psi_i(\cdot)$ from $\mathbb{W}_m^{1,\infty}(0,t_f)$ and a non-zero (Lagrange) vector $a = (a_1,\dots,a_{r-1})^T \in \mathbb{R}^{r-1}$ such that

$$\begin{aligned}&\frac{d}{dt}\psi_i(t) = -A_{q_i}^T(t)\psi_i(t) + S_{q_i}(y_i(t) + \theta_{q_i}\eta(t-t_i)) \\ &\text{a.e. on } \left[t_{i-1}^{opt}, t_i^{opt}\right] \\ &\psi_r(t_f) = -S_f\left(y_r^{opt}(t_f) + \theta_{q_r}\right)\end{aligned} \tag{5}$$

and $\psi_i(t_i^{opt}) = \psi_{i+1}(t_i^{opt}) + a_i b_{q_i,q_{i+1}}$, where $i = 1,\dots,r-1$. Moreover, for every admissible control $u(\cdot) \in \mathscr{U}$ the partial Hamiltonian

$$\begin{aligned}&H_{q_i}(t,y,v,\psi) = \left\langle \psi_i, A_{q_i}(t)(y_i + \theta_{q_i}\eta(t-t_i)) + B_{q_i}(t)u_i\right\rangle - \\ &\frac{1}{2}(y_i + \theta_{q_i}\eta(t-t_i))^T S_{q_i}(y_i + \theta_{q_i}\eta(t-t_i)) - \frac{1}{2}u_i^T R_{q_i}u_i\end{aligned}$$

satisfies the following maximization conditions

$$\begin{aligned}&\max_{v\in U\times\Theta} H_{q_i}\left(t, y^{opt}, v, \psi(t)\right) = \\ &H_{q_i}\left(t, y^{opt}(t) + \sum_{i=1}^{r}\theta_{q_i}^{opt}\eta(t-t_i), u^{opt}(t), \psi(t)\right).\end{aligned} \tag{6}$$

Here are $t \in \left[t_{i-1}^{opt}, t_i^{opt}\right)$, $i = 1,\dots,r$. Moreover, we use the notation $\psi(t) := \sum_{i=1}^{r}\beta_{[t_{i-1}^{opt}, t_i^{opt})}\psi_i(t)$ for all $t \in [0,t_f]$.

To avoid that a trajectory of the given hybrid system (3) has a sliding mode behavior with respect to a switching hyperplane, we will assume that the system fulfils the additional condition:

$$\alpha\dot{y}_i(t_i) + (1-\alpha)\dot{y}_{i+1}(t_i) \notin M_{q_i,q_{i+1}}; \quad \forall\alpha \in \mathbb{R},\ i = 1,\dots,r$$

The last relation is equivalent to the following

$$\begin{aligned}&b[(\alpha A_i(t_i) + (1-\alpha)A_{i+1}(t_i))(y_i(t_i) + \theta_{q_i}) + \\ &\alpha B_i u_i(t_i) + (1-\alpha)B_{i+1}u_{i+1}] + c \neq 0\end{aligned}$$

where the value of y_i is determined by the Cauchy formula

$$\begin{aligned}&y_i(t_i) = \Phi(t_i,t_0)x_0 + \sum_{j=2}^{i}\Phi(t_i,t_j)\theta_{j-1} + \\ &\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}\Phi(t_i,t_j)\Phi_j(t_j,\tau)B_j(\tau)u_j(\tau)\,\mathrm{d}\tau,\end{aligned}$$

where $\Phi(\cdot)$ is a "hybrid" fundamental matrix in the sense of initial value problem ((3)).

$$\begin{aligned}&\Phi(t_i,t_j) = \prod_{k=j}^{i}\Phi_{k+1}(t_{k+1},t_k) := \\ &\Phi_i(t_i,t_{i-1})\Phi_{i-1}(t_{i-1},t_{i-2})\Phi_{i-2}(t_{i-2},t_{i-3})\dots\Phi_{j+1}(t_{j+1},t_j)\end{aligned}$$

and $\Phi_i(\cdot)$ is the transfer function of the subsystem from the location q_i.

Using the correspondence between the solutions $x^{opt}(\cdot)$ and $y^{opt}(\cdot)$ of the initial value problems (1) and (3), see [6] we are now able to formulate the necessary optimality conditions for the original problem (2).

Theorem 4: Under the conditions of Theorem 1 there exist a function $p_i(\cdot)$ from $\mathbb{W}_m^{1,\infty}(0,t_f)$ and a non-zero (Lagrange) vector $a = (a_1,\dots,a_{r-1})^T \in \mathbb{R}^{r-1}$ such that

$$\begin{aligned}&\frac{d}{dt}p_i(t) = -A_{q_i}^T(t)p_i(t) + S_{q_i}x_i^{opt}(t) \text{ a.e. on } \left[t_{i-1}^{opt}, t_i^{opt}\right] \\ &p_r(t_f) = -S_f x_r^{opt}(t_f)\end{aligned} \tag{7}$$

and $p_i(t_i^{opt}) = p_{i+1}(t_i^{opt}) + a_i b_{q_i,q_{i+1}}$, where $i = 1,\dots,r-1$. Moreover, for every admissible control $u(\cdot) \in \mathscr{U}$ the Hamiltonian

$$\begin{aligned}&H_{q_i}(t,x,u,p) = \left\langle p_i, A_{q_i}(t)x_i + B_{q_i}(t)u_i\right\rangle - \\ &\frac{1}{2}x_i^T S_{q_i}x_i - \frac{1}{2}u_i^T R_{q_i}u_i\end{aligned}$$

satisfies the following maximization conditions

$$\begin{aligned}&\max_{v\in U\times\Theta} H_{q_i}\left(t, x^{opt}(t), v^{opt}(t), p(t)\right) = \\ &H_{q_i}\left(t, x^{opt}(t), v^{opt}(t), p(t)\right),\ t \in \left[t_{i-1}^{opt}, t_i^{opt}\right),\end{aligned} \tag{8}$$

where $i = 1, \ldots, r$ and $p(t) := \sum_{i=1}^{r} \beta_{[t_{i-1}^{opt}, t_i^{opt})} p_i(t)$ for all time instants $t \in [0, t_f]$.

Note that the function H is a real Hamiltonian in the sense of the auxiliary OCP (4). On the other side this function does not represent the Hamiltonian of the original OCP (2). As shown in [15] the "full" Hamiltonian (in the sense of the auxiliary problem (4)), namely, the function

$$\tilde{H}^{opt}(t) := \sum_{i=1}^{r} \beta_{[t_{i-1}^{opt}, t_i^{opt})}(t) H_{q_i}(t, y^{opt}(t), v^{opt}(t), \psi(t))$$

computed for optimal pair $(v^{opt}(\cdot), X^{opt}(\cdot))$ and for the corresponding adjoint variable $\psi(\cdot)$ is a continuous function. Using the equivalent representation of the initial impulsive hybrid system from (2), we can write this fact in the following form:

$$\begin{aligned} &\langle \psi_i(t), A_{q_i}(t) x_i^{opt}(t) + B_{q_i}(t) u_i^{opt}(t) \rangle - \\ &\langle \psi_{i+1}(t), A_{q_{i+1}}(t) x_{i+1}^{opt}(t) + B_{q_{i+1}}(t) u_{i+1}^{opt}(t) \rangle = \\ &\frac{1}{2} x_i^T(t) S_{q_i} x_i(t) + \frac{1}{2} u_i^T(t) R_{q_i} u_i(t) - \\ &\frac{1}{2} x_{i+1}^T(t) S_{q_{i+1}} x_{i+1}(t) - \frac{1}{2} u_{i+1}^T(t) R_{q_{i+1}} u_{i+1}(t) \end{aligned} \tag{9}$$

for all $i = 1, \ldots r-1$ and $t \in [0, t_f]$. Finally, note that the correct Hamiltonian in the sense of the original OCP (2) does not posses this continuity properties.

III. The Riccati-Formalism for Impulsive Hybrid Linear Quadratic OCPs

In this section we extend the well know Riccati techniques to the LQ hybrid OCP of the type (2). First Let us consider the linear boundary value problem (3), (5) for $U \equiv \mathbb{R}^m$. The maximization condition (6) from the above MP (Theorem 3) with respect to the first variable u of the full control vector v implies

$$u_i^{opt}(t) = R_{q_i}^{-1}(t) B_{q_i}^T(t) \psi_i(t), \; t \in [t_{i-1}^{opt}, t_i^{opt}] \tag{10}$$

Using this representation of an optimal control and the basic facts from the theory of linear differential equations, we now compute (similar to [8], [11]) an optimal control $u^{opt}(\cdot)$ for (4) in the form of a partially linear feedback control law

$$u^{opt}(t) = -\sum_{i=1}^{r} \beta_{[t_{i-1}, t_i)}(t) C_i(t) (y_i(t) + \theta_{q_i} \eta(t - t_i)) \tag{11}$$

where $C_i(t) = R_{q_i}^{-1}(t) B_{q_i}^T(t) P_i(t)$ is a partial gain matrix. For a LQ hybrid OCP governed by a linear hybrid system with autonomous location transitions we can find the matrix P_i as a solution to the differential equation of the Riccati type. This equation (written for every current location q_i) gives a rise to the dynamical behavior of this matrix. As shown in [7] the global Riccati matrix given by the following relation $P(t) = \sum_{i=1}^{r} \beta_{[t_{i-1}, t_i)}(t) P_i(t)$ is a discontinuous matrix-function. Analogously to the above-mentioned case of a hybrid LQ problem in the absence of state jumps, for every location $q_i \in Q$ and for almost all $t \in (t_{i-1}^{opt}, t_i^{opt})$ we obtain the following Riccati-type equation

$$\begin{aligned} &\dot{P}_i(t) + P_i(t) A_{q_i}(t) + A_{q_i}^T(t) P_i(t) - \\ &P_i(t) B_{q_i}(t) R_{q_i}^{-1}(t) B_{q_i}^T(t) P_i(t) + S_{q_i} = 0. \end{aligned} \tag{12}$$

Note that equation (12) describes the dynamical behavior of the matrix $P_i(\cdot)$. In contrast to the classical LQ-theory, this behavior is initially determined on some open time intervals $(t_{i-1}^{opt}, t_i^{opt})$. Equation (12) makes it possible to compute a partial Riccati matrix $Pi(\cdot)$ only in the case if $y_i(t) + \theta_{q_i} \eta(t - t_i) \notin \mathrm{Ker}(\mathrm{Ricc}_i(t))$ and $y_i(t) \neq -\theta_{q_i} \eta(t - t_i)$ for all $t \in (t_{i-1}^{opt}, t_i^{opt})$. Here $\mathrm{Ricc}_i(\cdot)$ is the right hand side of the Riccati equation (12).

For the time instants $t_i \in \tau$ we have the following jump condition (see [7]):

$$\left[P_{i+1}(t_i^{opt}) - P_i(t_i^{opt}) \right] (y_i(t_i^{opt}) + \theta_i^{opt}) = a_i b_{q_i, q_{i+1}}. \tag{13}$$

A symmetric discontinuous global Riccati matrix, see [7], $P(t) = \sum_{i=1}^{r} \beta_{[t_{i-1}, t_i)}(t) P_i(t)$ which satisfies all equations (12) and the boundary (terminal) condition $P(t_f) = S_f$ determines the optimal feedback dynamics of (3). Note that the corresponding global Riccati matrix-function is in general a discontinuous function even in the case of a LQ-type OCP governed by hybrid systems with autonomous location transitions (see [4]). In the case of a LIHS the discontinuity of the global matrix $P(\cdot)$ is also caused by the state jump of the amplitude θ_i in the location $q_i \in Q$. Clearly, the above partially linear feedback control function u^{opt} is a strongly discontinuous (piecewise continuous) function of x. Using the relation between variables x and y, we now are able to rewrite the optimal feedback control in the sense of the original impulsive hybrid OCP (2)

$$u^{opt}(t) = -C(t) x(t) = -\sum_{i=1}^{r} \beta_{[t_{i-1}, t_i)}(t) C_i(t) x_i(t)$$

where C_i is the same partial gain matrix as in the case of problem (4) and $P_i(\cdot)$ is the solution to the above Riccati matrix equation (12). Finally, note that similarly to (4) the optimal feedback control strategy for the original impulsive hybrid OCP (2) is an piecewise continuous function of x. The discontinuity points of this function are determined by switching times $t_i \in \tau$ as well as by the state jumps $\theta_i \in \Theta$.

The maximization condition (6) from Theorem 3 with respect to the second variable θ of the full control vector v are equivalent to the following sequence of finite dimensional maximization problems

$$\begin{aligned} &\max_{\theta} H_{q_i}(t_i^{opt}, Y^{opt}, (u^{opt}(t_i^{opt}), \theta), \psi(t_i^{opt})) \\ &\text{subject to } \|\theta_{q_i}\| \leq \Theta_{q_i}, \; i = 1, \ldots, r-1 \end{aligned} \tag{14}$$

Since the functions H_{q_i} are a concave (quadratic with respect to θ) functions, the optimal solution for every problem (14) can belongs to the interior $\mathrm{int}\{\mathcal{F}\}$ of the admissible set

$$\mathcal{F} := \{ \xi_{q_i} \in \mathbb{R}^n \; : \; ||\xi_{q_i}|| \leq \Theta_{q_i} \}$$

or satisfies the condition $\left\|\theta_{q_i}^{opt}\right\| = \Theta_{q_i}$. The auxiliary quadratic optimization problem ((14)) can be solved by well-known effective algorithms from the theory of the quadratic programming (see e.g., [14] for details). Note that the maximization problem from Theorem 3 is determined as a minimization problem on the Cartesian product $U \times \Theta$. Since the first variable of the control vector v takes the value in full space $\mathbb{R}^m$ and the second value is a finite dimensional component, we can separate the above maximization problem for the Hamiltonian H and consider two independent maximization procedures.

IV. Numerical Approach to Optimization of Impulsive Hybrid Systems

Relations (13), (12) and the affine restrictions

$$m_{q_i,q_{i+1}}(x) = b^T_{q_i,q_{i+1}} x + c_{q_i,q_{i+1}} = 0,$$

provide a basis for an effective numerical treatment for the LQ problem (2). As evident the Riccati-type equations (12) determine the Riccati matrix $P_i(t)$ for every t from open time intervals $(t_{i-1}^{opt}, t_i^{opt})$. The continuity property of the function $\tilde{H}^{opt}(\cdot)$ introduced above makes it possible to derive the necessary conditions for computing the value $P_i(t_i^{opt})$ for $i = 1,...,r-1$. Using the continuity formula (9), we obtain the following equation

$$\begin{aligned}&(y_i^{opt}(t_i^{opt}) + \theta_{q_i}^{opt})^T (P_i(t_i^{opt}) D_{q_i}(t_i^{opt}) P_i(t_i^{opt}) - \\ &2P_i(t_i^{opt}) A_{q_i}(t_i^{opt}) - F_{q_i,q_{i+1}}(t_i^{opt}))(y_i^{opt}(t_i^{opt}) + \theta_{q_i}^{opt}) = 0,\end{aligned} \tag{15}$$

where

$$\begin{aligned}F_{q_i,q_{i+1}}(t) = 2\left(S_{q_{i+1}}(t) - S_{q_i}(t)\right) + \\ P_{i+1}(t) D_{q_{i+1}}(t) P_{i+1}(t) + 2P_{i+1}(t) A_{q_{i+1}}(t)\end{aligned}$$

and $D_{q_i}(t) = B_{q_i}(t) R^{-1}_{q_i}(t) B^T_{q_i}(t)$. Note that from the jump condition (13) and nontriviality of the vector of Lagrange multipliers a it follows that there is at least one index i such that $y^{opt}(t_i^{opt}) + \theta_{q_i}^{opt} \neq 0$. Moreover, under the following assumption $y_i(t) + \theta_{q_i}\eta(t - t_i) \notin \mathrm{Ker}(\mathrm{RiccAlg}_i(t))$ equation (15) is equivalent to the classical partial algebraic Riccati equation

$$P_i(t_i^{opt}) D_{q_i}(t_i^{opt}) P_i(t_i^{opt}) - 2P_i(t_i^{opt}) A_{q_i}(t_i^{opt}) - F_{q_i,q_{i+1}}(t_i^{opt}) = 0.$$

Here $\mathrm{RiccAlg}_i(t)$ denotes the right hand side of the above equation. Evidently, (15) and the presented algebraic Riccati equation describe the discontinuous impulsive "dynamics" of state jumps to the optimal switching times.

We now are able to summarize a general conceptual computational algorithm for the numerical treatment of the optimal partially linear feedback control in the auxiliary hybrid LQ problem (4). Note that in the algorithm presented below an approximating control $v^{appr}(\cdot)$ to $v^{opt}(\cdot)$, the corresponding trajectory $y^{appr}(\cdot)$ and the sequence τ^{appr} to τ^{opt} are assumed to be given. The elements of τ^{appr} approximate the optimal switching times $t_i^{opt} \in \tau^{opt}$ for every $i = 1,...,r-1$. The control $v^{appr}(\cdot)$ the trajectory $y^{appr}(\cdot)$, the sequence τ^{appr} and the associated sequence of the corresponding locations can be obtained in various ways, for instance, with help of the gradient-based algorithms proposed in [4], or using the optimality zone algorithms from [15].

Algorithm 1:
1) Consider an initial pair $(v^0(\cdot), y^0(\cdot))$, where $v^0 \equiv v^{appr}$ and $y^0 \equiv y^{appr}$, an initial sequence $\tau^0 \equiv \tau^{appr}$, the corresponding sequence of locations and the terminal condition $P(t_f) = S_f$ for a given LIHS. Set $k = 1$ and $l = 1$.
2) With help of the inverted-time integrating procedure compute the value $P_r(t_{r-1}^{opt})$ of the partial Riccati matrix P_r as a solution of (12). Using (15) we calculate the Riccati matrix $P_{r-1}(t_{r-1}^{opt})$.
3) By the inverted-time integrating solution define $P_{r-k}(t_{r-k-1}^{opt})$, increase k by one. If $k = r-1$, then go to Step 4. Otherwise go to Step 2.
4) Complete all partial Riccati matrices $P_i(\cdot)$ and define the corresponding partial gain matrices

   $$C_i(t) = R^{-1}_{q_i}(t) B^T_{q_i}(t) P_i(t).$$

   Compute the quasi-optimal (in the sense of the above approximations) piecewise feedback control function from (11).
5) Complete all the partial Hamiltonian functions $H_{q_i}(t_i^{l-1}, Y^{l-1}, (u^l(t_i^{l-1}), \theta), \psi(t_i^{l-1}))$ for $i = 1,...,r$. Here $\psi(\cdot)$ is a solution of the corresponding adjoint system (5). Solve the auxiliary maximization problem (14) and compute θ^l as a solution to (14) with $H_{q_i}(t_i^{l-1}, Y^{l-1}, (u^l(t_i^{l-1}), \theta), \psi(t_i^{l-1}))$.
6) Using the obtained quasi-optimal feedback control low and the solution to (14), compute the corresponding trajectory $y^l(\cdot)$ and define the hybrid trajectory pair $Y^l(\cdot)$ for problem (4). Determine the new approximating sequence τ^l from the conditions

   $$t_i^l := \min\{t \in [0, t_f] \ : \ y^l(t) \bigcap M_{q_i,q_{i+1}} \neq \emptyset\}.$$

   where $i = 1,...,r-1$. Finally, increase l by one and go to Step 2.

Example 1: Let us consider a LIHS with one switching. The dynamics of the system is given by two linear equations associated with the corresponding location

$$\begin{aligned}\dot{x}_1 &= x_1(t) + u_1(t) + \theta\delta(t - t_1) & \forall t \in [0, t_1] \\ \dot{x}_2 &= -x_2(t) + 2u_2(t) & \forall t \in [t_1, t_f]\end{aligned}$$

where t_1 is a switching time and $x(0) = -0.04$. Lets considered various switching manifolds which are an affine-linear manifolds to show the different values that can take θ

$$M^{(1)}_{1,2}(x) = x + 0.3,\ M^{(2)}_{1,2}(x) = x + 0.1,\ M^{(3)}_{1,2}(x) = x + 0.05,$$

and the value of the state jump is determined by the condition $||\theta|| \leq 0.1$. Our aim is to minimize the quadratical cost function

$$J(u(\cdot), x(\cdot)) = \frac{1}{2}\sum_{i=1}^{2} \int_{t_i-1}^{t_i} (x_i^2(t) + u_i^2(t))dt$$

Since the presented example deals with the one-dimensional model, the Algorithm 1 can be sufficiently simplified. Note that in this one-dimensional situation a solution of (14) can be found from the condition

$$\max_{\theta\in[-0.1,0.1]} H_{q_i}\left(t_1, Y^{opt}, (u^{opt}(t_1), \theta), \psi(t_1)\right).$$

Hence $\theta^{opt} \in [-0.1, 0.1]$ and an optimal value can be found by an easy sorting procedure. Applying the simplified variant of the proposed Algorithm 1 we obtain the next optimal cost values for each switching rule

$$J^{opt^{(1)}} = 0.0791,\ J^{opt^{(2)}} = 0.0054,\ J^{opt^{(3)}} = 4.8455 \times 10^{-4}$$

with optimal jumps $\theta^{opt^{(1)}} = 0.1$, $\theta^{opt^{(2)}} = 0.1$, $\theta^{opt^{(3)}} = 0.05$, respectively. Comparison between each of the trajectories is shown in the Fig. 1 and the optimal control for the first switching rule is presented in Fig. 2.

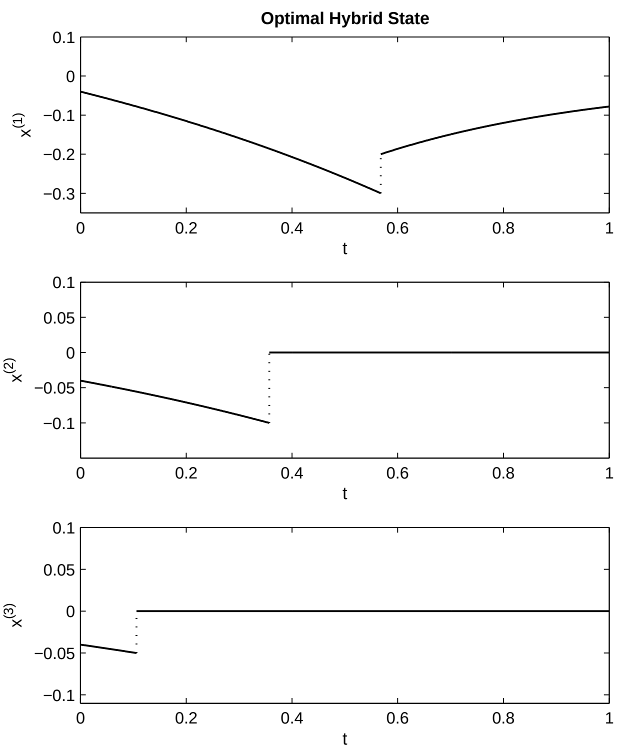

Fig. 1. Optimal trajectory for 3 different switching rules

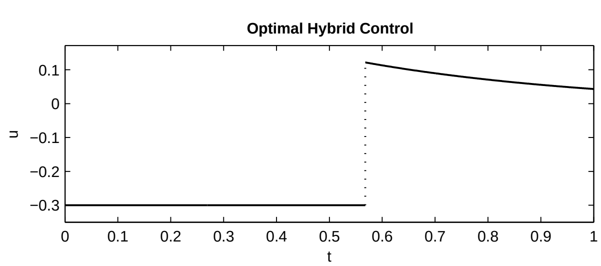

Fig. 2. Optimal Control for $M_{1,2}^{(1)}$

Now lets consider the following matrix example

Example 2: Let us consider a LIHS with one switching. The dynamics of the system is given by two linear equations associated with the corresponding location

$$\dot{x}_1 = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} x_1(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_1(t) + \theta\delta(t - t_1)$$
$$\forall t \in [0, t_1]$$
$$\dot{x}_2 = \begin{bmatrix} -0.5 & 0 \\ 0 & -3 \end{bmatrix} x_2(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_2(t)$$
$$\forall t \in [t_1, t_f]$$

where t_1 is a (unknown) switching time and $x(0) = (1, -1)^T$. The switching manifold is an affine-linear manifold

$$M_{1,2}(x) = x_1^1 + x_1^2 - 3$$

and the value of the state jump is determined by the condition $||\theta|| \leq 1$. Our aim is to minimize the quadratical cost function

$$J(u(\cdot), x(\cdot)) = \frac{1}{2}\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i} (x_i^2(t) + u_i^2(t))dt$$

Applying the proposed Algorithm 1 we obtain an optimal cost of $J^{opt} = 0.3153$ and a optimal jump of $\theta^{opt} = (-1, 0)^T$. The behavior of the system is shown in Fig. 3.

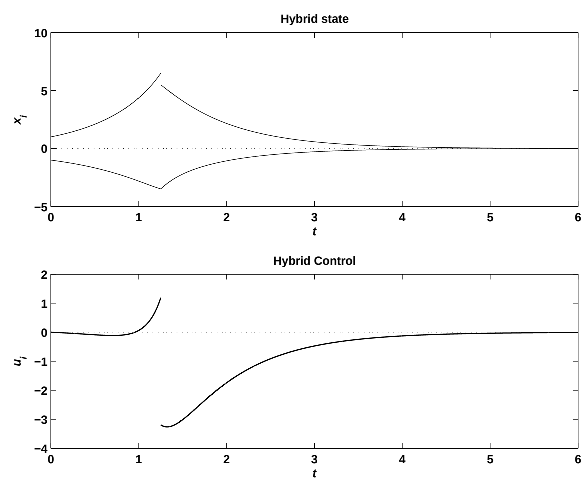

Fig. 3. Optimal trajectory for the ILHS

The above computational results are obtained using the MATLAB® package.

V. Conclusion

In this contribution, we establish the natural relationship between the hybrid MP and the Bellman DP approach for a family of impulsive hybrid LQ problems. Using a simple transformation of the original impulsive control system, we formulate necessary optimality conditions for the above-mentioned LQ problem and also derive the associated Riccati-type equation. This Riccati-type formalism provides a basis for an implementable numerical algorithm which can be applied to the original impulsive hybrid LQ problem under consideration. In different to the conventional LQ-based computational schemes, the proposed numerical method for the hybrid LQ-optimization uses the hybrid trajectory information. In some specific cases one can apply a simple version of Algorithm 1 and omit the forward integration procedure. Note that a simplified version of the proposed algorithm was applied to an illustrative example. The consistency analysis of the proposed numerical method and the corresponding (theoretical or implementable) convergence results belong to further works. Finally, note that the theoretical and computational approaches presented in this paper can be applied to some other classes of hybrid LQ-type OCPs.

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